A ONE-DIMENSIONAL PROOF OF THE HÖLDER INEQUALITY USING THE HEAT EQUATION

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BRIEF. This paper displays an alternative approach to proving the Holder Inequality using supersolutions of the Heat Equation for a one-dimensional case.

The Hölder inequality is an inequality in mathematics that states that the integral of the product of two function is bounded by the product of their integrals raised to a power. In this paper presents an interesting proof of this inequality using supersolutions of the heat equation. This proof is based on the monotonicity formula for the heat equation presented in Tobias Colding's lecture notes at MIT.

1. INTRODUCTION.

In mathematics, the heat equation is a partial differential equation that models how heat diffuses through a region. A function u, solves the heat equation if

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t). \qquad (1.1)$$

The fundamental solution of the heat equation is called the heat kernel. It is defined by

$$H(x,y,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}. \qquad (1.2)$$

We can prove that this is a solution by substituting in both sides of the equation and proving equality.

Any solution $u: \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ of the heat equation (1.1) is determined by it's initial values $u(\cdot\,,0)$ by

$$u(x,t) = \int_{\mathbb{R}^n} H(x,y,t)u(y,0)dy, \qquad (1.3)$$

where H is the heat kernel (1.2). We define f and g to be supersolutions of the heat equation when

$$\partial_t f \geq \Delta f, \partial_t g \geq \Delta g \qquad (1.4)$$

In general, the heat equation has many applications in physics and mathematics. Brownian Motion, Black-Scholes equation, and Schrodinger's equation are heat type equations. More recently, the heat equation has also been the basis for methods in machine learning. Graph Laplacian methods for example are based on the theory behind the heat equation [1].

The Hölder inequality is an inequality involving integrals named after the German mathematician Otto Hölder. It is a tool used in the study of Lebesgue spaces and can produce powerful results. Most notably it can be used to prove the triangle inequality. The standard

proof for the Hölder Inequality involves Young's Inequality for for Products which states that if $p, q \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$ then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad (1.5)$$

for non-negative values of a and b. Furthermore, equality is achieved when $a^p = b^q$. Therefore, if we have positive functions $f(x)$ and $g(x)$ then

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q} \qquad (1.6)$$

Now let $X = \left(\int_S |f|^p d\mu\right)^{\frac{1}{p}}$ and let $Y = \left(\int_S |g|^q d\mu\right)^{\frac{1}{q}}$. We now set $a = \frac{f(x)}{X}$ and $b = \frac{g(x)}{Y}$, therefore we have,

$$\frac{|f(x)g(x)|}{XY} \leq \frac{|f(x)|^p}{pX^p} + \frac{|g(x)|^q}{qY^q} \qquad (1.7)$$

Integrating on both sides gives us,

$$\frac{1}{XY}\int |f(x)g(x)|d\mu \leq \frac{1}{pX^p}\int |f^p|d\mu + \frac{1}{qY^p}\int |g^q|d\mu \quad (1.8)$$

X^p and Y^q will be equal to $\int f^p d\mu$ and $\int g^q d\mu$ respectively. Therefore, this inequality reduces to

$$\int |f(x)g(x)| \leq (\int |f(x)|^p)^{\frac{1}{p}}(\int |g(x)|^q)^{\frac{1}{q}} \qquad (1.9)$$

This is the Hölder Inequality [2].

The main motivation behind this paper is to show an alternate proof of the Hölder Inequality involving the Heat Equation, and by extension will show us the relationship between these two seemingly unrelated topics. We will do this by using the supersolutions of the heat equation to prove the Hölder inequality. To do this we will start by proving a lemma; we will then proceed by defining the Central Limit Theorem, and will use both the results of the lemma and the Central Limit Theorem to prove the Hölder inequality.

2. PROOF OF THE HÖLDER INEQUALITY

A supersolution, in context of the heat equation, is a function that satisfies the heat equation and is greater than or equal to the actual solution at every point in the domain of space and time. The Maximum Principle states that that any solution of the heat equation is bounded by its initial condition. To prove the existence of supersolutions for the heat equation, the maximum principle can be applied to show that the supersolution bounds the actual solution [3].

