Analysis and management of periodic review, order-up-to level inventory systems with order crossover

ABSTRACT

In this paper we investigate the (*R*, *S*) periodic review, order-up-to level inventory control system with stochastic demand and variable leadtimes. Variable leadtimes can lead to order crossover, in which some orders arrive out of sequence. Most theoretical studies of order-up-to inventory systems under variable leadtimes assume that crossovers do not occur and, in so doing, overestimate the standard deviation of the realized leadtime distribution and prescribe policies that can inflate inventory costs. We develop a new analytic model of the expected costs associated with this system, making use of a novel approximation of the realized (reduced) leadtime standard deviation resulting from order crossovers. Extensive experimentation through simulation shows that our model closely approximates the true expected cost and can be used to find values of *R* and *S* that provide an expected cost close to the minimum cost. Taking account of, as opposed to ignoring, crossovers leads, on average, to substantial improvements in accuracy and significant cost reductions. Our results are particularly useful for managers seeking to reduce inventory costs in supply chains with variable leadtimes.

Keywords: Periodic Review Inventory Models, Leadtime Variability, Order Crossover

1. INTRODUCTION AND MOTIVATION

The past several decades have seen explosive growth in global supply chains, as an increasing number of firms have sought to reduce costs by taking advantage of the low landed unit costs obtained by sourcing components or finished products offshore. As Stalk (2006) has observed, part of what has made offshore sourcing attractive are recent advances in the logistics of containerized shipping, which have reduced port-to-port transit times across the Pacific. However, delays in the offshore supply chain have yielded alarming increases in end-to-end leadtimes and leadtime variability. The major causes of such delays include port congestion (queueing delays, customs, security measures, loading and unloading times for mammoth container carriers) and coordination of shipments with inter-modal land transit.

An often overlooked side effect of longer, more variable supply chain leadtimes is an increase in the likelihood of order crossover and its effect on inventory costs (Robinson et al., 2001; Riezebos, 2006). Order crossover occurs when orders arrive out of sequence, such as when an order *N*, placed later than another order *M*, arrives prior to order *M*. The occurrence of crossovers is not restricted to long, global supply chains. As Srinivasan et al. (2011) have noted, crossovers can occur in practice for a variety of reasons, not necessarily related to supply chain length. For example, the adoption of lean, JIT systems brings a reduction in batch sizes, leading to more frequent ordering that increases the likelihood of crossover. Other contributing factors to crossovers include shipment consolidation, variable transportation times due to the use of multiple suppliers, and rescheduling of orders due to production constraints.

In this paper we examine the effect of order crossover on the (*R*, *S*) periodic review, order-up-to level inventory control system under stochastic demand and/or variable leadtimes. As Srinivasan et al. (2011) have observed, order crossover poses theoretical difficulties in the prescription of optimal policies for (*R*,*S*) inventory systems with variable leadtimes—that is, an (*R*,*S*) policy is, in general, not optimal under these conditions. However, as the (*R*,*S*) inventory system is commonly used in supply chains to facilitate the coordination of product flows, and since theoretical approaches to this problem have either ignored the effect of crossovers or have contrived model constructions that prohibit crossovers in order to preserve model optimality, it is important to explore how order crossover affects the costs associated with the (*R*,*S*) system in practice, especially since Robinson et al. (2001) found that the effect of crossovers on inventory costs can be large even in those cases in which crossover is relatively unlikely. We develop a new analytic model of the expected costs for an (*R*, *S*) system that includes both stochastic demand and leadtimes (Gross and Soriano, 1969). Alternate systems for handling stochastic demand and leadtimes are addressed by Bagchi et al., 1986; Kumar and Arora, 1992; Song et al., 2000; and Ayanso et al., 2006. Our model incorporates a novel approach to estimate the realized (reduced) leadtime standard deviation resulting from order crossovers, and we demonstrate through extensive simulation experiments that our model closely approximates expected cost under a wide range of cost, demand, and leadtime parameter values. The model can be used relatively easily to select an appropriate (*R, S*) policy, and by taking advantage of the crossover effect, we obtain policies that tend to have lower values of *R* and *S* as well as lower expected cost than the best (*R, S*) policy chosen using models that ignore crossover. Under situations in which crossovers are unlikely, our policies are identical to the traditional (*R, S*) policy.

Although crossovers may be problematic in theory, in practice they are surprisingly beneficial. For a given leadtime distribution, the occurrence of crossovers leads to an effective, or realized, leadtime distribution with a smaller standard deviation (Hayya et al., 2011). Conventional (*R, S*) policies that ignore crossovers will overestimate the realized leadtime standard deviation and thereby incur cost penalties due to excessive inventory levels and safety stocks. Our modeling approach exploits the effect of crossovers to develop least-cost or lower-cost (*R, S*) policies.

In the next section we discuss our approximation of the cost effect of order crossover and present our analytic model of expected cost. In Section 3 we use our model under a variety of cost parameter values and demand and leadtime distribution conditions to find values of (*R*, *S*) that minimize the approximated expected cost. Simulation experiments demonstrate how well our model approximates the true expected cost and the improvement in cost performance compared with that obtained using (*R*, *S*) models ignoring crossover. In Section 4 we show that one of the key assumptions of the model (namely independent leadtimes) can be relaxed. We conclude with a brief summary; technical details are given in the Appendix.

2. MODEL DEVELOPMENT

We consider the procuring of a single item with unit variable cost, *c* (see Table 1 for notation). Every *R* days the inventory is reviewed, and an order is placed (with a fixed cost *A*) to raise the inventory position to *S*. This order is available *L* days later, where the leadtime *L* is a random variable, independent of the order size and independent of other leadtimes. (The latter assumption is relaxed later in the paper.) The total relevant costs associated with the system are the sum of the inventory holding costs, the fixed costs of replenishments, and the shortage costs.

