The complex numbers $\mathbb{C}$ are the <u>field</u> formed by giving $\mathbb{R}\times\mathbb{R}$ the multiplication

$$(x_1,y_1)(x_2,y_2)=(x_1x_2-y_1y_2,\ x_1y_2+x_2y_1)$$

If we set $i=(0,1)$ then $i^2=-1$ and any $z\in\mathbb{C}$ can be uniquely written

$$z=x+iy \quad \text{for } x,y \text{ in } \mathbb{R}.$$

N.B. Unlike $\mathbb{R}$, $\mathbb{C}$ has a large number of field automorphisms.

But if the automorphism α fixes $\mathbb{R}$ then $\alpha(i)$ must be $\pm i$. By definition if $z=x+iy$

$$\bar{z}=x-iy$$

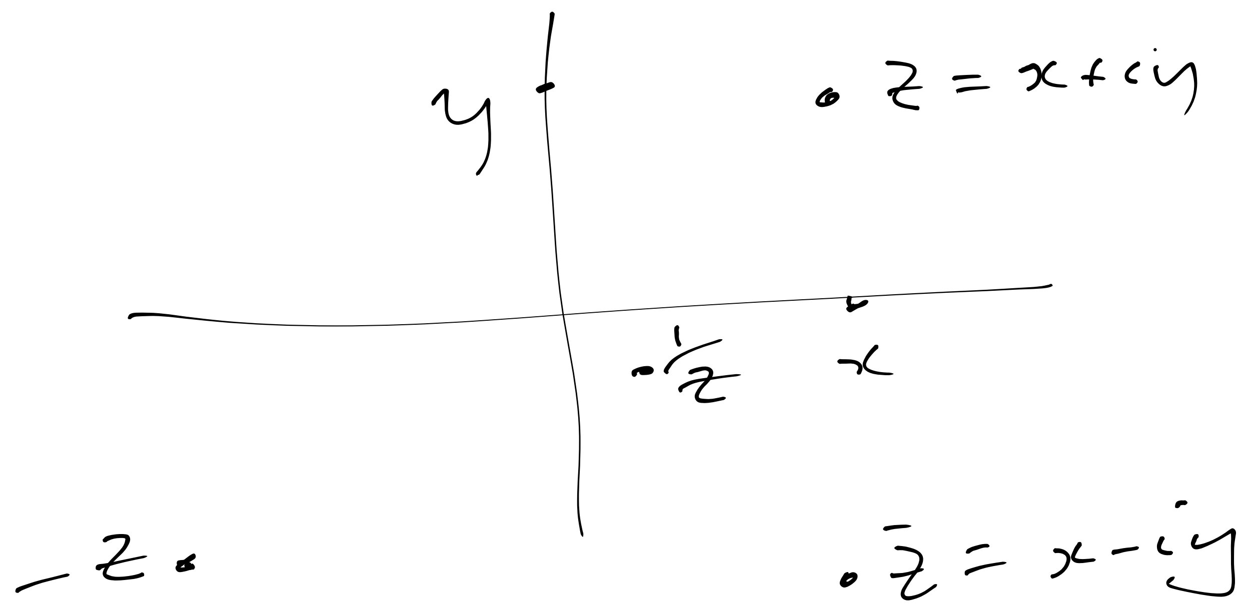
N.B. The complex numbers have a rich and deep structure. This is not just because of some algebraic magic arising from adding a square root of -1, but relies on the underlying deep structure of $\mathbb{R}$. ($\mathbb{Q}(i)$ is nothing special).

$$|z|=\sqrt{z\bar{z}}=\sqrt{x^2+y^2}=d(z,0)$$

If $z\neq 0$ $\quad z\cdot\left(\bar{z}/|z|^2\right)=1$ so $\mathbb{C}$ is a field with $z^{-1}=\bar{z}/|z|^2$.

$|z_1 - z_2|$ is the distance from z_1 to $z_2 = d(z_1, z_2)$
and $\mathbb{C}$ is a complete metric space with d.

<u>Pictures</u>



$Re(z) = \dfrac{z + \bar{z}}{2}$, $Im(z) = Re(-iz) = \dfrac{z - \bar{z}}{2i}$

$= x$ $\qquad\qquad$ $= y$

Addition is vector addition.
Multiplication is more obscure in (x, y)
but any $z = r(\cos\theta + i\sin\theta)$ (polar-coords)
with $r = |z| \geq 0$ and $0 \leq \theta < 2\pi$. θ is
unique if $z \neq 0$. r is called the modulus
or absolute value of z and θ is called
the <u>argument</u> of θ.
then $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$
so $z^n = r^n(\cos n\theta + i\sin n\theta)$ (de Moivre)

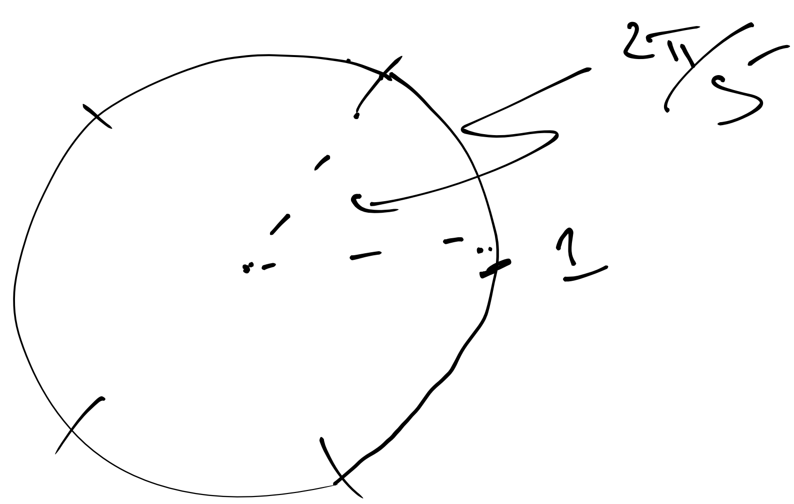
The unit circle $\mathbb{T} = \mathbb{T}^1 = \{z \mid |z| = 1\}$ is a closed subgroup of $\mathbb{C}^\times = \{z \mid z \neq 0\}$

