Assessing Knowledge of Mathematical Equivalence: A Construct Modeling Approach

Bethany Rittle-Johnson, Percival G. Matthews, Roger S. Taylor & Katherine L. McEldoon

PME-NA 2010

IES INSTITUTE OF EDUCATION SCIENCES | VANDERBILT PEABODY COLLEGE | NSF

Research Goals

Long Term:
Systematically study children's understanding of mathematical equivalence and the ways in which it develops.

Short Term:

- Develop a valid and reliable measure of students' level of understanding of mathematical equivalence.
- Create & validate a mathematical equivalence *Construct Map* (Wilson, 2005).

Why Math Equivalence?

- Push to re-conceptualize algebra as a continuous strand from elementary through high school (NCTM, 2000).
- Mathematical equivalence is an early developing & foundational concept in algebra
  - Principle that two sides of an equation represent the same value (also called equality). Symbolized by "="
  - Provides the foundation for two key algebra proficiencies
    - Understanding the equivalence of expressions & competence at performing same operation on both sides of an equation.

(e.g., Carpenter, et al., 2003; Kieran, 1992; Knuth, Stephens, McNeil, & Alibali, 2006; MacGregor & Stacey, 1997)

Children's View of Equivalence (They don't get it)

Bad News: 35 years of research indicates that a majority of first through sixth graders treat equations operationally (e.g., Weaver, 1973, Behr, Erlwanger, & Nichols, 1980; Perry, 1991; Alibali, 1999; Powell & Fuchs, 2010)

- Operational View
  - View "=" as a command to carry out arithmetic operations
  - 8 + 4 =_ + 5, most get 12 (add to equal) or 17 (add all) (Falkner, Levi & Carpenter, 1999)
- Relational View
  - View "=" as meaning two sides of an equation have the same value

Assessing Equivalence Knowledge

A *Measurement* Gap

• Despite its critical importance, no standard measure of equivalence knowledge and no evidence for the validity of the measures.

• "Without conducting and reporting validation work on key independent and dependent variables, we cannot know the extent to which our instruments tap what they claim to. And without this knowledge, we cannot assess the validity of inferences drawn from studies." (Hill & Shih, 2009, p. 248).

• Indeed, less than 20% of studies published in the Journal for Research in Mathematics Education over the past 10 years reported on the validity of the measures (Hill & Shih, 2009).

5

Goal of the Study

1. Develop a valid and reliable measure of students' understanding of mathematical equivalence.
2. Use a *Construct Modeling* approach (Wilson, 2005)
   - Develop and test a *construct map* – a representation of the continuum of knowledge that people are thought to progress through.

6

Equivalence Construct Map

Sophistication ↑

Level	Description	Core Equation Structure(s)
Level 4: Comparative Relational		
Level 3: Basic Relational		
Level 2: Flexible Operational		
Level 1: Rigid Operational		

7

Equivalence Construct Map

Level	Description	Core Equation Structure(s)
Level 4: Comparative Relational		
Level 3: Basic Relational		
Level 2: Flexible Operational		
Level 1: Rigid Operational	Define equal sign operationally. Only successful with equations with an operations-equals-answer structure.	Operations-equals-answer structure: a + b = c

8

Equivalence Construct Map

Level	Description	Core Equation Structure(s)
Level 4: Comparative Relational		
Level 3: Basic Relational	Successful with operations on both sides of the equal sign. Recognize and generate relational definition of the equal sign.	Operations on both sides: e.g., a + b = c + d
Level 2: Flexible Operational		
Level 1: Rigid Operational	Define equal sign operationally. Only successful with equations with an operations-equals-answer structure.	Operations-equals-answer structure: a + b = c

9

Equivalence Construct Map

Level	Description	Core Equation Structure(s)
Level 4: Comparative Relational		
Level 3: Basic Relational	Successful with operations on both sides of the equal sign. Recognize and generate relational definition of the equal sign.	Operations on both sides: e.g., a + b = c + d
Level 2: Flexible Operational	Successful with atypical equation structures that remain compatible with an operational view of the equal sign.	Operations on right *or* no operations: c = a + b & a = a
Level 1: Rigid Operational	Define equal sign operationally. Only successful with equations with an operations-equals-answer structure.	Operations-equals-answer structure: a + b = c

10

Equivalence Construct Map

Level	Description	Core Equation Structure(s)
Level 4: Comparative Relational	Compares the expressions on the two sides of the equal sign. Recognizes relational definition as the best definition.	Operations on both sides with multi-digit numbers or multiple instances of a variable.
Level 3: Basic Relational	Successful with operations on both sides of the equal sign. Recognize and generate relational definition of the equal sign.	Operations on both sides: e.g., a + b = c + d
Level 2: Flexible Operational	Successful with atypical equation structures that remain compatible with an operational view of the equal sign.	Operations on right *or* no operations: c = a + b & a = a
Level 1: Rigid Operational	Define equal sign operationally. Only successful with equations with an operations-equals-answer structure.	Operations-equals-answer structure: a + b = c

11

~~*Knowledge change is gradual and dynamic, not stages.~~

Level 4 Example

- *Without adding* 89 + 44, can you tell if the statement is true or false?
  - 89 + 44 = 87 + 46. Explain why.