To determine appropriate choices for the review period *R* and order-up-to level *S*, we develop an analytic expression for the expected total cost. (We treat *R* as a decision variable, but our approach can easily be modified to handle a pre-specified *R* value that could be determined based on transportation economies, manufacturing schedules or multi-item order coordination requirements.) We comment on our modeling assumptions and consider their validity subsequently, but first we address the important issue of order crossovers caused by the random leadtimes. Incidentally, Hayya and Harrison (2010) incorporate crossovers for continuous review, order point/order quantity models under deterministic demand and exponentially distributed leadtimes. As well, for the case of discrete demand and leadtime distributions Srinivasan et al. (2011) use simulation search to find the best value of *S*. (In their study, *R* is assumed to be given.)

2.1 Crossovers Caused by Random Leadtimes

As crossovers can have an important effect, particularly on expected shortage costs, various methods for dealing with them have been proposed, including stochastic dynamic programming (Riezebos and Gaalman, 2009) and setting *S* based on the distribution of the shortfall (the gap at the start of a period between the inventory position and *S*) – see Bradley and Robinson (2005) and Robinson and Bradley (2008).

We account for order crossover through the use of an effective leadtime, a concept introduced by Hayya et al. (2008). Consider a sequence of consecutive orders numbered 1, 2, 3, etc., placed at times t_1, t_2, t_3, etc. Let the chronologically-ordered arrival times be denoted by $\tau_1, \tau_2, \tau_3,$ etc. Note that, due to possible crossovers, τ_i does not necessarily represent the arrival time of order *i*. Thus it is the effective leadtimes, defined as $\mathcal{L}_1 = \tau_1 - t_1, \mathcal{L}_2 = \tau_2 - t_2, ..., \mathcal{L}_i = \tau_i - t_i, ...,$ and not the original *L*s, which determine costs. Moreover, although $\mu_{\mathcal{L}} = \mu_L$, $\sigma_{\mathcal{L}}$ can be much smaller than σ_L, as shown in Hayya et al. (2008) using order statistics and in Bradley and Robinson (2005) by means of an analytic result through an upper bound on the number of outstanding orders.

In order to incorporate effective leadtimes into our model, we first used simulation to estimate $\sigma_{\mathcal{L}}$ for various values of R under the assumption of independent Gamma-distributed leadtimes. (We discuss the use of the Gamma leadtime distribution in Section 3.1 below.) We then fit the analytic relationship

$$\frac{\sigma_{\mathcal{L}}}{\sigma_L} = 1 - 0.8758 \exp\left(-1.0898 R \big/ \sigma_L\right) \qquad (1)$$

to the results (see Figure 1) using a minimax criterion. This expression has the appealing limiting behavior of $\sigma_{\mathcal{L}}/\sigma_L \to 1$ as $R/\sigma_L \to \infty$, i.e., there is no adjustment to variability when crossovers become negligible. Moreover, the expression is more parsimonious than the piecewise linear function of Hayya et al. (2009), and it provides a very close fit for broader parameter ranges. (Although (1) was developed using Gamma-distributed leadtimes, the same structural form with only slightly different coefficient values also holds for Lognormal and Weibull leadtime distributions.)

Note that the presence of crossovers implies that an (R, S) policy is not an optimal policy in that an ordering decision at the beginning of a period should, in principle, take account of the timing of each of the outstanding orders. Srinivasan et al. (2011) investigate such policies through the use of Markov decision processes and simulation search. In this paper we stay with the widely used and much more easily implemented (R, S) policy.

2.2 Cost Expression and Normalization

Let D represent the daily demand with mean μ_D and standard deviation σ_D. The effective leadtime $\mathcal{L}$, which replaces L throughout the analytic model, has mean $\mu_{\mathcal{L}}$ and standard deviation $\sigma_{\mathcal{L}}$. If X signifies the demand during the key "protection interval" $R+\mathcal{L}$, X is then the random sum of $R+\mathcal{L}$ daily random demands. Assuming that D and $\mathcal{L}$ are independent, the mean of X, μ_X, and its standard deviation σ_X are, respectively (see Silver et al. (1998)), $(\mu_{\mathcal{L}}+R)\mu_D$ and $\sqrt{(\mu_{\mathcal{L}}+R)\sigma_D^2+\mu_D^2\sigma_{\mathcal{L}}^2}$. Similarly let Y represent the demand during the effective leadtime. Then μ_Y and σ_Y are given, respectively, by $\mu_{\mathcal{L}}\mu_D$ and $\sqrt{\mu_{\mathcal{L}}\sigma_D^2+\mu_D^2\sigma_{\mathcal{L}}^2}$.

If a replenishment with fixed cost A takes place at each review, and full backordering occurs with a charge of Bc per unit short, then the expected total cost per day is approximately

$$\begin{aligned} EC(R,S) = rc\frac{R}{2}\mu_D + \frac{A}{R} + \frac{Bc}{R}\left[\int_S^{\infty}(X_0-S)\,f_X(X_0)\,dX_0 - \int_S^{\infty}(Y_0-S)\,f_Y(Y_0)\,dY_0\right] \\ +rc\left(S-\mu_X\right)+\frac{rc}{R}\int_0^{\infty}\left[\int_{\mathcal{L}_0}^{R+\mathcal{L}_0}\left(\int_S^{\infty}(W_0-S)\,f_{W|t}(W_0)\,dW_0\right)dt\right]f_{\mathcal{L}}(\mathcal{L}_0)d\mathcal{L}_0 \end{aligned} \quad (2)$$

where r is the inventory holding charge (\$/\$/day), $f_X(X_0)$, $f_Y(Y_0)$ and $f_{\mathcal{L}}(\mathcal{L}_0)$ are the probability density functions of X, Y, and $\mathcal{L}$, and $f_{W|t}(W_0)$ is the probability density function of the demand in a period of length t. The first two terms on the right hand side of (2) are the cycle stock holding cost and the fixed cost of replenishments, and the expression in the first large brackets is the expected units short per cycle (de Kok, 1990; Hadley and Whitin, 1963). On the second line the first term is the safety stock holding cost, and the last expression adjusts for the expected level of backorders at a random point in time, based on Hadley and Whitin (1963), which we have extended to include variable leadtimes.

Here are reasons why (2) is not an exact expression for the expected cost. In general, the distribution $f_{\mathcal{L}}(\mathcal{L}_0)$ is not known exactly, hence must be approximated by a convenient analytic form, which will be discussed below. Furthermore, if orders cross, not just the arrival times but also the order quantities will cross, and thus the arriving order size may not match the desired order size. We return to this latter issue in Section 3.2.