By de Moivre, the nth roots of unity for $n \in \mathbb{N}$ are

$$\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$$

for $k = 0, 1, 2, \ldots n-1$

geometrically they are (e.g. 5th roots of unity)



Hence any $z \neq 0$ has n distinct nth roots: if $z = r(\cos\theta + i\sin\theta)$, its nth roots are $\left\{ r^{1/n} \left(\cos\theta/n + i \sin\frac{\theta}{n}\right) \omega \;\middle|\; \omega \text{ an nth root of } 1 \right\}$

Functions of a complex variable

The most obvious functions on $\mathbb{R}^2$ are the polynomials $\sum a_{mn} x^m y^n$, $a_{m,n} \in \mathbb{R}$.

These functions can be made complex-valued by letting $a_{m,n} \in \mathbb{C}$ ("complexifying") to obtain an algebra $\mathrm{Poly}(\mathbb{C})$ of complex valued functions. It's a *-algebra under

$$f^*(z) = \overline{f(z)}$$

It contains the elements "z" $= x+iy$ and $\bar{z} = x - iy$.

Exercise Show that any element of $\mathrm{Poly}(\mathbb{C})$ can be uniquely expressed as

$$\sum a_{mn} z^m \bar{z}^n$$

for some $a_{mn} \in \mathbb{C}$.

eg $yx + y^2 = \left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right) + \left(\frac{z-\bar{z}}{2i}\right)^2$

One may define operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ on $\mathrm{Poly}\,\mathbb{C}$ by $\frac{\partial}{\partial z}\left(\sum a_{mn} z^m \bar{z}^n\right) = \sum m a_{mn} z^{m-1} \bar{z}^n$

and $\frac{\partial}{\partial \bar{z}}\left(\sum a_{mn} z^m \bar{z}^n\right) = \sum n a_{mn} z^m \bar{z}^{n-1}$

<u>Check</u> $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

calculus

$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$

$\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$

<u>exercise</u> prove $\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

<u>Defn.</u> An element p in $\mathrm{Poly}(\mathbb{C})$ is called a <u>polynomial in z</u> if

$\frac{\partial p}{\partial \bar{z}} = 0 \iff p = \sum_{n=0}^{k} a_n z^n \iff \frac{\partial p}{\partial x} = -i\frac{\partial p}{\partial y}$

$\iff p \in \ker \frac{\partial}{\partial \bar{z}}$.

Möbius transformations

are functions of the form $z \longmapsto \dfrac{az+b}{cz+d}$

for $a, b, c, d \in \mathbb{C}$, with $ad - bc \neq 0$.

To avoid domain problems one introduces the

<u>Riemann sphere</u> $\mathbb{CP}^1$ (complex projective space)

$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ with the rule $\dfrac{1}{0} = \infty$

and $\dfrac{a\infty + b}{c\infty + d} = \dfrac{a}{c}$.

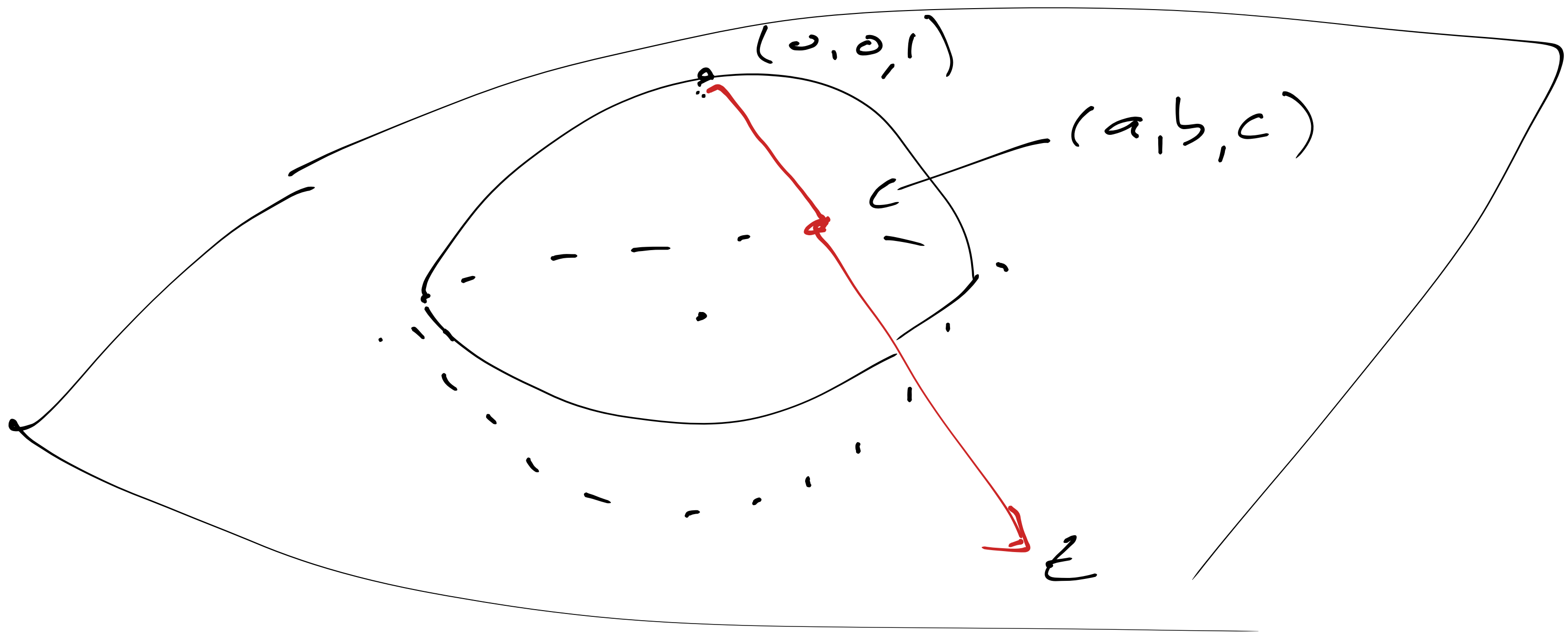
<u>Better</u> $\mathbb{CP}^1$ is the space of <u>1 dimensional</u> <u>subspaces</u> of $\mathbb{C}^2$ and the map $z \to \dfrac{az+b}{cz+d}$ is the action induced by the linear transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\mathbb{C}^2$. To see this, any 1 dimensional subspace except $\{(0, z) \mid z \in \mathbb{C}\} = \infty$, is of the form $\mathbb{C}(z, 1)$ (obvious).

