First homework assignment (a refresher on linear algebra)

1) What is a <u>basis</u> of a vector space?

2) Why does every vector space have a basis?

3) If $T: V \to W$ is a linear map between finite dimensional vector spaces, what does it mean for T to be invertible? Show that if $\dim V = \dim W$ and T is surjective then T is injective.

4) Construct an explicit <u>exact sequence</u>

$$0 \to V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \xrightarrow{T_3} V_4 \xrightarrow{T_4} V_5 \xrightarrow{T_5} 0$$

of (non-zero) finite dimensional vector spaces. Compare $\dim V_1 + \dim V_3 + \dim V_5$ to $\dim V_2 + \dim V_4$.
Does equality of these dimensions imply that an arbitrary choice of T_i's defines an exact sequence?

5) Review the definition of $V \oplus W$ for vector spaces and $S \oplus T$ for linear maps $S: V_1 \to V_2$, $T: W_1 \to W_2$ between them.

6) How does the index of an operator behave under direct sums?

7) With the notation of exercise 4, construct something a bit less than an exact sequence – a complex – namely construct vector spaces and maps so that $T_{i+1} \circ T_i = 0$ but $\ker T_{i+1}$ is strictly larger than $\mathrm{Im}(T_i)$. Calculate the cohomology groups $\ker T_i / \ker T_{i-1} = H^i$ and verify in your example that

$$\sum_{i=1}^{5} (-1)^i \dim V_i = \sum_{i=1}^{5} (-1)^i \dim H^i .$$