In some circumstances the purchaser may have to pay for pipeline inventory. For example, if payment terms are Free-On-Board, there will be an additional expected cost component of $rcL\mu_D$. This term is independent of R and S and so does not affect their selection, but it will affect total inventory costs.

We express the order-up-to level as follows:

$$S = \mu_X + k\sigma_X \tag{3}$$

where, as usual, k is a safety factor, and define the ratio between shortage and holding costs as

$$\rho = B / r \tag{4}$$

In addition we define

$$w = \sqrt{\frac{2A}{\mu_D cr}} \tag{5}$$

where w is the deterministic equivalent EOQ, expressed in days of demand, also known as the Wilson number.

Equation (2) can be normalized to reduce the number of parameters by dividing by the cost of holding one day's worth of demand in inventory for a period of one day, i.e., $\mu_D cr$. We also substitute (3), (4) and (5) to obtain the Normalized Expected Total Cost (*NEC*) as

$$NEC(R,k)=\frac{EC(R,S)}{\mu_D cr}=\frac{R}{2}+\frac{w^2}{2R}$$
$$+\frac{\rho}{R\mu_D}\left[\int_{\mu_X+k\sigma_X}^{\infty}[X_0-(\mu_X+k\sigma_X)]\,f_X(X_0)\,dX_0-\int_{\mu_X+k\sigma_X}^{\infty}[Y_0-(\mu_X+k\sigma_X)]\,f_Y(Y_0)\,dY_0\right] \quad (6)$$
$$+\frac{k\sigma_X}{\mu_D}+\frac{1}{R\mu_D}\int_0^{\infty}\left[\int_{\mathcal{L}_0}^{R+\mathcal{L}_0}\left(\int_{\mu_X+k\sigma_X}^{\infty}(W_0-(\mu_X+k\sigma_X))\,f_{W|t}(W_0)\,dW_0\right)dt\right]f_{\mathcal{L}}(\mathcal{L}_0)d\mathcal{L}_0$$

which holds for any distributions of X, *Y*, and $W\,|\,t$. The units for *NEC* are "days holding cost"— units that managers, who typically refer to inventory costs over time periods, would use.

Empirical studies such as Strijbosch et al. (2002) have found that the Gamma distribution provides a good fit for total demand over a specified period of time. While one study (Tyworth and O'Neill, 1997) has concluded that employing a Normal approximation for leadtime demand is fairly robust, we find that this is not the case for demand distributions in supply chains with high levels of leadtime uncertainty. These distributions tend to be asymmetric over non-negative values, with a long right-hand tail. Due to the flexibility provided by its shape parameter, the Gamma family of distributions provides significantly better fits than the Normal; in particular, the latter can result in significant probabilities of negative values of demand for small values of μ_L/σ_L (Chopra et al., 2004; Tadikamalla, 1984).

For Gamma distributed *X* and *Y* it is shown in the Appendix that

$$NEC(R,k)=\frac{R}{2}+\frac{w^2}{2R}$$
$$+\sqrt{(\mu_{\mathcal{L}}+R)v_D^2+\mu_{\mathcal{L}}^2v_{\mathcal{L}}^2}\left(k+\frac{\rho}{R}G_{sG}(\alpha,\alpha+k\sqrt{\alpha})\right)-\sqrt{\mu_{\mathcal{L}}v_D^2+\mu_{\mathcal{L}}^2v_{\mathcal{L}}^2}\,\frac{\rho}{R}G_{sG}\left(\gamma,\frac{\delta}{\beta}\left(\alpha+k\sqrt{\alpha}\right)\right) \quad (7)$$
$$+\frac{1}{R\mu_D}\int_0^\infty\left[\int_{\mathcal{L}_0}^{R+\mathcal{L}_0}\left(\int_{\mu_X+k\sigma_X}^\infty(W_0-(\mu_X+k\sigma_X))\,f_{W|t}(W_0)\,dW_0\right)dt\right]f_{\mathcal{L}}(\mathcal{L}_0)d\mathcal{L}_0$$

where v_D and $v_{\mathcal{L}}(=\sigma_{\mathcal{L}}/\mu_{\mathcal{L}})$ are the coefficients of variation of the (daily) demand and the effective leadtime, $\alpha=\frac{\mu_X^2}{\sigma_X^2}=\frac{(\mu_{\mathcal{L}}+R)^2}{(\mu_{\mathcal{L}}+R)v_D^2+\mu_{\mathcal{L}}^2v_{\mathcal{L}}^2}$ and $\gamma=\frac{\mu_Y^2}{\sigma_Y^2}=\frac{\mu_{\mathcal{L}}^2}{\mu_{\mathcal{L}}v_D^2+\mu_{\mathcal{L}}^2v_{\mathcal{L}}^2}$ are the shape parameters of the X and Y distributions, and $\beta=\frac{\mu_X}{\sigma_X^2}=\frac{\mu_{\mathcal{L}}+R}{(\mu_{\mathcal{L}}+R)\mu_Dv_D^2+\mu_D\mu_{\mathcal{L}}^2v_{\mathcal{L}}^2}$ and $\delta=\frac{\mu_Y}{\sigma_Y^2}=\frac{1}{\mu_Dv_D^2+\mu_D\mu_{\mathcal{L}}v_{\mathcal{L}}^2}$ are their scale parameters. Note that $\sigma_{\mathcal{L}}$ is found through the use of (1). $G_{sG}(\alpha,u)$ is the standardized linear loss function, as defined in terms of incomplete Gamma functions by (A2) in the Appendix. We note that expression (7) would somewhat simplify if demands and leadtimes were assumed to be Normally distributed, but then its cost performance would increasingly deteriorate with uncertainty.

For the special case in which the leadtime is Gamma distributed but demand per period is constant, that is, when $v_D = 0$, X follows a shifted Gamma distribution (with minimum value μ_DR), and there is considerable simplification of the *NEC* expression. Details are available, on request, from the first author.

