List of topics for the preliminary exam in algebra

1 Basic concepts

1. Binary relations. Reflexive, symmetric/antisymmetryc, and transitive relations. Order and equivalence relations. Equivalence classes.
2. Partially ordered sets. Smallest, largest, minimal, and maximal elements in a poset. Chain conditions. Zorn's Lemma.

2 Group theory

1. Groups. Definitions, basic properties, and examples.
2. Subgroups. Normal subgroups. Examples: normalizer, centralizer, and center. Subgroups generated by a subset, generating sets of a group. Subgroups of cyclic groups.
3. Homomorphisms of groups and their properties. Monomorphisms, epimorphisms, isomorphisms, and automorphisms. Classification of cyclic groups up to isomorphism.
4. Kernels of homomorphisms. Cosets and quotient groups. Homomorphism theorems.
5. The Lagrange theorem.
6. Direct products and direct sums of groups. Basic properties and examples.
7. Symmetric groups. Parity of a permutation. Decomposing a permutation into a product of transpositions. Cycle decomposition of a permutation. Computing parity of a product of disjoint cycles. Alternating groups.
8. Group actions on sets. Transitive, faithful, free, and regular actions. Stabilizers. Isomorphic actions. Classification of transitive and regular actions up to isomorphism. The Cayley theorem.
9. The orbit-stabilizer theorem. Application: every finite p-group has non-trivial center. Groups of order p^2.
10. Sylow's theorems. Examples of Sylow subgroups in cyclic groups, $GL_n(\mathbb{Z}_p)$, and S_p for a prime p. Application: Wilson's theorem.

11. Extensions of groups. Split extensions.
12. Classification of groups of order pq and other "small" groups.
13. Abelian groups. Free abelian groups and their bases. Subgroups of free abelian groups.
14. The decomposition theorem for finitely generated abelian groups.
15. Simple groups. Simplicity of A_n for $n \geq 5$.
16. Nilpotent groups and their basic properties. Examples: finite p-groups and $UT_n(R)$, where R is a commutative unital ring.
17. Nilpotency criterion for finite groups in terms of Sylow subgroups.
18. Solvable groups and their basic properties. Solvability of triangular matrix groups. The Burnside problem on torsion groups and its solution for solvable groups.
19. Free groups: the constructive definition and the universal property.
20. Normal closures and groups presentations. Examples.
21. Cayley graphs.
22. Subgroups of free groups. The Nielsen-Schreier formula.

3 Basic ring and field theory

1. Rings. Basic properties and examples. Zero divisors, nilpotent elements, and idempotents. Invertible elements and the group of units of a ring. Group rings. Direct products of rings. Subrings.
2. Integral domains and fields. Fields of quotients. Characteristic of a field.
3. Ring homomorphisms, ideals, and quotient rings.
4. Isomorphism theorems for rings.
5. Simple rings and ring quotients by maximal ideals.

4 Polynomials over fields and Galois theory

1. Rings of polynomials.
2. Polynomials over a field. Division algorithm. Applications: little Bézout's theorem, the number of roots of a polynomial over a field, polynomials versus polynomial functions. Groups of units of finite fields.

3. Divisibility in rings; *gcd* and *lcm*. Computing the *gcd* and *lcm* of a set of polynomials over a field. Application: all ideals in $K[x]$, where K is a field, are principal.

4. Reducible and irreducible polynomials over a field. Decomposing a polynomial into a product of irreducible polynomials. Quotients by ideals generated by irreducible polynomials.