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} wz \\ w \end{pmatrix} = w \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = \dfrac{w}{cz+d} \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$.

(provided $cz+d \neq 0$, and if $cz + d = 0$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{C} \begin{pmatrix} z \\ 1 \end{pmatrix} = \infty$)

<u>Corollary</u> Composition of Möbius transformations is matrix multiplication.

The Riemann sphere is also visible by stereographic projection



Exercise

$$z = \frac{a+ib}{1-c}$$

$\mathbb{CP}^1$ is also called $\overline{\mathbb{C}}$, $\mathbb{C}_\infty$, the one point compactification of $\mathbb{C}$.

Since $ad-bc \neq 0$ one can always normalize $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $ad-bc=1$ without changing the Möbius transformation.

Similarly one can multiply $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by ± 1 without affecting the Möbius transformation, but that's it. So the Möbius group is

$$\left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \det A = 1 \right\} \Big/ \{\pm 1\} = PSL_2(\mathbb{C}).$$

The action of Möbius on $\mathbb{C}P^1$ is <u>sharply triply transitive</u>, i.e. given any two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) in $\mathbb{C}P^1$ there is a unique Möbius transformation sending $z_1 \to w_1$, $z_2 \to w_2$, $z_3 \to w_3$

To prove this, by group theory, it suffices to show that $(0, 1, \infty)$ can be sent to any (z_1, z_2, z_3). But consider

$$\begin{pmatrix} z_2-z_3 & 0 \\ z_2-z_1 & z_1-z_3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z_2-z_1 \end{pmatrix}\begin{pmatrix} 1 & -z_1 \\ 0 & 1 \end{pmatrix}$$

(if any of $z_1 z_2 z_3$ is ∞, move it)

Check
$$z_1 \to 0 \to 0 \to 0$$
$$z_2 \to z_2 - z_1 \to 1 \to 1$$
$$z_3 \to z_3 - z_1 \to \frac{z_3-z_1}{z_2-z_1} \to \infty$$

Moreover stabilizer of $(0, 1, \infty)$ is
$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ \underset{0}{\underline{b=0}}, \ \underset{\infty}{\underline{c=0}}, \ \underset{1}{\underline{a=d}}, ad=1 \right\} / \{\pm 1\}$$

= trivial.

the image of (z_1, z_2, z_3, z_4) after $(z_1, z_2, z_3, z_4) \to (0, 1, \infty, \cdot)$

$$\frac{z_4-z_1}{z_2-z_1} \cdot \frac{z_2-z_3}{z_4-z_3} \quad \leftarrow \text{the } \underline{\text{cross ratio}}$$

Corollary Möbius group is generated by

$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Proof if $g \in PSL_2(\mathbb{C})$ let $g(0)=z_1, g(1)=z_2, g(\infty)=z_3$
use above 3 matrices to get back to $(0,1,\infty)$.

Corollary Möbius group sends {lines and circles} $\circlearrowleft$.

Proof $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ is translation .. obvious

$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ is scaling + rotation, obvious.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: $z \to 1/z$. A line/circle has equation

$|z-z_0|^2 = r^2$ $\quad a|z|^2 + bz + \bar{b}\bar{z} + c = 0$ $\quad a, c \in \mathbb{R}$ (straight line also $\uparrow$ $bz + \bar{b}\bar{z} + c = 0$)

$\frac{1}{z}$ sends it to $\frac{a}{|z|^2} + \frac{b}{z} + \frac{\bar{b}}{\bar{z}} + c = 0$