While the first two terms of (7) are convex in R, the others may not be (Silver and Robb, 2008). Indeed, even in the case of deterministic leadtimes, Liu and Song (2012) have shown that the average cost function is not jointly convex in R and S. As we are unable to prove convexity of

NEC for Gamma-distributed X , *Y*, and $W \mid t$, we conduct a two-dimensional grid search to find the best (in terms of minimizing *NEC*) combination of reorder period and order-up-to level (denoted by *R** and *S**, respectively). Note that it is straightforward to evaluate (7) using, e.g., Mathematica®, numerically integrating the last component. (A copy of the Mathematica code is available from the first author.)

2.3 A Simple Heuristic

We also evaluated the performance of a heuristic to select *S*, given *R*, that does not require integration. As shown in the Appendix, by neglecting two terms in (7) – the second Gamma term, which is likely to be considerably smaller than the first one, and the backorder term, which is usually relatively small – we can obtain a closed-form approximation for the value of *k*, and hence *S*, for a given value of *R* as follows:

$$k^{ap} = \frac{1}{\sqrt{\alpha}}\, p_{sG\geq}^{-1}\left(\alpha, \frac{R}{\rho}\right) - \sqrt{\alpha}\ , \qquad (8)$$

where $p_{sG\geq}^{-1}(\alpha, 1-q)$ is the *q*th quantile of the standard Gamma distribution with parameter α (Fortuin, 1980). Note that k^{ap} depends upon the shape parameter α but not upon the scale parameter of the Gamma distribution of *X*, and α depends only upon *R*, μ_L, v_L, and v_D and not upon μ_D (see (A1) in the Appendix).

For given *R*, we evaluate k^{ap} from (8) and then use it to calculate *S*(*R*) from (3). The heuristic's performance is evaluated in Section 3.3.

3. SIMULATION EXPERIMENTS AND RESULTS

In this section we present the results of simulation experiments used to examine the performance of both our analytic model and the simple heuristic. We first discuss the simulation model and the parameter set chosen for the experiments. After presenting the results of the experiments, we discuss their implications concerning the importance of taking crossovers into account through the incorporation of the effective leadtime.

3.1 Simulation Model and Selection of Parameter Values for Experiments

Simulation was employed in order to provide an estimate of the true normalized expected daily cost of operating at a given choice of (R, S). The simulation model was developed in ARENA (Version 10, Rockwell Software, Inc.). In the simulation model, orders arrive at the beginning of the day, after which an order is placed if needed; the day's demand then occurs.

Based on our experiences with industrial data sets and on the literature, in the simulations we modeled both daily demand and the leadtime with Gamma distributions (e.g., Fortuin 1980; Cachon 1999; Strijbosch et al. 2002). Note that this does not satisfy the analytic model's assumption that X and Y have Gamma distributions, since in general a random sum of Gamma variables is not Gamma distributed. As we observed that high-variance Gamma random variates generated by ARENA were inaccurate, variates were instead generated in Mathematica and input to ARENA to drive the simulation. (All results are available from the first author upon request.)

To test the analytic model, we first employed a grid search in Mathematica to find values of (R^*, S^*) and the resulting *NEC* for 243 cases determined by setting each of five parameters (w, ρ, v_D, μ_L, and v_L) at one of three values. (Along with Gross and Soriano (1969), we found that the mean demand had very limited impact on *NEC*, and so μ_D was set to 100 for all experiments.) For the grid search we set the minimum allowable safety factor to zero (see Silver et al. 1998 and Teunter et al. 2010, who also make this assumption), since in a practical context many managers would be reluctant to have negative safety stock. (This constraint turned out to be binding in only a few cases.) The CPU time required for the grid search will depend upon the range of R and k values considered and the fineness of the grid. Searching over all integer values of R from 1 to 200 and from $k = 0$ to 6 in increments of 0.1 required under two minutes of CPU time per case on a 2.40GHz processor.

Concerning the use of (μ_L, v_L) as leadtime parameters rather than (μ_L, σ_L), our observations of leadtime data in contexts including domestic supply and international shipping (Fang et al., 2012) suggest that the relationship between σ_L and μ_L is approximately linear, so that v_L is roughly constant. Thus one need only vary μ_L to reflect the realistic impact of changing the average leadtime. Similarly, a smaller value of v_L reflects the impact of uncertainty reduction, e.g., through switching to a more reliable supplier or by process improvement that reduces σ_L while holding μ_L constant. Again, to reflect a wide variety of industrial sourcing environments, we consider μ_L values of 4, 25, and 100 days and v_L values of 0, 0.25, and 0.5, the latter as in van der Heijden and de Kok (1998).

The remaining parameter settings were determined by extending the range beyond what we observed in industrial data sets in order to "stress test" the model under extreme conditions of ρ, w, and v_D. We set w to 1, 20, and 100 days and used ρ values of 100, 500, and 2000. The lowest value of ρ represents the situation of substantial depreciation (i.e., a high value of r), e.g., computer hardware (Kapuscinski et al., 2004), whereas the highest value of ρ reflects circumstances of very high loss of goodwill (i.e., a high value of B). The coefficient of variation of (daily) demand, v_D, was set to 0, 2.5, and 5, comparable to empirical observations (Silver and Robb, 2008).

3.2 Analytic Model Performance

We present results on two aspects of model performance: first, how closely our model approximates the expected cost at (*R**, *S**), and second, how well this expected cost compares with the expected cost of the globally minimum cost selection of *R* and *S*.

Comparison of the analytic model cost to the expected cost at the analytic model solution

For each of the 243 cases, after completing the grid search on (7) to find (*R**, *S**) and the corresponding approximate *NEC*, we used simulation to estimate the actual expected cost at (*R**, *S**), using a warm-up period of 1 million days and a run of 25 million days. Each case required less than 10 minutes to run. (The average 95% confidence interval half-width on expected cost was 0.2%, with a maximum value of 0.66 %.)

Table 2 presents a summary of the accuracy of the analytic model cost, averaged within each parameter value. The absolute percentage differences between the analytic and simulated values of *NEC* average just 0.9%. Moreover, 90% of the 243 cases show cost differences of no greater than 2%, indicating that the analytic model is quite accurate most of the time.

