1. Suppose \( d \) is a metric for the topological space \( X \).
   a) Prove \( d : X \times X \to \mathbb{R} \) is continuous. (The space \( X \times X \) has the product topology induced by the metric \( d \) on \( X \).)
   b) Prove that if \( A \) and \( B \) are non-empty, compact and disjoint, then there is a point \( a \in A \) and a point \( b \in B \) such that \( d(a, b) = d(A, B) \). (Here \( d(A, B) = \text{g.l.b.}\{d(a, b) : a \in A \text{ and } b \in B\} \).)

2. Definition. Let \( H \) be the set of all sequences \((a_1, a_2, \ldots)\) of real numbers such that \( \sum_{i=1}^{\infty} a_i^2 < \infty \). A topology for this set is defined by the metric \( d((a_1, a_2, \ldots), (b_1, b_2, \ldots)) = \left[ \sum_{i=1}^{\infty} (a_i - b_i)^2 \right]^{\frac{1}{2}} \). The pair \((H, d)\) is called Hilbert space. The Hilbert cube \( Q \) is the subset of \( H \) consisting of all sequences \((a_1, a_2, \ldots)\) such that \( |a_i| \leq 1/i \) for each \( i \).
   If \( X_i = [-1, 1] \), for \( i = 1, 2, \ldots \), then prove that \( \prod_{i=1}^{\infty} X_i \) (with the product topology) is homeomorphic to the Hilbert cube.

3. Recall that if \( X \) and \( Y \) are topological spaces then \( \text{Map}(X, Y) \) is the set of all continuous functions from \( X \) to \( Y \). A subbasis for our topology on \( \text{Map}(X, Y) \) is sets of the following form where \( K \) is compact in \( X \) and \( U \) is open in \( Y \):
   \[
   M(K, U) = \{ f \in \text{Map}(X, Y) : f(K) \subset U \}.
   \]
   This is called the compact open topology on \( \text{Map}(X, Y) \).
   (i) Show that \( \text{Map}(X, Y) \) (with this topology) is \( T_1 \) when \( Y \) is \( T_1 \).
   (ii) Assume that \( Y \) is \( T_3 \). Show that if \( K \) is compact in \( X \), \( U \) is open in \( Y \) and \( f \in M(K, U) \) then there is an open set \( W \) in \( \text{Map}(X, Y) \) such that \( f \in W \subset \overline{W} \subset M(K, U) \). (This is the basic idea behind showing \( \text{Map}(X, Y) \) is \( T_3 \).)
4. If $G$ is a simply connected topological group and $H$ is a discrete normal subgroup of $G$, then prove that $G$ is a covering space for $G/H ≡ \{ Hg : g \in G \}$.

Let $e$ and 1 represent the identities for $G$ and $G/H$ respectively. First show that $p : G \to G/H$ is open.

As $H$ is discrete, there is a neighborhood $U$ of $e$ such that $U \cap H = \{ e \}$. You may use the following:

**Fact.** There is a neighborhood $V_e$ of $e$ such that $V_e \circ V_e^{-1} \subset U$.

Show that $p(V_e)$ is evenly covered to complete the proof.

5. If $p : E \to X$ is a covering map and $E$ is path connected, prove that for each $x, y \in X$, the cardinality of $p^{-1}(x)$ and $p^{-1}(y)$ are the same.

6. Let $X = \mathbb{T}^2 \# \mathbb{T}^2$ (the connected sum of two tori) and $Y = \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$ (the connected sum of three projective planes). Work 4 of the following 5.

   i) Give presentations for: $\pi_1(X)$ and $\pi_1(Y)$
   
   ii) Compute $H_0(X)$, $H_1(X)$ and $H_2(X)$; $H_0(Y)$, $H_1(Y)$ and $H_2(Y)$.
   
   iii) What is the universal cover of $X$? What is the universal cover of $Y$?

   iv) There is a “covering space” theorem that allows you to compute $\pi_2(X)$, $\pi_2(Y)$. State the theorem and compute these groups.

   v) Let $W$ be the one point union of $\mathbb{P}^2$ and $\mathbb{T}^2$ joined at the point $x$. Let $U$ be the universal cover of $W$ and $p : U \to W$ the covering map. Give a detailed description of $U$. The points of $p^{-1}(x)$ should be prominent in this description.

7. i) Let $X$ be the connected sum of a torus, a projective plane and another torus $X = \mathbb{T}^2 \# \mathbb{P}^2 \# \mathbb{T}^2$. Use van Kampen’s Theorem to compute a presentation of $\pi_1(X)$.

   ii) Compute $H_1(X)$ (you may use any theorem we covered in the course as long as you state it correctly!) Finitely generated abelian groups are the direct sum of a free abelian group with a finite abelian group. Write your final answer in this form.