(modified from Jacobs, et al, 2007)

12

Tasks

1. Solving Equations items – abilities to solve open equations.
   - 8 + 4 = ☐ + 5
2. Structure of Equations items – knowledge of valid equation structures.
   - 3 + 5 = 5 + 3 True or False
3. Defining the Equal Sign items - explicit knowledge of the equal sign.
   - What does the equal sign mean?

(e.g., Alibali, 1999; Behr, Erlwanger, & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Li, Ding, Capraro, & Capraro, 2008; McNeil, 2007; Rittle-Johnson & Alibali, 1999; Weaver, 1973)

13

Assessment

- 37-item written assessment, using items from past research.
- Selected items so at least two per construct map level for each of the three common item types.
- Created **2 parallel forms** (to use as pretest and posttest in future research).

14

Data Source

- Assessment administered to 174 students in ten 2nd-6th grade classrooms (2 per grade).
  - School: Urban, parochial, serving working-to middle-class, predominantly Caucasian students.
- Administered twice in the fall, two weeks apart.

15

Evidence for Reliability & Validity

•**Good internal consistency:** Performance on individual items highly correlated with performance on other items**.**
- Form 1: α = .94 Form 2: α =.95

•**Good stability:** Performance at Time 1 very similar to performance at Time 2.
- Form 1: $r(26)$ = .94 Form 2: $r(26)$ = .95

• **Strong Content Validity**
- 4 mathematics education researchers rated each item as *important* (rating of 3) to *essential* (rating of 5) for tapping knowledge of equivalence, with a mean rating of 4.1.

16

Construct Validity

- Items tapped a single construct.
  - Confirmatory factor analysis confirmed the unidimensionality of the assessment, Bentler CFI = .98
- Items fit our construct map.
  - Rasch model – type of Item Response Theory (IRT) model.
    - Estimates the difficulty of each item and the ability of each student simultaneously.
  - Wright Map - graphical display of the results that helps us evaluate our construct map.

17

Interpreting the Wright Map

Difficulty/Ability

```
STUDENTS - LOGITS - ITEMS
        XXXXXX  1  14SOL.L3
           XXX  |  22STR.L3
           X M|    16SOL.L3
         XXXXX  |  14SOL.L3
                |  16SOL.L3
        XXXXXX  0M 13SOL.L3
            XX  |  13SOL.L3
             X  |  15SOL.L3
         XXXXX  |  17STR.L3
           XXX  |  20STR.L3
    XXXXXXXXXX -1  9STR.L2
                |
      XXXXXXXX  |  9STR.L2
        XXXXXX S|  3STR.L1
    XXXXXXXXXX  |  5DEF.L2
               -2S 8STR.L2
         XXXXX  |  8STR.L2
        XXXXXX  |  21STR.L3
                |
         XXXXX  |  6SOL.L2
               -3
```

Key:

- On left: X is one student
- On right: Individual items
  - Item #
  - Item Type
    - SOL is solve equation
    - STR is structure of equations
    - DEF is define equal sign
  - Level 1, 2, 3 or 4

18

Interpreting the Wright Map

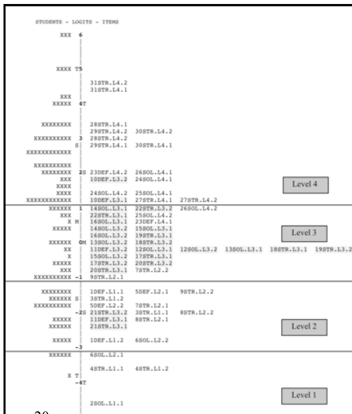
```
STUDENTS - LOGITS - ITEMS
        XXXXXX  1  14SOL.L3
           XXX  |  22STR.L3
           X M|    16SOL.L3
         XXXXX  |  14SOL.L3
                |  16SOL.L3
        XXXXXX  0M 13SOL.L3
            XX  |  13SOL.L3
             X  |  15SOL.L3
         XXXXX  |  17STR.L3
           XXX  |  20STR.L3
    XXXXXXXXXX -1  9STR.L2
                |
      XXXXXXXX  |  9STR.L2
        XXXXXX S|  3STR.L1
    XXXXXXXXXX  |  5DEF.L2
               -2S 8STR.L2
         XXXXX  |  8STR.L2
        XXXXXX  |  21STR.L3
                |
         XXXXX  |  6SOL.L2
```