The 18 largest absolute percentage errors (with a maximum error of 21.3%) all occurred for $v_D = 0$, and the seven largest of these occur when w=1 (leading to R=1). In addition, three of the four largest absolute percentage errors are for μ_L=100 and v_L=0.5 (i.e., σ_L =50). This combination of parameters—implying no variation in demand, 100 orders outstanding on average, and an enormous number of crossovers—would be highly unlikely in practice.

Most of the remaining errors in *NEC* stem from inaccuracies in estimating the effects of order crossover on the expected units short. The larger errors tend to occur for very small values of

R/σ_L, where the discrepancy between the fitted regression and the simulated values of σ_{L} is greatest (see Figure 1). If necessary, one could develop a more accurate regression in this region. We also observed from a histogram of the effective leadtimes that, as the number of crossovers gets very large (again for very low R/σ_L), the distribution more closely fits a Normal than a Gamma. An intuitive explanation is that the total demand, *X*, is the sum of *R* plus $\mathcal{L}$ Gamma variables, where $\mathcal{L}$ is random. For *R* large relative to $\mathcal{L}$, *X* is approximately the sum of *R* Gamma variables and thus is approximately Gamma distributed. However, when *R* is small relative to $\mathcal{L}$, *X* is predominantly a random sum of Gamma variates which, as mentioned earlier, is not well fitted by a Gamma distribution. Thus, we examined the impact of using the simulated $\sigma_{\mathcal{L}}$ (as a surrogate of a more accurate regression) and a Normal distribution for each of *X* and *Y* where $R/\sigma_L \le 0.2$. This drastically reduces the largest absolute percentage errors.

Variability in order sizes could, in principle, increase errors in the model's estimate of the expected units short due to order and arrival sizes not matching (as orders cross), especially in cases of higher v_D and lower *R*. However, an extensive, supplementary numerical investigation identified that, at least for the range of parameters considered here, shortage costs are primarily influenced by when orders arrive and not by the magnitude of the individual orders received.

We investigated two other possible sources of inaccuracy in estimating the expected units short. In the simulation, generated leadtimes are rounded to (positive) integer days. Since for v_D=0 and an effective leadtime $\mathcal{L}_0$, the total demand *X* in $R+\mathcal{L}_0$ is $\mu_D\left(R+\mathcal{L}_0\right)$, the generated total demand has a discrete distribution with values of $\mu_D\left(R+1\right), \mu_D\left(R+2\right)$, etc., whereas the analytic model assumes a continuous Gamma distribution. However, we found that incorporation of a discrete leadtime in the model actually caused a slight deterioration in performance.

Finally, another source of discrepancy was that in the simulation the generated daily demand is rounded to the nearest integer, including zero, making it possible that no demand occurs in the entire review cycle and hence no replenishment takes place. We fine-tuned the analytic model to correct for this by multiplying the order cost term (the first term in (7)) by $1-\left[\text{Prob}\left\{D<0.5\right\}\right]^{R}$, which is readily computed *a priori* for a given demand distribution and value of R. However, this adjustment proved unimportant except in cases with highly variable demand and w=1 (implying an R of 1 or 2), for which it still amounted to less than 0.9% of *NEC*.

Comparison of the expected cost at the analytic model solution to that of the best solution

It is also of interest to compare the cost performance of (R*, S*) to that of the best (R, S) pair found by two-dimensional grid search using the simulation model, since, due to the analytic model's approximation of the true cost surface, the cost of operating at (R*, S*) may differ from that of the optimal (R, S). The first column of Table 3 presents the percentage cost penalty between the simulated cost at (R*, S*) and the minimum cost solution found with the simulation grid search, averaged by parameter value. The average percentage cost penalty is only 0.5%, and all but 9 cases (out of the 243) have percentage cost penalties of no more than 2%. Similar to the results on the errors due to the approximation of *NEC,* the percentage cost penalties are highest when $v_D = 0$, and three of the four largest absolute percentage errors are for μ_L=100 and v_L=0.5. In summary, these results provide additional evidence that the analytic model is highly effective.

3.3 The Performance of the Simple Heuristic

The additional cost penalty incurred by using the heuristic described in Section 2.3, rather than the analytic model, is summarized in the second column of Table 3. The heuristic performs quite well, with an average percentage cost penalty of only 2.2%.

However, we observed that the penalty can be high (up to 34%) for cases in which w =1 (which generally implies small, frequent orders—that is, small R values). In this case the difference between the two Gamma terms in (7) is small, and the heuristic (which ignores the second Gamma term) tends to overstate expected penalty costs and thus mis-specify the (R, S) pair.

3.4 The Importance of Incorporating Effective Leadtimes

We performed additional tests to evaluate the importance of using the effective leadtime as a means of accounting for crossovers. It is clear from Figure 1 that in cases where R/σ_L exceeds 10 or so, the values of $\sigma_{\mathcal{L}}$ and σ_L are essentially equal because the likelihood of crossovers is negligible, hence the effective leadtime concept is irrelevant. For the 125 (out of 243) cases where there was a nontrivial difference between $\sigma_{\mathcal{L}}$ and σ_L, we replaced $\sigma_{\mathcal{L}}$ in (7) with the original leadtime standard deviation σ_L (i.e., ignoring crossovers) and used a grid search to find R and S. Not taking account of crossovers when $\sigma_{\mathcal{L}} < \sigma_L$ leads to higher values of both R and S. For a given value of R, using σ_L, instead of $\sigma_{\mathcal{L}}$, results in a larger value of S. The increase in R results from the associated effect in the ρ/R factor in the shortage term of (7). Looking at this from the opposite perspective, when we properly take account of crossovers we see that R is decreased which, in turn, leads to even more crossovers, which is beneficial from a cost standpoint.

We then evaluated each solution by simulation. The cost at the (R, S) pair found by using (7) (with $\sigma_{\mathcal{L}}$) was, on average, 11% lower than the cost at the (R, S) pair found with σ_L. In 30 cases the cost improvement exceeded 10%, and the largest decrease observed was 82%. Using the effective leadtime produced the greatest percentage benefit in cases with small w (which implies low R, i.e., frequent orders), highly-variable leadtimes (i.e., high σ_L, which is the product of μ_L and v_L), and low demand variation. Table 4 displays the relationship between w, probability of crossover, and percentage cost improvement from using the effective leadtime. When w = 1, we observe the highest average probability of crossover (low values of R / σ_L) and also the greatest improvement from using the effective leadtime. The likelihood of crossover and potential for improvement each decrease with increasing values of w (i.e., larger values of R / σ_L).

