MODERN ALGEBRA $-January, 2016$

**1.** Let $p$ be a prime number. Prove that every group $G$ of order $p^3$ has an Abelian normal subgroup $N$ such that the factor group $G/N$ is cyclic.

**2.** (a) Are the groups $\mathbf{Z}_{72} \oplus \mathbf{Z}_{54}$ and $\mathbf{Z}_{18} \oplus \mathbf{Z}_{216}$ isomorphic ?

(b) Let $A$ and $B$ be free Abelian groups with bases $(a_1, a_2)$ and $(b_1, b_2, b_3)$, respectively. Is $A$ isomorphic to the group $B/C$, where $C$ is the cyclic group generated by $10b_1 - 5b_2 + 15b_3$ ?

**3.** Prove that arbitrary group of order 2015 is solvable.

**4.** Prove that the ring of Gaussian integers $\mathbf{Z}[i] = \{a + bi \mid a, b \in \mathbf{Z}\}$ has no homomorphic images of order 7.

**5.** Let $E$ be the splitting field of a polynomial $f(x) \in \mathbf{Q}[x]$ with $degree(f(x)) \leq 4$. Prove that the Galois group $Gal(E/\mathbf{Q})$ is not a cyclic group of order 6.

**6.** Suppose the extension $E$ of the field $\mathbf{Q}$ is obtained by adjoining to $\mathbf{Q}$ of finitely many square roots of integers from $\mathbf{Z}$. Prove that for arbitrary subfield $F$ of $E$, the extension $F/\mathbf{Q}$ is normal.

**7.** How many non-isomorphic semisimple complex alebras of dimension 10 do exist ? Explain, please.

**8.** Let $\Phi : S_3 \to GL(X)$ be an $n$-dimensional complex representation of the symmetric group $S_3$, where $n$ is an odd integer. Prove that there exists a non-zero vector $x \in X$ which is an eigenvector for every representation operator $\Phi(g)$, $g \in S_3$.