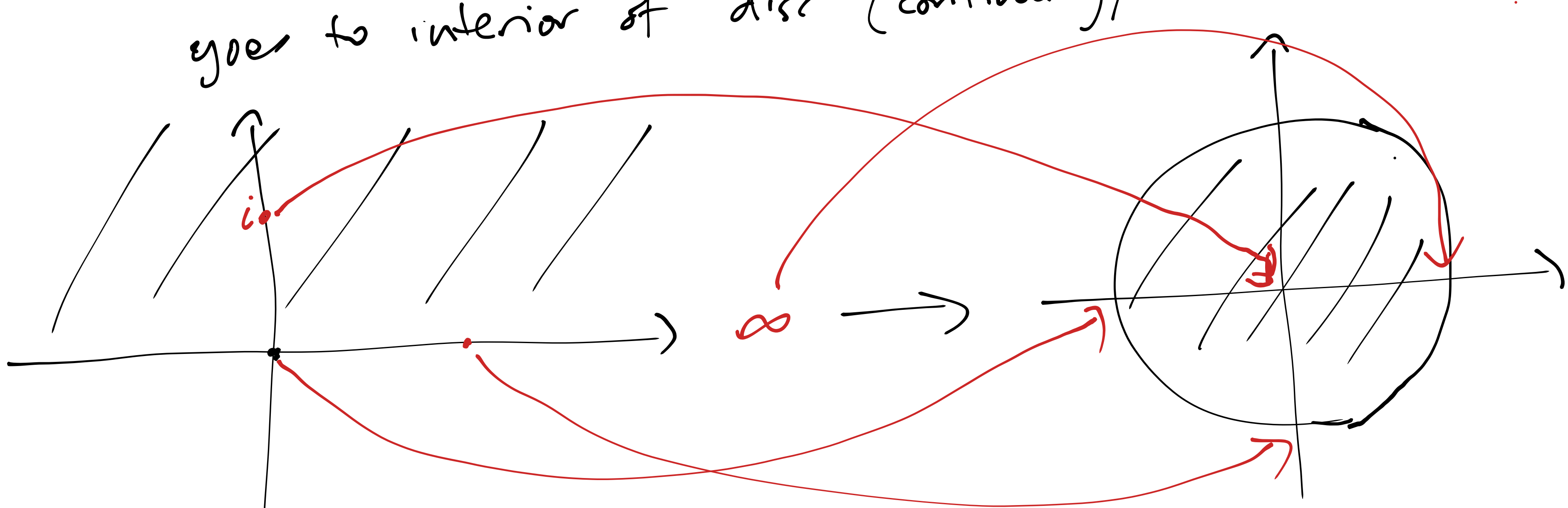
$\to a + b\bar{z} + \bar{b}z + c|z|^2 = 0$

Cayley transform $\frac{z-i}{z+i}$

$0 \to -1, \quad 1 \to \frac{1-i}{1+i} = \frac{\cos\pi/4 - i\sin\pi/4}{\cos\pi/4 + i\sin\pi/4} = -i, \quad \infty \to 1$

so $\mathbb{R} \to$ unit circle, $+i \to 0$

so upper $\frac{1}{2}$ plane $\mathbb{H} = \{z \mid \mathrm{Im}(z) > 0\}$
goes to interior of disc (continuity/connectedness)



If a Möbius transformation preserves $\mathbb{R}$ then it is given by a matrix in $GL_2(\mathbb{R})$ by the above. Conversely $PSL_2(\mathbb{R})$ acts on $\mathbb{R}$ and since $i \to i$ under $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $PSL_2(\mathbb{R})$ maps $\mathbb{H}$ to $\mathbb{H}$.

Conjugating by $\begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix}$ will give an action of $PSL_2(\mathbb{R})$ on the unit disc (and circle).

<u>Exercise</u> Write an explicit formula for the action of $SL_2(\mathbb{R})$ on the unit disc.

e.g. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is rotation by 2θ.

$$\left\{\begin{aligned} &\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c+is & s-ic \\ c-is & s+ic \end{pmatrix}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \\ &= 2i\begin{pmatrix} c+is & 0 \\ 0 & c-is \end{pmatrix} \end{aligned}\right.$$

What is $\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$?

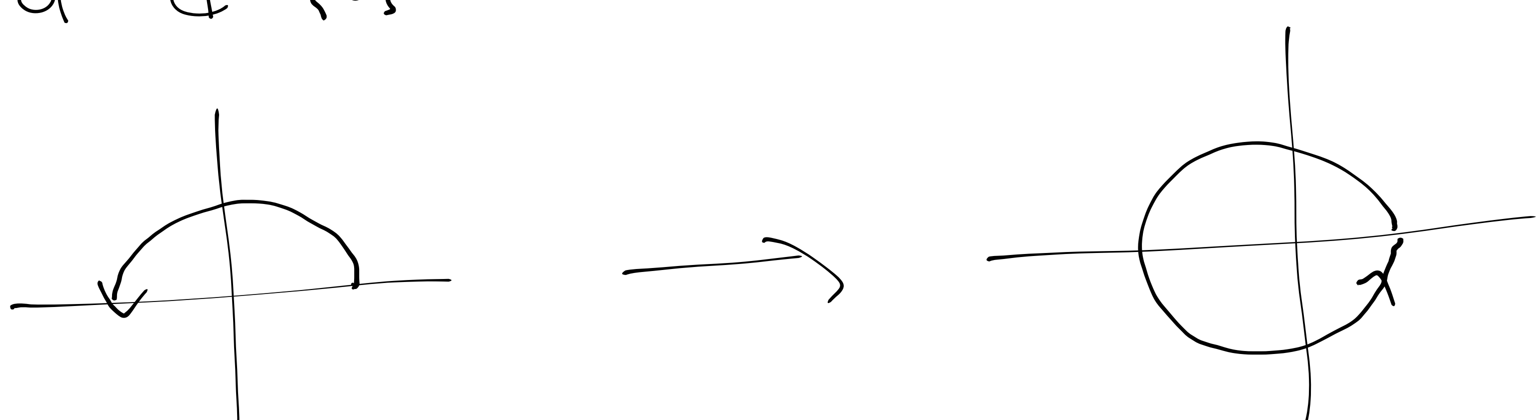
In general $GL_2(\mathbb{R})$ splits into 2 connected components according to the sign of the determinant. The other connected component maps $\mathbb{H}$ to $\overline{\mathbb{H}}$.

. More generally, a <u>rational function</u> of z is a ratio $\frac{P(z)}{Q(z)}$ of two polynomials determines a function from $\mathbb{CP}^1$ to itself.