Furthermore, as there are two sources of variability present, namely in D and L, reducing the uncertainty in L by using $\sigma_{\mathcal{L}}$ instead of σ_L is most beneficial from a percentage cost reduction standpoint when v_D is low, because we are then reducing the primary source of uncertainty. However, note that absolute costs increase with v_D; hence, even if the percentage cost reduction is lower, the absolute cost reduction can still be appreciable.

In additional experimentation with R / σ_L ranging between 0.2 and 0.5, w = 20, and moderate values of v_D = 0.5 to 1.5, substantial cost improvements of 12% to 43% were obtained. In summary, the use of effective leadtimes to incorporate crossovers can lead to major improvements in the accuracy of the estimated inventory cost and, with appropriately modified values of R and S, to significant cost reductions.

4. EXTENSION TO AUTOCORRELATED LEADTIMES

To this point we have assumed independent leadtimes, partly because it facilitated examination of the effects of changing a number of parameter values (v_D, μ_L, v_L, w, and ρ). In addition, independent leadtimes have been implicitly assumed in much of the literature on order-up-to policies and in applications of these policies to practical operating systems. However, we now show that our general approach can be easily adapted to handle leadtimes that display autocorrelation, which in practice may be more prevalent than independent leadtimes. In particular, positive autocorrelation would indicate that a factor (such as supplier slowdowns, port congestion, or natural or other disasters) has a similar effect on consecutive leadtimes.

Fortunately, even with autocorrelated *L*s the concept of effective leadtimes is still applicable, and the only modification required is how $\sigma_{\mathcal{L}}$ is determined. Given a value of the autocorrelation coefficient, we can use the procedure described in Section 2.1, but simulating serially correlated Gamma-distributed leadtimes (Cario and Nelson, 1996) rather than independent leadtimes, to obtain an analytic relationship for $\sigma_{\mathcal{L}} / \sigma_L$ that replaces expression (1). We did this using an autocorrelation coefficient of 0.67, based upon data from an industrial application. The following form provided an excellent fit to the empirically generated $\sigma_{\mathcal{L}}$ values:

$$\frac{\sigma_{\mathcal{L}}}{\sigma_L} = 1 - 0.8078 \exp\left({}^{-2.953R}\!\big/_{\sigma_L} \right). \tag{9}$$

It can easily be shown that the right-hand side of (9) is greater than that of (1) for all values of R / σ_L, suggesting that, as expected, positive autocorrelation reduces order crossover.

We explored the performance of our approach by again setting $\mu_D = 100$ and, based upon the industrial source of the autocorrelated leadtime data, using $v_D = 1.75, \mu_L = 36, v_L = 0.2,$ and $\rho = 100$; although in the specific case *w* was approximately 25 days, we modified this parameter value to $w = 1$ in order to obtain a nontrivial probability of order crossover. We then performed a grid search to identify (*R**, *S**) under each of three experimental conditions: (a) taking neither the crossovers nor the correlation into account, but simply using the original σ_L; (b) taking the crossovers but not the correlation into account, by using $\sigma_{\mathcal{L}}$ as derived from the regression relationship given in (1); and (c) taking both the crossovers and the correlation into account, by using $\sigma_{\mathcal{L}}$ derived from the relationship in (9). We then ran a simulation at each (*R**, *S**) using Gamma leadtimes with an autocorrelation of 0.67 and also ran a simulation search to find the lowest possible cost.

We found that the percentage cost penalties due to using the analytic solutions rather than the best solution were (a) 2.5%, (b) 0.10%, and (c) 0.03%. Thus even using the effective leadtime relationship based on independent leadtimes provided a substantial improvement in cost performance, and the introduction of the autocorrelation-based relationship further improved performance. We also found, as expected, that the autocorrelation reduced the likelihood of crossovers—from 42% in the independent-leadtimes case to 40%. However, the best simulated expected cost per period for the autocorrelated case was only 3% higher than that of the independent case. In summary, it is straightforward to extend the analytic model (with its associated benefits) to the case of autocorrelated leadtimes.

5. SUMMARY AND CONCLUSIONS

In this paper we have developed a new analytic model of the inventory-related costs of the (*R*, *S*) inventory control system, which is commonly used in supply chains. The model incorporates Gamma-distributed demand and leadtimes and includes an approximation to account for the impact of order crossovers, which can occur with variable leadtimes. Because crossovers tend to reduce the variability of the realized leadtime distribution, we capture this effect in our model by replacing the given leadtime standard deviation with an effective leadtime standard deviation that adjusts for the likelihood of crossover. We found that, using the effective leadtime distribution, the analytic model performed very well under a variety of conditions that encompass most supply chain environments.

Moreover, we showed that taking account of order crossovers by using the effective leadtime provides much more accurate estimates of the inventory cost than using (*R*, *S*) inventory models that ignore crossovers. A traditional (*R*, *S*) inventory model that ignores crossovers overestimates leadtime variability and therefore prescribes policies with higher safety stock levels than needed, inflating costs. Our model exploits the crossover effect and yields (*R*, *S*) policies with lower *R* and *S* values appropriate for the lower realized level of uncertainty; these policies have lower average inventory levels and lower expected total cost. Under conditions in which crossovers are highly unlikely (that is, when $\sigma_{\mathcal{L}}$ and σ_L are essentially equal), our model applies *without modification* and provides results identical to those obtained using a traditional (*R*, *S*) model. The model is relatively easy to implement, requiring the calculation of expression (7) for each potential value of *R* and *k*; this is straightforward to do in Mathematica or a similar package. Note that (7) incorporates the analytic relationship between $\sigma_{\mathcal{L}}$ and σ_L that is given in (1); as this expression holds for any Gamma-distributed leadtime, it is not necessary to recalculate the expression's coefficients. We found that the same structural form holds for Lognormal or Weibull-distributed leadtimes but with slight changes in the values of the coefficients.