5. Existence of the splitting field of a polynomial. Algebraically closed fields.

6. The fundamental theorem of algebra. Classification of irreducible polynomials over $\mathbb{R}$.

7. Irreducibility of polynomials over $\mathbb{Z}$ and $\mathbb{Q}$. The Gauss lemma and Eisenstein's criterion.

8. Algebraic and transcendental extensions. Degrees of extensions. Classification of simple extensions.

9. Ruler-and-compass construction. The problems of doubling the cube and constructing the regular pentagon.

10. Splitting fields: existence and uniqueness.

11. Classification of finite fields and their subfields. Frobenius lemma, automorphisms of finite fields.

12. Algebraic closures. Existence and uniqueness of the algebraic closure of a field.

13. Normal extensions: equivalent definitions and examples. Galois groups. Representing the Galois group of the splitting field of a polynomial by permutations.

14. Separable polynomials. Separability of irreducible polynomials over finite fields and fields of characteristic 0. Separable extensions: examples and non-examples.

15. Simple extensions and Artin's primitive element theorem.

16. Galois extensions. The Galois group of a polynomial. Order of the Galois group of a finite extension. Examples of extensions with Galois groups $\mathbb{Z}_2^n$, S_3, S_5.

17. The fundamental theorem of Galois theory.

18. Solvability of algebraic equations in radicals and Galois groups. Unsolvability of equations of degree 5 in radicals.

19. The inverse Galois problem. Examples of extensions with Galois group $\mathbb{Z}_n$. Theorems of Hilbert and Shafarevich (without proof).

5 Advanced topics in ring theory

1. Euclidean domains and division algorithm. Examples: rings of polynomials, Gaussian integers.
2. Principal ideal domains. Every Euclidean ring is a PID. $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a PID but not Euclidean.
3. Irreducible and prime elements in integral domains. Associates. Unique factorisation domains. Examples of integral domains with non-unique factorization.
4. Rings of polynomials over UFDs. Relation between PIDs and UFDs.
5. Noetherian rings. The Hilbert basis theorem.
6. Finite division rings. The little Weddeburn Theorem.

6 Modules

1. Modules over unital rings. Submodules. Basic properties and examples. Homomorphisms of modules, kernels and quotients.
2. Free modules over PIDs and their submodules. Torsion free modules.
3. The decomposition theorem for finitely generated modules over PID's. The Jordan normal form of a matrix over an algebraically closed field. Minimal and characteristic polynomials of a matrix. The Cayley-Hamilton Theorem.
4. Injective and projective modules.

7 Associative algebras

1. Associative algebras over fields. Subalgebras, ideals, homomorphisms, and quotients.
2. Dimension of an algebra. Finite dimensional algebras and their matrix representations.
3. Division algebras. Finite dimensional division algebras and zero divisors. Finite dimensional division algebras over algebraically closed fields.
4. Real division algebras. Quaternions. Frobenius Theorem.
5. Central simple algebras. Simplicity of matrix algebras. The Artin-Weddeburn Theorem.
6. Free associative algebras.

8 Topics in Representation theory

1. Linear and matrix representations of groups. Invariant subspaces and subrepresentations. Irreducible representations. The sum of representations.
2. Group representations and modules, Maschke's theorem.
3. Equivalent representations. The number of non-equivalent irreducible complex representations of a finite group and their dimensions.
4. Constructing irreducible complex representations of finite abelian groups and of some "small" finite groups.

9 Topics in universal algebra

1. Lattices and Complete lattices.
2. Algebras (in the sense of universal algebra).
3. Homomorphisms and subalgebras. Congruence relations. Homomorphism theorems.
4. Varieties of Algebras.
5. Birkhoffs variety theorem (illustrated and proved in the context of group theory).

10 Recommended textbooks

Most topics are covered in either of the following three books:

[1] J. Rotman, A First Course in Abstract Algebra, Pearson, 2005.

[2] J. B. Fraleigh, A First Course in Abstract Algebra. Pearson, 2002.

[3] J. Gallian, Contemporary Abstract Algebra. Brooks Cole, 2012.

Good references for some additional topics and different perspectives are

[4] S. Lang, Algebra. Springer, 2005.

[5] B. L. van der Waerden, Modern Algebra. Frederick Ungar Publishing Co., 1949.