ANALYSIS PRELIMINARY EXAM, AUGUST 2016

VANDERBILT UNIVERSITY

Name:

Directions. Please read carefully the following directions:

- This exam contains eight questions. **Choose five, and only five, problems to be graded.** Indicate clearly in the table below which problems you have chosen.
- **Only the problems you have chosen will be graded.**
- If you do not indicate clearly which problems have been chosen, it will be up to the graduate committee to choose them.
- If the statement of a problem is not clear (for instance, you think that the question is ambiguous, the notation is confusing, etc), state clearly how you interpret it, and then solve it accordingly.
- If you invoke a theorem without providing its proof, write its statement clearly.
- While there is not an absolute standard to decide which results you should establish in order to answer the questions versus those that you can quote without providing a proof, you are expected to demonstrate mathematical knowledge of the subject, and provide proofs for the questions that you are being specifically asked.
- By a **measure** it is meant a positive real measure unless stated otherwise.

Question	**Points**	**Grade?**
1		
2		
3		
4		
5		
6		
7		
8		
Total:		

Question 1. All measure related statements in this problem refer to the Lebesgue measure, which will be denoted by μ. Let $\{f_j\}_{n=1}^{\infty} \subset L^1(\mathbb{R})$ be a sequence such that $\{f_j\}_{n=1}^{\infty}$ converges almost everywhere to a function f. Assume that for every $\varepsilon > 0$, there exist a measurable set $A \subseteq \mathbb{R}$, a non-negative function $h \in L^1(\mathbb{R})$, and an integer $J \geq 1$ such that

$$\int_{A^c} |f_j| \, d\mu \leq \varepsilon$$

for every $j \geq J$, and $|f_j(x)| \leq h(x)$ for every $x \in A$ and every $j \geq J$ (A^c is the complement of A). Prove that $f \in L^1(\mathbb{R})$ and that f_j converges to f in $L^1(\mathbb{R})$.

Question 2. All measure related statements in this problem refer to the Lebesgue measure.

(a) Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and $g \in L^1(\mathbb{R}^n)$. Prove that the convolution $f * g$ of f and g is well-defined, $f * g \in L^p(\mathbb{R}^n)$, and

$$\| f * g \|_p \leq \| f \|_p \| g \|_1 .$$

(b) Prove that smooth (i.e., infinitely differentiable) functions are dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. For this you can assume, without proof, that the continuous functions with compact support are dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

Question 3. All measure related statements in this problem refer to the Lebesgue measure, which will be denoted by μ.

(a) Prove that if $f \in L^1(\mathbb{R}^n)$ and $K \subset \mathbb{R}^n$ is a compact set, then

$$\lim_{|x|\to\infty} \int_{x+K} |f| \, d\mu = 0,$$

where

$$x + K = \{x + y \,|\, y \in K\}.$$

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a uniformly continuous function and assume that $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty)$. Prove that

$$\lim_{|x|\to\infty} f(x) = 0.$$

Question 4. All measure related statements in this problem refer to measures on $\mathbb{R}^n$.

(a) State the definition of a Borel measure.
(b) State the definition of the s-dimensional Hausdorff measure on $\mathbb{R}^n$, $0 \leq s \leq n$.
(c) Prove that the s-dimensional Hausdorff measure is a Borel measure. To do this you can state and use, without proof, a well-known criterion to determine when a given measure is Borel.

Question 5. Let X be a locally compact Hausdorff topological space. Denote by $\mathscr{K}(X;\mathbb{C})$ the space of complex valued compactly supported continuous functions on X. Denote by $\mathscr{K}(X,A;\mathbb{C})$ the space of all $f \in \mathscr{K}(X;\mathbb{C})$ such that $\operatorname{supp}(f) \subseteq A$, where $\operatorname{supp}(f)$ denotes the support of f. For each compact set $K \subseteq X$, endow $\mathscr{K}(X,K;\mathbb{C})$ with the topology of uniform convergence. Endow $\mathscr{K}(X;\mathbb{C})$ with the inductive limit of locally convex topologies given by $\mathscr{K}(X,K;\mathbb{C})$ as K ranges over all compact sets of X.

(a) Prove that a linear form μ on $\mathscr{K}(X;\mathbb{C})$ defines a complex Radon measure on X if and only if for each $K \subseteq X$, there exists a constant M_K such that for every $f \in \mathscr{K}(X;\mathbb{C})$ with $\operatorname{supp}(f) \subseteq K$, we have
$$|\mu(f)| \leq M_K \sup_{x \in X} |f(x)|,$$
where $|\cdot|$ is the absolute value in $\mathbb{C}$.

(b) State the definition of a positive Radon measure.

(c) Let $\mathscr{K}(X;\mathbb{R})$ be defined as in (a), but with $\mathbb{C}$ replaced by $\mathbb{R}$. Prove that any positive linear form on $\mathscr{K}(X;\mathbb{R})$ defines a Radon measure on X.

Question 6. Let X be a locally compact σ-compact Hausdorff topological space. Let μ and ν be Radon measures on X. Assume that that ν has a density relative to μ, i.e., that there exists a locally μ-integrable function g on X such that $\nu = g\mu$. Prove that the following holds. For any $f : X \to \mathbb{R}$, $f \geq 0$, which is μ-integrable and ν-integrable, and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if h is a function satisfying $0 \leq h \leq f$ and $\int^* h\, d\mu \leq \delta$, then $\int^* h\, d\nu \leq \varepsilon$, where $\int^*$ denotes the upper integral.

Question 7. Let X be a locally compact metric space. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of Radon measures on X, and μ be a given Radon measure on X. Recall that we say that the sequence $\{\mu_n\}_{n=1}^{\infty}$ converges to μ in the vague topology if

$$\lim_{n\to\infty} \int f \, d\mu_n = \int f \, d\mu$$

for any function f on X that is continuous with compact support. Assume that $\{\mu_n\}_{n=1}^{\infty}$ converges to μ in the vague topology. Prove that for all Borel sets $A \subseteq X$ that are contained in a compact set and that satisfy $\mu(\partial A) = 0$, it holds that $\mu_n(A) \to \mu(A)$ as $n \to \infty$.

Question 8. Let X be a normed vector space and $\{x_n\}_{n=1}^{\infty} \subset X$ a sequence. Recall that the formal series $\sum_{n=1}^{\infty} x_n$ is called convergent if the sequence of the partial sums $\sum_{n=1}^{N} x_n$ converges in X as $N \to \infty$, and absolutely convergent if the sequence of the partial sums $\sum_{n=1}^{N} \| x_n \|$ converges in $\mathbb{R}$ as $N \to \infty$. Prove that X is a Banach space if and only if every absolutely convergent series is convergent.