This work has important practical implications for managers of supply chains. Developing trends suggest that supply chains will experience order crossover with increasing frequency. The popularity of JIT-induced shorter order cycles and smaller batch sizes and the growing length, complexity, and uncertainty in global supply chains are factors that combine to increase leadtime variability and raise the likelihood of order crossovers. For these environments we provide managers with an inventory model that captures the beneficial effects of crossovers and that yields policies which reduce their inventory costs. In addition, the model is easily adapted to incorporate autocorrelated leadtimes and also applies in a multi-product environment.

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APPENDIX

Development of Normalized Expected Cost Expression

For Gamma distributed *X* we have $f_X(X_0) = \dfrac{\beta^{\alpha} X_0^{\alpha-1} e^{-\beta X_0}}{\Gamma(\alpha)}$, where the shape parameter

$$\alpha = \frac{\mu_X^2}{\sigma_X^2} = \frac{(\mu_{\mathcal{L}} + R)^2}{(\mu_{\mathcal{L}} + R)v_D^2 + \mu_{\mathcal{L}}^2 v_{\mathcal{L}}^2} \tag{A1}$$

and the scale parameter $\beta = \dfrac{\mu_X}{\sigma_X^2} = \dfrac{\mu_{\mathcal{L}} + R}{(\mu_{\mathcal{L}} + R)\mu_D v_D^2 + \mu_D \mu_{\mathcal{L}}^2 v_{\mathcal{L}}^2}$. Equivalently, $\mu_X = \dfrac{\alpha}{\beta}$, $\sigma_X = \dfrac{\sqrt{\alpha}}{\beta}$

and $v_X = \dfrac{1}{\sqrt{\alpha}}$, where v_X is the coefficient of variation of *X*. It can be shown (Fortuin, 1980) that

$$\int_{\mu_X + k\sigma_X}^{\infty} (X_0 - (\mu_X + k\sigma_X))\, f_X(X_0)\, dX_0 = \sigma_X G_{sG}(\alpha, \alpha + k\sqrt{\alpha})$$

where the standardized Gamma linear loss function

$$G_{sG}(\alpha, u) = \left[\Gamma(\alpha+1, u) - u\Gamma(\alpha, u)\right] / \left(\sqrt{\alpha}\,\Gamma(\alpha)\right) \tag{A2}$$

with the (upper) incomplete Gamma function $\Gamma(\alpha, u) = \int_u^{\infty} e^{-t} t^{\alpha-1}\, dt$ and the complete Gamma function $\Gamma(\alpha) = \Gamma(\alpha, 0) = \int_0^{\infty} e^{-t} t^{\alpha-1}\, dt$. Similarly, for Gamma distributed *Y* we have $f_Y(Y_0) = \dfrac{\delta^{\gamma} Y_0^{\gamma-1} e^{-\delta Y_0}}{\Gamma(\gamma)}$, where $\gamma = \dfrac{\mu_Y^2}{\sigma_Y^2} = \dfrac{\mu_{\mathcal{L}}^2}{\mu_{\mathcal{L}} v_D^2 + \mu_{\mathcal{L}}^2 v_{\mathcal{L}}^2}$ and $\delta = \dfrac{\mu_Y}{\sigma_Y^2} = \dfrac{1}{\mu_D v_D^2 + \mu_D \mu_{\mathcal{L}} v_{\mathcal{L}}^2}$. Then it can be shown that

$$\int_{\mu_X+k\sigma_X}^{\infty}\left[Y_0-(\mu_X+k\sigma_X)\right]f_Y(Y_0)dY_0=\sigma_Y G_{sG}\left[\gamma,\frac{\delta}{\beta}(\alpha+k\sqrt{\alpha})\right].$$

Thus (6) can be expressed as

$$NEC(R,k)=\frac{R}{2}+\frac{w^2}{2R}+\frac{\sigma_X}{\mu_D}\left(k+\frac{\rho}{R}G_{sG}(\alpha,\alpha+k\sqrt{\alpha})\right)-\frac{\sigma_Y}{\mu_D}\frac{\rho}{R}G_{sG}\left(\gamma,\frac{\delta}{\beta}(\alpha+k\sqrt{\alpha})\right)$$
$$+\frac{1}{R\mu_D}\int_0^{\infty}\left[\int_{\mathcal{L}_0}^{R+\mathcal{L}_0}\left(\int_{\mu_X+k\sigma_X}^{\infty}(W_0-(\mu_X+k\sigma_X))\,f_{W|t}(W_0)\,dW_0\right)dt\right]f_{\mathcal{L}}(\mathcal{L}_0)d\mathcal{L}_0. \tag{A3}$$

Substituting $\dfrac{\sigma_X}{\mu_D}=\sqrt{(\mu_{\mathcal{L}}+R)v_D^2+\mu_{\mathcal{L}}^2v_{\mathcal{L}}^2}$ and $\dfrac{\sigma_Y}{\mu_D}=\sqrt{\mu_{\mathcal{L}}v_D^2+\mu_{\mathcal{L}}^2v_{\mathcal{L}}^2}$ into (A3) results in (7).

Determining an Approximate Best Value of S for a Given Value of R

For a given value of *R* we may approximate the best value of *S* as follows. We first recognize that the fourth term in (7) comes from the integral involving *Y* in (2) which is likely to be much smaller than the integral involving *X* in (2). Thus we ignore the fourth term in (7). We also ignore the fifth (last) term in (7), and use $\dfrac{dG_{sG}(\alpha,\alpha+k\sqrt{\alpha})}{dk}=-p_{sG\geq}(\alpha,\alpha+k\sqrt{\alpha})$, where $p_{sG\geq}(\alpha,u)$ is the right-hand tail area of the standardized Gamma distribution. Therefore, $\dfrac{dNEC(R,k)}{dk}$ is approximately equal to $\dfrac{\sigma_X}{\mu_D}\left(1-\dfrac{\rho}{R}p_{sG\geq}(\alpha,\alpha+k\sqrt{\alpha})\right)$. Setting the derivative to zero, we have $\dfrac{\rho}{R}p_{sG\geq}\left(\alpha,\alpha+k\sqrt{\alpha}\right)=1$, or

$$p_{sG\geq}(\alpha,\alpha+k\sqrt{\alpha})=\frac{R}{\rho}\ . \tag{A4}$$

For a given value of *R*,we can solve (A4) for k to obtain the safety factor that minimizes the approximate version of *NEC*:

$$k^{ap} = \frac{1}{\sqrt{\alpha}} p_{sG\geq}^{-1}\left(\alpha, \frac{R}{\rho}\right) - \sqrt{\alpha} \ ,$$

where $p_{sG\geq}^{-1}(\alpha, 1-q)$ is the qth quantile of the standard Gamma distribution with parameter α.