- Mean Student Ability: 0.7 logits
- Level 3 solve item 3 + 4 = □ + 5 (13SOL.L3) has difficulty of -.04 logits.
  - Student of average ability expected to get correct 68% of time.
- Level 2 solve item 8 = 6 + □ (6SOL.L2.2) has difficulty of -2.8 logits.
  - Student of average ability expected to get correct 97% of time

$$Pr(success) = \frac{1}{1+e^{-(\theta - \delta)}}$$

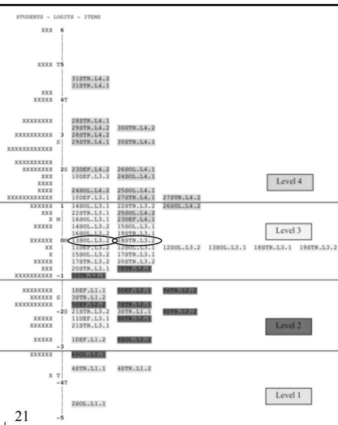
19

Wright Map



- Most items fit the expected level of difficulty
- Level 3 items highlighted

20



Wright Map

- Spearman's rho between hypothesized difficulty level and empirically estimate, $\rho(62) = .84$, $p < .01$
- As predicted, equation structure mattered more than item type:
  - 13SOL.L3.2:
    - 3 + 4 = ☐ + 5
  - 18STR.L3.2
    - Is
      "7 + 6 = 6 + 6 + 1" true or false

Analysis of Individual Items

```
XXXXXXXX  2S 23DEF.L4.2  26SOL.L4.1
     XXX   |  10DEF.L3.2  24SOL.L4.1
    XXXX   |
    XXXX   |  24SOL.L4.2  25SOL.L4.1
XXXXXXXXXXX   |  10DEF.L3.1  27STR.L4.1  27STR.L4.2
  XXXXXX  1  14SOL.L3.1  22STR.L3.2  26SOL.L4.2
     XXX   |  22STR.L3.1  25SOL.L4.2
       X M|  16SOL.L3.1  23DEF.L4.1
   XXXXX   |  14SOL.L3.2  15SOL.L3.1
           |  16SOL.L3.2  19STR.L3.1
  XXXXXX  0M 13SOL.L3.2  18STR.L3.2
```

- Can explore items that are not of predicted difficulty
- E.g. Should explicit definition be Level 4?
  - 10DEF.L3.2 "What does the equal sign mean?"

Equivalence Construct Map

Level	Description	Core Equation Structure(s)
Level 4: Comparative Relational	Compares the expressions on the two sides of the equal sign. *Generates relational definition* and recognizes it as the best definition.	Operations on both sides with multi-digit numbers or multiple instances of a variable
Level 3: ~~Basic~~ Implicit Relational	Successful with operations on both sides of the equal sign. Recognize ~~and generate~~ relational definition of the equal sign.	Operations on both sides: e.g., a + b = c + d
Level 2: Flexible Operational	Successful with atypical equation structures that remain compatible with an operational view of the equal sign.	Operations on right *or* no operations: c = a + b & a = a
Level 1: Rigid Operational	Define equal sign operationally. Only successful with equations with an operations-equals-answer structure.	Operations-equals-answer structure: a + b = c

Summary

- We developed a valid and reliable measure of students' knowledge of equivalence.
  - A 2nd experiment replicated these findings with a new sample of public school children.
- Construct map captures shifts in knowledge of equivalence over grade levels.
  - Incorporate flexible operational view as transition.
  - Distinguish implicit from explicit relational understanding. Capture developing comparative thinking.

Benefits of a Construct Modeling Approach

- Can sequence items to determine factors that increase difficulty of items.
  - Permits testing of whether performance on specific items fit our expectations.
- Produces a criterion-referenced measure that is particularly appropriate for assessing the effects of an intervention on individuals (Wilson, 2005). (Our current research)
- Knowing where individual students are on the construct map could help educators modify and differentiate their instruction to meet individual student needs.

For More Information

http://peabody.vanderbilt.edu/earlyalgebra.xml