Lemma 2.1. *Let f and g be positive supersolutions of the heat equation. In other words, $(\partial_t - \Delta)f \geq 0$ and $(\partial_t - \Delta)g \geq 0$. If $p, q \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$ then $(\partial_t - \Delta)\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right) \geq 0$, and equality implies $f = cg$ for some $c \in \mathbb{R}$.*

Proof. Let's define u to be a function such that $u = \log\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)$. We take the exponential of both sides, thus giving us $e^u = \left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)$. Therefore $(\partial_t - \Delta)\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right) = (\partial_t - \Delta)e^u$. Let's now define a function $e^{u(x,t)}$. If we take the partial of this function with respect to t, we get $\partial_t e^{u(x,t)} = e^{u(x,t)}\,\partial_t u(x,t)$. Now the Laplacian of this function will be given as, $\Delta e^{u(x,t)} = \frac{\partial^2 e^{u(x,t)}}{\partial x^2}$. We can rewrite this using Chain Rule as $\frac{\partial}{\partial x}\left(e^{u(x,t)}\frac{\partial u(x,t)}{\partial x}\right)$. We now apply product rule giving us

$\left(\frac{\partial}{\partial x}e^{u(x,t)}\right)\frac{\partial u(x,t)}{\partial x}+e^{u(x,t)}\frac{\partial^2 u(x,t)}{\partial x^2}$. This can be simplified to $e^{u(x,t)}\left|\frac{\partial u(x,t)}{\partial x}\right|^2+e^{u(x,t)}\frac{\partial^2 u(x,t)}{\partial x^2}$. We now can factor out the term $e^{u(x,t)}$ thus giving us that

$$\Delta e^{u(x,t)}=e^{u(x,t)}\left(\Delta u(x,t)+\left|\frac{\partial u(x,t)}{\partial x}\right|^2\right) \tag{2.2}$$

Applying this method to our original expression, $(\partial_t-\Delta)e^u$, where $u=\log\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)$, we get

$$(\partial_t-\Delta)\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)=(\partial_t-\Delta)e^u=e^u\left((\partial_t-\Delta)u-\left|\frac{\partial u}{\partial x}\right|^2\right) \tag{2.3}$$

We can expand this as

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\log\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)=\frac{\partial}{\partial x}\left(\frac{1}{p}\log f+\frac{1}{q}\log g\right)=\frac{1}{pf}\frac{\partial f}{\partial x}+\frac{1}{qg}\frac{\partial g}{\partial x} \tag{2.4}$$

From that we know that $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2}{\partial x^2}\log\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)$. From here, we get that $\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{1}{pf}\frac{\partial f}{\partial x}+\frac{1}{qg}\frac{\partial g}{\partial x}\right)$. This is equal to $\frac{1}{p}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{1}{q}\frac{\partial}{\partial x}\left(\frac{\frac{\partial g}{\partial x}}{g}\right)=\frac{1}{p}\left(\frac{\frac{\partial^2 f}{\partial x^2}}{f}-\frac{\left(\frac{\partial f}{\partial x}\right)^2}{f^2}\right)+\frac{1}{q}\left(\frac{\frac{\partial^2 g}{\partial x^2}}{g}-\frac{\left(\frac{\partial g}{\partial x}\right)^2}{g^2}\right)$. Now we compute $\partial_t(u)=\partial_t\left(\log f^{\frac{1}{p}}g^{\frac{1}{q}}\right)=\frac{\partial}{\partial t}\left(\frac{1}{p}\log f+\frac{1}{q}\log g\right)=\frac{1}{p}\frac{\partial}{\partial t}[\log(f)]+\frac{1}{q}\frac{\partial}{\partial t}[\log(g)]=\frac{1}{p}\frac{\partial}{\partial t}[f]+\frac{1}{q}\frac{\frac{\partial}{\partial t}}{g}[g]$. Therefore, we have

$$\Delta u=\frac{1}{p}\left(\frac{\Delta f}{f}-\frac{|\nabla f|^2}{f^2}\right)+\frac{1}{q}\left(\frac{\Delta g}{g}-\frac{|\nabla g|^2}{g^2}\right) \tag{2.5}$$

and

$$\partial_t u=\frac{\frac{\partial}{\partial t}[f]}{pf}+\frac{\frac{\partial}{\partial t}[g]}{qg} \tag{2.6}$$