Rational functions can be composed but they do not form a group. On the other hand, <u>iteration</u> of rational maps is exceedingly interesting, even in the very simple case $Q=1$, $P(z)=z^2+c$ for $z\in\mathbb{C}$.

Even z^2 is already interesting.

Away from 0 it is a <u>double covering</u> of $\mathbb{C}\setminus\{0\}$ to itself.



- What is the meaning of $\sqrt{z}$?
- Iterate z^2 on $\mathbb{T}^1$. Periodic orbits, dense orbits. orbit$(\cos 1 + i\sin 1) = \{\cos 2^n + i \sin 2^n\}$

Is $\{2^n \bmod 2\pi\}$ dense in $[0, 2\pi]$?

$\left\{2^n \cdot \frac{2\pi k}{l} \bmod 2\pi\right\}$ always an l th root of unity.

In another direction, functions can be obtained from polynomials in z by taking limits. Most simple are the power series in the variable z (or $z - z_0$).

The notion of $\sum_{n=0}^{\infty} a_n z^n$ is obvious (it exists when $\lim_{k \to \infty} \sum_{n=0}^{k} a_n z^n$ exists)

and the general picture is done at least twice in undergraduate courses:

Given $\sum_{n=0}^{\infty} a_n z^n$ there is a radius of convergence R so that

(i) $\sum a_n z^n$ only converges if $z = 0$
$\iff R = 0$

(ii) If $R > 0$ $\sum a_n z^n$ converges if $|z| < R$ and may or may not converge if $|z| = R$

(iii) If $R = \infty$, $\sum_{n=0}^{\infty} a_n z^n$ always converges.

(iv) $R = \left(\overline{\lim} \, |a_n|^{\frac{1}{n}} \right)^{-1}$ and absolute convergence inside disc, uniform as well.

(absolute convergence $\overset{\text{why?}}{\Rightarrow}$ convergence)

Elementary functions may be defined in this way:

$$\exp(z) = e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad \sin z = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2n+1}}{(2n+1)!}$$

Power series can be added, composed and multiplied. The estimates are not too hard and covered in an undergraduate course

in particular

$$\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} a_n b_{m-n}\right) z^m$$

and we get

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \left(= \mathrm{Re}(e^{iz}) \text{ if } z \in \mathbb{R} \right.$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \left. = \mathrm{Im}(e^{iz}) \text{ if } z \in \mathbb{R} \right)$$

<u>Exercise</u> Show $e^{z_1+z_2} = e^{z_1} e^{z_2}$

So $e^{x+iy} = e^x(\cos y + i \sin y)$

which we could take as the definition of e^z.

always remember

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \quad \text{if } \theta \in \mathbb{R}$$

Complex differentiation

Recall that a function $f: U \to \mathbb{R}^n$, $U \underset{\text{open}}{\subset} \mathbb{R}^m$

is said to be differentiable at $x \in \mathbb{R}^m$ if

there is a linear transformation $L: \mathbb{R}^m \to \mathbb{R}^n$

with $\lim_{y \to 0} \frac{\| f(x_0+y) - f(x_0) - L(y) \|}{\|y\|} = 0$

recall: the matrix of the linear map $L = Df_{x_0}$

is $\left.\frac{\partial y_i}{\partial x_j}\right|_{x_0}$, all partial derivates exist.

And if $\frac{\partial y_i}{\partial x_j}$ exist $\boxed{\text{and are continuous}}$ $\boxed{\text{the class } C^1}$

at x_0 then f is differentiable at x_0

Calculus intuition then f "looks like" Df in a neighborhood of x_0

Now Let $f: U \to \mathbb{C}$, $U \underset{\text{open}}{\subset} \mathbb{C}$. We say

f is (complex) differentiable at $z_0 \in U$ if

$$\lim_{\substack{w \in \mathbb{C},\, w \to 0 \\ w \neq 0}} \left| \frac{f(z_0+w) - f(z_0)}{w} - f'(z_0) \right| = 0$$

for some $f'(z_0) \in \mathbb{C}$.

(i.e. $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $0 < |w| < \delta$
$\Rightarrow \left| \frac{f(z_0+w) - f(z_0)}{w} - f'(z_0) \right| < \varepsilon$)

e.g. $f(z) = z$ $\qquad \frac{f(z_0+w) - f(z_0)}{w} = 1$

so $f'(z)$ exists $\forall z$.

<u>Now</u> $\left| \frac{f(z_0+w) - f(z_0)}{w} - f'(z_0) \right|$

$= \frac{1}{|w|} \left| f(z_0+w) - f(z_0) - f'(z_0) \cdot w \right|$

So since <u>multiplication by $f'(z_0)$</u> is (real) linear, f is differentiable as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Moreover any real linear map $L : \mathbb{C} \to \mathbb{C}$ is multiplication by a complex number iff it is $\mathbb{C}$-linear, or $L(iz) = iL(z)$.

Hence if f is (real) diff'able, it is

$\boxed{\text{complex diff'able} \iff Df_{z_0} \text{ is complex linear}}$

In terms of partial derivatives this means simply that the matrix of partial derivatives must commute with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \leftarrow$ multiplication by i.

if $f(x+iy) = u(x,y) + i v(x,y)$ this means that, at z_0

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial u}{\partial y} \\ -\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} \quad \text{i.e.} \quad \boxed{\begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{array}}$$

These are the famous Cauchy-Riemann equations.

We say that f is <u>holomorphic in Ω</u>, $\Omega \subseteq \mathbb{C}$ open if it is holomorphic at every point of Ω.

This point of view makes one thing completely transparent - the equivalence between holomorphicity and conformality.