Note that k^{ap} may be obtained using a reverse lookup in Microsoft® Excel, viz.,

$$\frac{1}{\sqrt{\alpha}} GAMMAINV\left(1 - \frac{R}{\rho}, \alpha, 1\right) - \sqrt{\alpha} \ .$$

In our own experimentation we utilized Mathematica, as problems have been observed with inverse statistical functions in Excel (McCullough and Wilson, 2005).

Table 1: Some notation

A	\$	Fixed cost of each replenishment
B	-	Charge per unit short expressed as a multiple of c
c	\$/unit	Unit variable cost of an item
D	units/day	Demand per day, distributed with mean and standard deviation (μ_D, σ_D)
EC	\$/day	Expected total cost per unit time
k	-	Safety factor
L	days	Replenishment leadtime, with mean and standard deviation (μ_L, σ_L)
$\mathcal{L}$	days	Effective leadtime, with mean and standard deviation $(\mu_{\mathcal{L}}, \sigma_{\mathcal{L}})$
NEC	days	Normalized expected total cost per unit time
r	\$/\$/day	Inventory holding charge
R	days	Reorder interval
S	units	Order-up-to level, $S = \mu_X + k\sigma_X$
v_D	-	Coefficient of variation of (daily) demand, $v_D = \sigma_D/\mu_D$
v_L	-	Coefficient of variation of replenishment leadtime, $v_L = \sigma_L/\mu_L$
$v_{\mathcal{L}}$	-	Coefficient of variation of effective leadtime, $v_{\mathcal{L}} = \sigma_{\mathcal{L}} / \mu_{\mathcal{L}}$
w	days	The Economic Order Quantity expressed as a time interval, also known as the Wilson number, $w = \sqrt{\frac{2A}{\mu_D c r}}$
$W \mid t$	units	Demand during an interval of known length t
X	units	Demand during the "protection interval" $R + \mathcal{L}$, distributed with mean and standard deviation (μ_X, σ_X). Coefficient of variation $v_X = \frac{\sigma_X}{\mu_X}$
Y	units	Demand during the random effective leadtime $\mathcal{L}$, distributed with mean and standard deviation (μ_Y, σ_Y)
α	-	The shape parameter of the X Gamma distribution
β	-	The scale parameter of the X Gamma distribution
γ	-	The shape parameter of the Y Gamma distribution
δ	-	The scale parameter of the Y Gamma distribution
ρ	days	Shortage to holding cost ratio, $\rho = B/r$ (i.e., the cost of holding one unit in inventory for ρ days is equal to the cost of a shortage of one unit)

Table 2: Accuracy of the model in estimating the expected cost of (R, S) policies

		Percentage difference, analytic versus simulated *NEC*	Absolute percentage difference, analytic versus simulated *NEC*
w	**1**	0.7%	1.8%
	20	0.1%	0.7%
	100	0.0%	0.2%
ρ	**100**	0.0%	0.8%
	500	0.3%	0.9%
	2000	0.5%	1.0%
v_D	**0.0**	1.0%	2.0%
	2.5	0.1%	0.3%
	5.0	-0.3%	0.4%
μ_L	**4**	0.2%	0.5%
	25	-0.2%	0.9%
	100	0.8%	1.3%
v_L	**0.00**	-0.1%	0.2%
	0.25	0.0%	1.0%
	0.50	0.9%	1.5%
Average		**0.3%**	**0.9%**

Note: Results are averages across the 81 (=3^4) experiments in which the parameter in the first column was set to the value shown in the second column. The last row gives the overall averages across all 243 experiments.

Table 3: Performance of the analytic and heuristic approaches relative to the minimum cost (R, S) policy

		Percentage cost penalty due to using the analytic solution	**Additional percentage cost penalty due to using heuristic**
w	**1**	1.0%	6.0%
	20	0.2%	0.5%
	100	0.3%	0.0%
$\boldsymbol{\rho}$	**100**	0.2%	2.8%
	500	0.4%	2.0%
	2000	1.0%	1.7%
$\boldsymbol{v_D}$	**0.0**	1.1%	0.9%
	2.5	0.1%	2.7%
	5.0	0.3%	3.0%
$\boldsymbol{\mu_L}$	**4**	0.4%	1.2%
	25	0.2%	1.8%
	100	1.0%	3.5%
$\boldsymbol{v_L}$	**0.00**	0.1%	1.2%
	0.25	0.5%	1.9%
	0.50	0.9%	3.4%
Average		**0.5%**	**2.2%**

Note: Results are averages across the 81 ($=3^4$) experiments in which the parameter in the first column was set to the value shown in the second column. The last row gives the overall averages across all 243 experiments.

Table 4: Probability of crossover when σ_L is used to determine *R* and *S*, and percentage cost reduction due to using $\sigma_{\mathcal{L}}$

		Probability of crossover using σ_L	Percentage reduction in cost using $\sigma_{\mathcal{L}}$
w	**1**	30.5%	20.2%
	20	10.4%	5.1%
	100	1.3%	0.4%
Average		**17.1%**	**2.2%**

Note: These results are for 125 (out of 243) cases with nontrivial difference between σ_L and $\sigma_{\mathcal{L}}$. Results are averages across the cases in which *w* was set to the value shown in the second column. The last row gives the overall averages across all 125 experiments.

Figure 1: The relationship between R/σ_L and $\sigma_{\mathcal{L}}/\sigma_L$, based on simulated Gamma leadtimes with various values of μ_L and ν_L, including the fitted curve given in (1)