Now that we have $\partial_t u$ and Δu by computation of terms we get

$$(\partial_t-\Delta)u=\frac{1}{p}\left(\frac{(\partial_t-\Delta)f}{f}+\frac{|\nabla f|^2}{f^2}\right)+\frac{1}{q}\left(\frac{(\partial_t-\Delta)g}{g}+\frac{|\nabla g|^2}{g^2}\right) \tag{2.7}$$

Therefore, since we know that f and g are supersolutions of the heat equation, we know that (2.7) will be $\geq\frac{1}{p}\frac{|\nabla f|^2}{f^2}+\frac{1}{q}\frac{|\nabla g|^2}{g^2}$. Now from 2.3 we have $(\partial_t-\Delta)\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)=(\partial_t-\Delta)e^u=\ e^u\left((\partial_t-\Delta)u-\left|\frac{\partial u}{\partial x}\right|^2\right)\geq e^u(\partial_t-\Delta)u=\frac{1}{p}\left(\frac{(\partial_t-\Delta)f}{f}+\frac{|\nabla f|^2}{f^2}\right)+\frac{1}{q}\left(\frac{(\partial_t-\Delta)g}{g}+\frac{|\nabla g|^2}{g^2}\right)\geq\frac{1}{p}\frac{|\nabla f|^2}{f^2}+\frac{1}{q}\frac{|\nabla g|^2}{g^2}-\left|\frac{1}{p}\frac{\nabla f}{f}+\frac{1}{q}\frac{\nabla g}{g}\right|$ (2.8) Now by (2.3), we have $(\partial_t-\Delta)e^u=e^u\left((\partial_t-\Delta)u-\left|\frac{\partial u}{\partial x}\right|^2\right)\geq\ e^u\frac{1}{p}\left(\frac{(\partial_t-\Delta)f}{f}+\frac{|\nabla f|^2}{f^2}\right)+\frac{1}{q}\left(\frac{(\partial_t-\Delta)g}{g}+\frac{|\nabla g|^2}{g^2}\right)\geq\frac{1}{p}\frac{|\nabla f|^2}{f^2}+\frac{1}{q}\frac{|\nabla g|^2}{g^2}-\left|\frac{1}{p}\frac{\nabla f}{f}+\frac{1}{q}\frac{\nabla g}{g}\right|^2$. Thus, we have, $e^{-u}(\partial_t-\Delta)e^u\geq\frac{1}{pq}\left(\frac{|\nabla f|^2}{f^2}+\frac{|\nabla g|^2}{g^2}-\frac{2}{fg}\langle\nabla f,\nabla g\rangle\right)$

Thus, this expression reduces down to $\frac{1}{pq}f^{\frac{1}{p}}g^{\frac{1}{q}}\left|\nabla\log\left(\frac{f}{g}\right)\right|^2$. Since f and g are positive supersolutions of the heat equation, this expression will be ≥ 0 and when equality does occur that implies that $\left|\nabla\log\left(\frac{f}{g}\right)\right|^2=0$. Therefore, $\log\left(\frac{f}{g}\right)=c$ so by taking the exponential on both sides we arrive to the expression $f=e^c g$ when equality holds.

3. THE CENTRAL LIMIT THEOREM

The Central Limit Theorem tells us that if we have a function $u(x,t)$ which solves the heat equation and we define the function

$$v(x,t)=t^{\frac{n}{2}}u(\sqrt{t}x,t) \tag{3.1}$$

then v converges to the Gaussian function

$$v(x,t)\to(4\pi)^{\frac{-n}{2}}e^{\frac{-|x|^2}{4}}\int\ u_0(y)dy \tag{3.2}$$

as $t\to\infty$ [4]. We will use this property and Lemma 2.1 to prove the Hölder Inequality.

