Topology Preliminary Exam, Fall 2018

Instructions: You have **3** hours to complete this exam. Work EXACTLY **6** of the following **7** problems. Each problem is to be worked on a separate sheet of paper with the problem number clearly listed at the top of the page. Clearly cross out work you do not want graded. All problems carry the same value, **16** points.

If you must use a main theorem from the course, be sure to state it correctly when you use it.

1. Suppose X is a compact metric space. Show for any open cover $\mathcal{U}$ of X there is a Lebesque number $l(\mathcal{U})$ for $\mathcal{U}$. I.e. there is number $l > 0$ such that for any $A \subset X$ with diameter(A) $\leq l$, there exists $U \in \mathcal{U}$ such that $A \subset U$.

2. Suppose Y is a metric space, and $f : [0,1] \to Y$ is continuous and onto. Given $\epsilon > 0$ prove there exists $\delta > 0$ such that if x and y are points of Y and $d(x,y) < \delta$ then there is a path from x to y in $B(x,\epsilon)$, the ϵ-ball about x in Y.

   Definition. A space X is T_3 if, given any closed set $D \subset X$ and point $x \in X - D$, there exist disjoint open sets U and V such that $D \subset U$, and $x \in V$. The space X is T_4 if, for each pair of disjoint closed sets A and B in X, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$. A space that is both T_3 and T_1 is called *regular*. A space that is both T_1 and T_4 is called *normal*.

   Definition. If $\mathcal{T}$ is a topology for X, and there is a countable basis for $\mathcal{T}$, then X, with this topology, is called a *second countable space*.

3. Prove that each second countable regular space is normal.

   Definition. If X and Y are topological spaces then $Map(X,Y)$ is the set of all continuous functions from X to Y. A subbasis for our topology on $Map(X,Y)$ is sets of the following form where K is compact in X and U is open in Y:

   $$M(K,U) = \{f \in Map(X,Y) : f(K) \subset U\}.$$

   This is called the *compact open topology* on $Map(X,Y)$.

4. Let X be a space, and let $\mathrm{Hom}(X)$ be the space of homeomorphisms of X onto itself (i.e. $\mathrm{Hom}(X) \subset \mathrm{Map}(X, X)$). $\mathrm{Hom}(X)$ is a group under the operation of composition. If X is compact and T_2 (and hence regular), then

   ($\mathrm{Hom}(X)$, composition, compact-open topology) is a topological group.

   Prove:

   i) The function $F : Hom(X) \times Hom(X) \to Hom(X)$, defined by $(g, h) \to g \circ h$, is continuous, and

   ii) the function $I : Hom(X) \to Hom(X)$ by $g \to g^{-1}$ is continuous.

   NOTE. If $(\hat{X}, p)$ is a covering space for X then assume X and $\hat{X}$ are path connected and locally path connected T_2-spaces.

5. Suppose $(\hat{X}, p)$ is a covering space for X, and α is a path in X. Let $x_0 = \alpha(0)$ and suppose that $\tilde{x}_0 \in p^{-1}(x_0)$, then there is a unique path $\tilde{\alpha}$ in $\hat{X}$ with $\tilde{\alpha}(0) = \tilde{x}_0$, and $p\tilde{\alpha} = \alpha$. (I.e. Prove the Unique Path Lifting Theorem.)

6. i) Up to homeomorphism, there are only two possible universal covers for the closed 2-dimensional surfaces. They are a) ______________ and b) ______________.

   ii) If the universal cover of a surface M is given by part a) above then the cardinality of $\pi_1(M)$ is ______________. If the universal cover of a surface M is given by part b) above then $\pi_1(M)$ has cardinality ______________.

   iii) Let $\mathbb{P}^2$ be the projective plane and S^1 a circle. Prove that any continuous function $f : \mathbb{P}^2 \times \mathbb{P}^2 \to S^1 \times S^1 \times S^1$ is homotopically trivial.

7. Let X be the wedge (one point union) of a circle and a projective plane.

   i) Compute $\pi_1(X)$, $\chi(X) =$ the Euler characteristic of X and $H_i(X)$ for $i \in \{0, 1, 2, 3\}$. Explain your answers.

   ii) Draw a picture and describe in words the universal cover of this X. Your picture should highlight all points in the universal cover that are mapped to the wedge point.