For a non-zero linear transformation of $\mathbb{R}^2$ to be conformal it obviously (check) must be a composition of a rotation and a scaling, i.e. it must be complex linear, or multiplication by a complex number QED ! So $\frac{z-i}{z+i}$ maps $\mathbb{H}$ conformally to disc.

Note that $f'(z) = M_{f'(z)} 1 = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{df}{dz}$$

Just as in elementary calculus we have

$$(fg)' = f'g + fg' \quad (f+g)' = f' + g'$$

and $$(f/g)' = \frac{f'g - g'f}{g^2} \leftarrow \text{usual caveat.}$$

$$(f \circ g)'(z) = f'(g(z))g'(z)$$

Thus polynomials in z, for instance can be differentiated as usual.

Note also that we may define the operators on differentiable $\mathbb{C}$-valued functions $C^1(\Omega)$ by $\frac{\partial}{\partial z} f = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}$ $\quad \frac{\partial}{\partial \bar{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}$

and $f \in C^1$ is holomorphic $\Leftrightarrow$ $\underline{f \in \ker \frac{\partial}{\partial \bar{z}}}$

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0$$

For e^z, if it was defined as $e^x(\cos y + i\sin y)$

one has $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$ e $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^x \sin y$

so e^z is <u>holomorphic everywhere</u> i.e.

<u>entire</u> (definition), and $\frac{d}{dz} e^z = e^z$

<u>Or</u> one could use the theorem $\frac{d}{dz}\left(e^{z^2}\right) = 2ze^{z^2}$

(proved exactly as in elementary calculus)

Theorem A power series $\sum_{n=0}^{\infty} a_n z^n = f(z)$ is complex differentiable within its circle of convergence (interior) and

$$\frac{df}{dz} = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

This gives us a vast supply of holomorphic functions and of course conformal mappings.

Complex integration

The functions we will integrate will require no heavy measure theory. They'll all be continuous in fact except at a few points and improper integrals will be obvious.

Let $f: I \to \mathbb{C}$ be continuous
(I interval $[a,b]$ in $\mathbb{R}$)

then $\int_a^b f(x)\,dx = \int_a^b \operatorname{Re} f(x)\,dx + i\int_a^b \operatorname{Im} f(x)\,dx.$

Notes (i) $\int_a^b$ is linear

(ii) $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

(proof let $\lambda = \dfrac{\left|\int f(x)\,dx\right|}{\int_a^b f(x)\,dx}$

$$\left|\int f(x)\,dx\right| = \lambda \int_a^b f(x)\,dx = \int_a^b \lambda f(x)\,dx$$

$$= \int_a^b \operatorname{Re}(\lambda f(x))\,dx \leq \int_a^b |\lambda f(x)|\,dx$$

$$\leq |\lambda| \int_a^b |f(x)|\,dx$$

$$= \int_a^b |f(x)|\,dx.$$

(or by Riemann sums)

(iii) $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$ if $a < c < b$.

(iv) if $f_n \xrightarrow{\text{uniformly}} f$ on $[a,b]$

then $\int_a^b f_n\,dx \to \int_a^b f\,dx$

Recall

<u>Uniform convergence</u> on a <u>Set</u> S

$f_n \xrightarrow{\text{un}} S$ iff $\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| \to 0$

<u>or</u> $\forall \varepsilon > 0\ \exists N$ s.t. $|f_n(x) - f(x)| < \varepsilon\ \forall x, n \geq N$

<u>now proof</u> $\int_a^b (f - f_n)\,dx \leq \int_a^b |f - f_n|(x)\,dx$

$$\leq \sup |f - f_n|(x)\,(b-a)$$

$$\to 0.$$

Let's talk about curves.

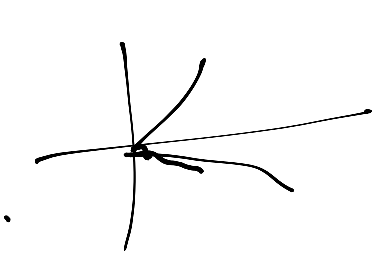
A (parametrized) curve is a cts. function $\gamma : [a,b] \to \mathbb{C}$.
c:

NB. continuous is not necessarily nice!

A smooth (parametrized) curve is one for which γ is C^1 and $\gamma' \neq 0$ on $[a,b]$.

- One thinks of the curve as a subset of $\mathbb{C}$. Two parametrisations $[a,b]: \gamma_1, \gamma_2 : [a',b']$ are equivalent if there is a C^1 function $\beta : [a,b] \to [a',b']$ with $\beta'(x) > 0 \ \forall x \in [a,b]$ (so there is a C^1 inverse) $\beta(a) = a'$, $\beta(b) = b$ with $\gamma_2 = \gamma_1 \circ \beta$. Note that the same subset ~~of~~ $\mathbb{C}$ can be parametrized in two inequivalent ways so we can talk about oriented curves as curves in $\mathbb{C}$ with an equivalence class of parametrizations so endpoints are "first" & "second".

- the hypothesis $\gamma' \neq 0$ is vital to avoid things like $\gamma(t) = (t^2, t^3)$ near 0.



- curves are also allowed to be *piecewise* smooth with definitions we can safely imagine. Continuity is still required.

examples: Smooth $\gamma(t) = e^{2\pi i t}$ $t \in [0,1]$

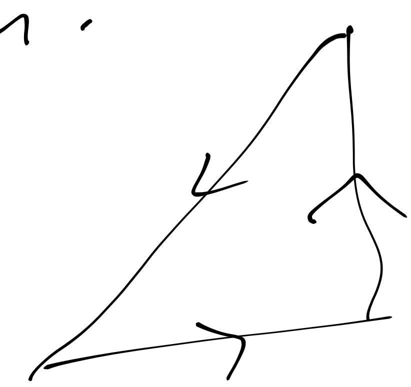
a curve is called closed if $\gamma(a) = \gamma(b)$ and simple if $\gamma(s) = \gamma(t) \Rightarrow s = t$ for $t \in (a,b)$.

positive orientation

piecewise smooth

$$\gamma(t) = \begin{cases} (t,0) & 0 \le t \le 1 \\ (1,t-1) & 1 \le t \le 2 \\ (3-t, 3-t) & 2 \le t \le 3 \end{cases}$$

piecewise smooth.



