Real Analysis Preliminary Exam, Summer 2015.

Instructions:

- **The exam duration is 3 hours.**
- **All answers must be properly justified.**
- **Give complete concise statements of any standard results that you use.**
- **Solve any 5 of the following 8 problems.**

1. a) Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with $|f(x) - f(y)| \leq |x - y|$, $\forall x, y \in \mathbb{R}$. If $E \subset \mathbb{R}$ is Lebesgue measurable, prove the image $f(E) = \{f(x) : x \in E\}$ is Lebesgue measurable.

b) Let $g : \mathbb{R} \to \mathbb{R}$ be continuous. If $B \subset \mathbb{R}$ is a Borel set, prove that the preimage $g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}$ is Lebesgue measurable. To do this, first show that $\Sigma = \{S \subset \mathbb{R} : g^{-1}(S) \text{ is Lebesgue measurable}\}$ is a σ-algebra.

2. a) State Fatou's Lemma and the Lebesgue Dominated Convergence Theorem.

b) Let f, f_n, g, g_n be nonnegative Lebesgue measurable functions on $\mathbb{R}^d$. Suppose $g, g_n \in L^1(\mathbb{R}^d)$, $f_n \to f$ a.e., $g_n \to g$ a.e., $0 \leq f_n \leq g_n$, and $\int g_n \to \int g$. Prove: $\int f_n \to \int f$.

c) Fix $1 < p < \infty$. Suppose $h_n, h, \varphi \in L^p(\mathbb{R}^d)$, $h_n \to h$ a.e., $|h_n| \leq \varphi$. Prove: $\|h_n - h\|_p \to 0$.

3. a) Suppose that f is absolutely continuous on $[0, 1]$, and let $V(x) = \mathrm{Var}(f, [0, x])$ denote the variation of f over $[0, x]$. Prove that V is absolutely continuous on $[0, 1]$.

b) For each of the following two statements, provide a proof if it is true, or provide a counterexample if it is false:

(i) If $E \subset \mathbb{R}^2$ is Lebesgue measurable, and the 1-dimensional Lebesgue measure of $E_y = \{x \in \mathbb{R} : (x, y) \in E\}$ equals zero for all $y \in \mathbb{R}$, then the 2-dimensional Lebesgue measure of E equals zero.

(ii) If $f : \mathbb{R}^2 \to \mathbb{R}$ is Lebesgue measurable, then for every fixed $y \in \mathbb{R}$, the function $f_y : \mathbb{R} \to \mathbb{R}$ defined by $f_y(x) = f(x, y)$ is Lebesgue measurable.

4. Fix $1 < p < \infty$. Let $K \in L^1(\mathbb{R})$ satisfy $\int K(x)\, dx = 1$ and define $K_\epsilon(x) = \frac{1}{\epsilon} K(\frac{x}{\epsilon})$. Given $f \in L^p(\mathbb{R})$ prove that $\lim_{\epsilon \to 0^+} \|f * K_\epsilon - f\|_p = 0$.

Remarks:

- Recall convolution is defined by $f * g(x) = \int f(x - y) g(y) dy = \int f(y) g(x - y) dy$.
- You may use strong continuity of translation in L^p, and may use basic properties of approximate identities and convolution, but should clearly state results that you use.

5. Fix $1 < p < \infty$. For each of the following statements, provide a proof if it is true, or provide a counterexample if it is false:

a) Convergence in $L^p(\mathbb{R})$ implies weak convergence in $L^p(\mathbb{R})$.

b) Weak convergence in $L^p(\mathbb{R})$ implies convergence in $L^p(\mathbb{R})$.

c) Convergence in $L^p(\mathbb{R})$ implies convergence in measure.

d) Convergence in measure implies convergence in $L^p(\mathbb{R})$.

6. a) State the Closed Graph Theorem and the Open Mapping Theorem.

b) Suppose that $A, B : L^2[0,1] \to L^2[0,1]$ are linear operators which satisfy

$$\forall f, g \in L^2[0,1], \quad \langle Af, g \rangle = \langle f, Bg \rangle.$$

Prove that A is a bounded operator.

c) Suppose that $\|\cdot\|$ and $|||\cdot|||$ are norms on a linear space X. Prove that if there exists $\alpha > 0$ such that

$$\forall x \in X, \quad \|x\| \leq \alpha\, |||x|||,$$

then there exists $\beta > 0$ such that

$$\forall x \in X, \quad |||x||| \leq \beta \|x\|.$$

7. a) State the Hahn-Banach Theorem for normed linear spaces over the scalar field $\mathbb{R}$.

b) Consider $L^\infty(\mathbb{R})$ as a Banach space over the scalar field $\mathbb{R}$. Prove that there exists a bounded linear functional T on $L^\infty(\mathbb{R})$ which is *not* given by integration against an element of $L^1(\mathbb{R})$. In other words, there exists $T \in (L^\infty(\mathbb{R}))^*$ that is not of the form $T(f) = \int fg\, dx$, for some $g \in L^1(\mathbb{R})$.

8. You may assume that all measure spaces in this problem are complete.

a) State a version of the Radon-Nikodym Theorem for measures.

b) Prove the Lebesgue Decomposition Theorem for measures. Namely, let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let ν be a σ-finite measure on $\mathcal{B}$. Prove that there exist measures ν_s and ν_{ac} on $\mathcal{B}$ such that:

$$\nu = \nu_s + \nu_{ac}, \quad \nu_{ac} << \mu, \quad \text{and} \quad \nu_s \perp \mu.$$