**Topology Preliminary Exam, Fall 2017**

**Instructions:** You have **3** hours to complete this exam. Work EXACTLY **6** of the following **7** problems. Each problem is to be worked on a separate sheet of paper with the problem number clearly listed at the top of the page. Clearly cross out work you do not want graded. All problems carry the same value, **20** points.

1. Suppose $d$ is a metric for the topological space $X$.

   a) Prove $d : X \times X \to \mathbb{R}$ is continuous. (The space $X \times X$ has the product topology induced by the metric $d$ on $X$.)

   b) Prove that if $A$ and $B$ are non-empty, compact and disjoint, then there is a point $a \in A$ and a point $b \in B$ such that $d(a, b) = d(A, B)$. (Here $d(A, B) = g.l.b.\{d(a, b)\colon\ a \in A \text{ and } b \in B\}$.)

2. **Definition**. Let $\mathcal{H}$ be the set of all sequences $(a_1, a_2, \ldots)$ of real numbers such that $\sum_{i=1}^{\infty} a_i^2 < \infty$. A topology for this set is defined by the metric $d((a_1, a_2, \ldots), (b_1, b_2, \ldots)) = [\sum_{i=1}^{\infty}(a_i - b_i)^2]^{\frac{1}{2}}$. The pair $(\mathcal{H}, d)$ is called *Hilbert space*. The *Hilbert cube* $\mathcal{Q}$ is the subset of $\mathcal{H}$ consisting of all sequences $(a_1, a_2, \ldots)$ such that $|a_i| \leq 1/i$ for each $i$.

   If $X_i = [-1, 1]$, for $i = 1, 2, \ldots$, then prove that $\Pi_{i=1}^{\infty} X_i$ (with the product topology) is homeomorphic to the Hilbert cube.

3. Recall that if $X$ and $Y$ are topological spaces then $Map(X, Y)$ is the set of all continuous functions from $X$ to $Y$. A subbasis for our topology on $Map(X, Y)$ is sets of the following form where $K$ is compact in $X$ and $U$ is open in $Y$:

   $$M(K, U) = \{f \in Map(X, Y) : f(K) \subset U\}.$$

   This is called the *compact open topology* on $Map(X, Y)$.

   (i) Show that $Map(X, Y)$ (with this topology) is $T_1$ when $Y$ is $T_1$.

   (ii) Assume that $Y$ is $T_3$. Show that if $K$ is compact in $X$, $U$ is open in $Y$ and $f \in M(K, U)$ then there is an open set $W$ in $Map(X, Y)$ such that $f \in W \subset \overline{W} \subset M(K, U)$. (This is the basic idea behind showing $Map(X, Y)$ is $T_3$).

4. If $G$ is a simply connected topological group and $H$ is a discrete normal subgroup of $G$, then prove that $G$ is a covering space for $G/H \equiv \{Hg : g \in G\}$.

   Let $e$ and 1 represent the identities for $G$ and $G/H$ respectively. First show that $p : G \to G/H$ is open.

   As $H$ is discrete, there is a neighborhood $U$ of $e$ such that $U \cap H = \{e\}$. You may use the following:

   **Fact.** There is a neighborhood $V_e$ of $e$ such that $V_e \circ V_e^{-1} \subset U$.

   Show that $p(V_e)$ is evenly covered to complete the proof.

5. If $p : E \to X$ is a covering map and $E$ is path connected, prove that for each $x, y \in X$, the cardinality of $p^{-1}(x)$ and $p^{-1}(y)$ are the same.

6. Let $X = \mathbb{T}^2 \# \mathbb{T}^2$ (the connected sum of two tori) and $Y = \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$ (the connected sum of three projective planes). Work 4 of the following 5.

   i) Give presentations for: $\pi_1(X)$ and $\pi_1(Y)$

   ii) Compute $H_0(X)$, $H_1(X)$ and $H_2(X)$; $H_0(Y)$, $H_1(Y)$ and $H_2(Y)$.

   iii) What is the universal cover of $X$? What is the universal cover of $Y$?

   iv) There is a "covering space" theorem that allows you to compute $\pi_2(X)$, $\pi_2(Y)$. State the theorem and compute these groups.

   v) Let $W$ be the one point union of $\mathbb{P}^2$ and $\mathbb{T}^2$ joined at the point $x$. Let $U$ be the universal cover of $W$ and $p : U \to W$ the covering map. Give a detailed description of $U$. The points of $p^{-1}(x)$ should be prominent in this description.

7. i) Let $X$ be the connected sum of a torus, a projective plane and another torus $X = \mathbb{T}^2 \# \mathbb{P}^2 \# \mathbb{T}^2$. Use van Kampen's Theorem to compute a presentation of $\pi_1(X)$.

   ii) Compute $H_1(X)$ (you may use any theorem we covered in the course as long as you state it correctly!) Finitely generated abelian groups are the direct sum of a free abelian group with a finite abelian group. Write your final answer in this form.