4. THE HÖLDER INEQUALITY

Theorem 4.1. *Let h_1 and h_2 be functions with compact support. Let $\frac{1}{p}+\frac{1}{q}=1$, where $(p,q)\in(1,\infty)$. The Hölder Inequality for integrals claims that*

$$\int_R\ |h_1h_2|\leq\left(\int_R\ |h_1|^p\right)^{\frac{1}{p}}\left(\int_R\ |h_2|^q\right)^{\frac{1}{q}} \tag{4.2}$$

and equality implies that $h_1=ch_2$ for some $c\in\mathbb{R}$. Notice that the Hölder Inequality is a more general form of the Cauchy-Schwartz Inequality.

Claim 4.3. *If f and g decay rapidly at infinity, in other words the functions approach 0 as we approach infinity, then*

$$\frac{d}{dt}\int_R\ f^{\frac{1}{p}}g^{\frac{1}{q}}\geq 0 \tag{4.4}$$

Proof of Claim. Since f and g decay at infinity, the divergence theorem and differentiation under the integral sign imply

$$\frac{d}{dt}\int_R\ f^{\frac{1}{p}}g^{\frac{1}{q}}=\int_R\ (\partial_t-\Delta)\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right) \tag{4.5}$$

We know that $(\partial_t-\Delta)\left(f^{\frac{1}{p}}g^{\frac{1}{q}}\right)\geq\frac{1}{pq}f^{\frac{1}{p}}g^{\frac{1}{q}}\left|\nabla\log\left(\frac{f}{g}\right)\right|^2$, thus by Lemma 2.1 we know that $\frac{1}{pq}f^{\frac{1}{p}}g^{\frac{1}{q}}\left|\nabla\log\left(\frac{f}{g}\right)\right|^2\geq 0$, therefore proving that

$$\frac{d}{dt}\int_R\ f^{\frac{1}{p}}g^{\frac{1}{q}}\geq 0 \tag{4.6}$$

Proof of Hölder inequality. Let f and g be the positive solutions of the heat equation starting at $|h_1|^p$ and $|h_2|^q$, respectively. Then, let $u(x,t)=t^{\frac{n}{2}}f(\sqrt{t}x,t)$ and let $v(x,t)=t^{\frac{n}{2}}g(\sqrt{t}x,t)$. By applying the Central Limit Theorem, we have that $u(x,t)$ converges to

$$(4\pi)^{\frac{-n}{2}}e^{\frac{-|x|^2}{4}}\int\ |h_1|^p \tag{4.7}$$

Similarly, $v(x,t)$ converges to

$$(4\pi)^{\frac{-n}{2}}e^{\frac{-|x|^2}{4}}\int\ |h_2|^q \tag{4.8}$$

Since $\int f^{\frac{1}{p}}g^{\frac{1}{q}}=\int u^{\frac{1}{p}}v^{\frac{1}{q}}$. Then from (4.6) and (4.7) we have as $t\to\infty$

$$\int\ f^{\frac{1}{p}}g^{\frac{1}{q}}=\int\ u^{\frac{1}{p}}v^{\frac{1}{q}} \tag{4.9}$$

$$\to\left(\int\ |h_1|^p\right)^{\frac{1}{p}}\left(\int\ |h_2|^q\right)^{\frac{1}{q}}\left(\int\ (4\pi)^{\frac{-n}{2}}e^{\frac{-|x|^2}{4}}\right)^{\frac{1}{p}+\frac{1}{q}}$$

We know that $\frac{1}{p}+\frac{1}{q}=1$, so this simplifies into

$$\int f^{\frac{1}{p}} g^{\frac{1}{q}} \rightarrow (\int |h_1|^p)^{\frac{1}{p}} (\int |h_2|^q)^{\frac{1}{q}} \quad (4.10)$$

From Claim 4.3 we know that $\frac{d}{dt}\int_R f^{\frac{1}{p}} g^{\frac{1}{q}} \geq 0$, therefore we have

$$\int |h_1 h_2| dx \leq (\int |h_1|^p)^{\frac{1}{p}} (\int |h_2|^q)^{\frac{1}{q}} \quad (4.11)$$

thus, proving the Hölder Inequality.

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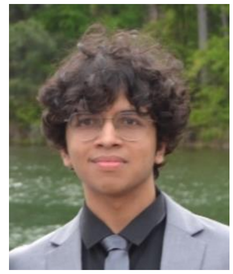

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