- there's no requirement of C^1 at the end points of the component intervals but we'll see that will be fine for integration.

- a and b could always be chosen as $[0,1]$.

<u>Definition</u> C an <u>oriented</u> piecewise smooth curve in $U \overset{\text{open}}{\subseteq} \mathbb{C}$

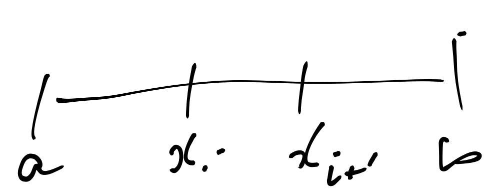
$f: U \to \mathbb{C}$ holomorphic.

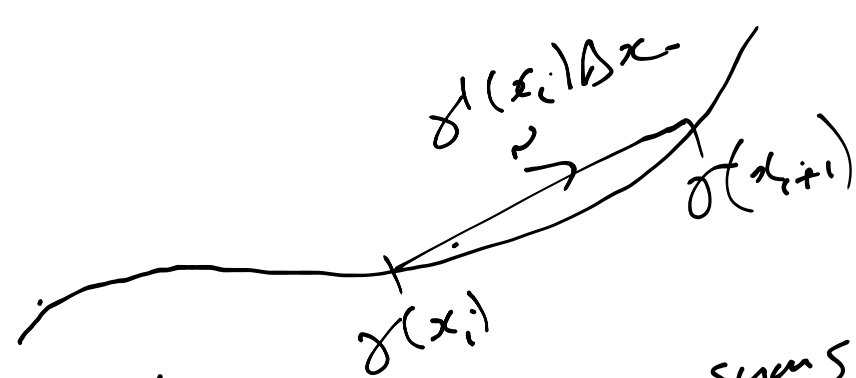
Then $\displaystyle\int_C f(z)\,dz \overset{\text{def}}{=} \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

for some paramatrization in the class for the orientation. (Observe: holomorphic $\Rightarrow$ continuous $\Rightarrow$ integral exists)

Indendence of Paramatrization (on each smooth part separately) follows from the change of variables formula but is probably better thought of in terms of Riemann sums:

$\xrightarrow{\text{contribution}}$ $f(\gamma(x_i))\,\underbrace{\gamma'(x_i)\,\Delta x_i}_{\text{a small vector from } \gamma(x_i) \text{ to } \gamma(x_2)}$





Any parametrisation will give Riemann sums for the same thing.

<u>Observe</u> (i) $\displaystyle\int_a^b |\gamma'(t)|\,dt$ = length of curve

(ii) If $f = g'$ in Ω (f has a primitive) $\displaystyle\int_C f\,dz = g(\text{second endpoint}) - g(\text{first endpoint})$

Examples

$$\boxed{\int_C (z-z_0)^n\, dz}$$

$\gamma(t) = Re^{it} + z_0$
$0 \le t \le 2\pi$

C = circle of radius R
positive orientation
centered at z_0

$\frac{d\gamma}{dt} = iRe^{it}$

so answer $= \int_0^{2\pi} iRR^n e^{int} e^{it}\, dt$

$$= i R^{n+1} \underbrace{\int_0^{2\pi} e^{i(n+1)t}\, dt}_{=0 \text{ unless } n=-1}$$

$$= \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

Note: we could avoid doing the integral for $n \neq -1$, using the fundamental theorem of calculus.

Thm If f has a primitive (on Ω) $\frac{dg}{dz} = f$
then $\int_C f\, dz = g(\gamma(b)) - g(\gamma(a))$.

Proof $\frac{d\, g\circ\gamma}{dt} = \frac{dg}{dz}\frac{dz}{dt} = f(\gamma(t))\gamma'(t)$
+ ordinary F.T.C. to Reg, Img

Now obvious estimate

$$\left|\int_C f(z)\,dz\right| \le \sup_{z\in C}|f(z)|\ \underline{\text{length}(C)}$$

So if $f_n \to f$ uniformly on $C \subseteq \Omega$

$$\int_C f_n(z)\,dz \to \int_C f(z)\,dz.$$

since $\left|\int_C f_n(z) - f(z)\,dz\right| \le \sup_{z\in C}|f_n - f|\ \text{length}(C) \to 0.$

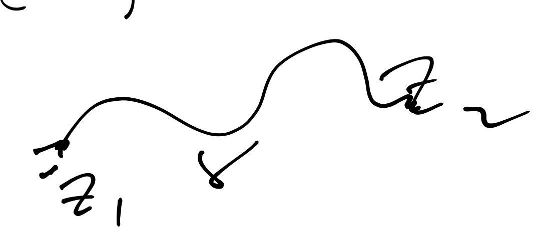
Corollary
$$\int_{\text{circle of radius } r} \left(\sum_{n=0}^{\infty} a_n z^n\right) dz = 0 \qquad r<R$$

- observe that if $-C$ is C with the opposite orientation then

$$\int_{-C} f(z)\,dz = -\int f(z)\,dz.$$

since if $\gamma : [0,1] \to \mathbb{C}$ parametrizes C, $\hat{\gamma}(t) = 1-t$ parametrizes $-C$.

- Corollary if Ω is connected (= path connected) and $f'(z) = 0$ on Ω then f = constant.



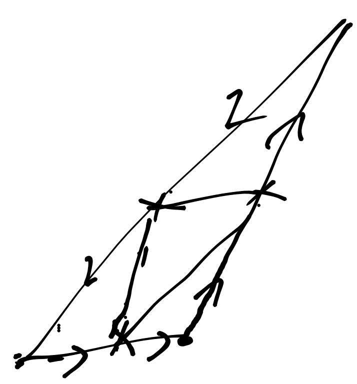
$$0 = \int_\gamma 0\,dz = \int_\gamma f'(z)\,dz = f(z_2) - f(z_1)$$

Cauchy-Goursat for a triangle

Let f be holomorphic in $\Omega \subset$ open, and T be an (oriented) "triangle (curve)" (with its interior) in Ω. Then

$$\int_{\Delta} f(z)\,dz = 0.$$

<u>Proof</u> Note any T splits into 4 triangles $T/4$ similar to T given by a scaling by 2 and a rotation/translation:



and $\int_{\partial T} f\,dz = \sum_{S \in T/4} \int_{\partial S} f\,dz$

Inductively construct a sequence T_i such that:

a) $T_0 = T$

b) $T_{i+1} \in T_i/4$

c) $\left| \int_{\partial T_{i+1}} f\,dz \right| \geq \frac{1}{4} \left| \int_{\partial T_i} f\,dz \right|$

Observe: $\operatorname{diam}(T_i) = \frac{1}{2^i} \operatorname{diam} T$

$\text{perimeter}(T_i) = 1/2^i \text{ perimeter } T$

$$\left| \int_{\partial T_i} f\,dz \right| \geq \frac{1}{4^i} \left| \int_{\partial T} f\,dz \right|$$

The T_i form a decreasing sequence of closed sets so $\bigcap_i T_i$ consists of a single point z_0.

Let N be such that there is an open ball B centred at z_0 with $T_N \subset B \subset \Omega$.

Then $T_n \subset B \subset \Omega$ $\forall n \geq N$ and in T_N we have

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + R(z)$$

with $\dfrac{R(z)}{z - z_0} \to 0$ as $z \to z_0$.

Now let $\varepsilon > 0$ be given and choose $n \geq N$ large enough so that

$$|R(z)| \leq \varepsilon |z - z_0| \quad \text{for } z \in T_n$$

Then $\int_{\partial T_n} f(z)\,dz = \int_{\partial T_n} R(z)\,dz$ since $f(z_0) + (z - z_0) f'(z_0)$ has a primitive and

$$\left| \int_{\partial T_n} f(z)\,dz \right| \leq \int_{\partial T_n} \varepsilon |z - z_0|\,dz$$

$$\leq \varepsilon \frac{1}{2^n} \operatorname{diam}(T_1) \frac{1}{2^n} \operatorname{perimeter}(T)$$

$$\leq \frac{\varepsilon}{4^n} \text{ (constant)}$$

So as soon as $\varepsilon < \frac{1}{\text{constant}}$ we contradict $(*)$. $\underline{QED}$ unless $\int_{\partial T} f(z)\,dz = 0$.

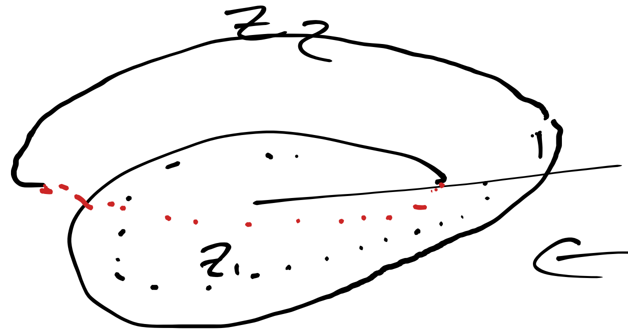
Interlude on $\sqrt{z}$, $\log z$, branches.

If we dec on one square root of z for $z=4$, say, we can certainly extend it to a holomorphic function in a neighborhood of 4. $\sqrt{r}\,e^{i\theta/2}$ for $-\pi<\theta<\pi$ say, which is holomorphic. But as we follow z around a circle from $\theta=0$ to $\theta=\pi$ and $\theta=0$ to $\theta=-\pi$, there is no choice of $\sqrt{z}$ if we want the function $\sqrt{z}$ to be continuous. Thus we cannot extend it to a continuous, let alone holomorphic, function of z.

By definition a branch for $\sqrt{z}$ is an open set Ω together with a holomorphic function f of $z\in\Omega$ for which $f(z)^2=z$. It is common to take $\mathbb{C}-\{z\,|\,Re(z)>0\}$ a "branch with cut" $f(z)=\sqrt{r}\,e^{i\theta/2}$ as a branch of $\sqrt{z}$. Exactly the same considerations apply to $\log z = \log r + i\theta$ and $\exp(z)$ except that $\log$ is not defined at 0.

One could also try to solve the problem by making $\sqrt{z}$ a "multiple valued" function by choosing both square roots, or $\log z = \{\log r + i\theta + 2n\pi i \,|\, n\in\mathbb{Z}\}$.

But the approach with a future is that of Riemann surfaces where, say for $\sqrt{z}$, we take two copies of $\mathbb{C}$, $\{z_1\}$ and $\{z_2\}$ and declare $r_1e^{i\theta_1}$ with θ_1 near 2π to be distant from $r_1e^{i\theta_1}$ for θ_1 near zero, and identify $r_1e^{i2\pi}$ with $r_2e^{i\theta_2}$. The same for z_2, and ... This can be done with a bit of sleight of hand but is a little difficult to visualize since it can't be

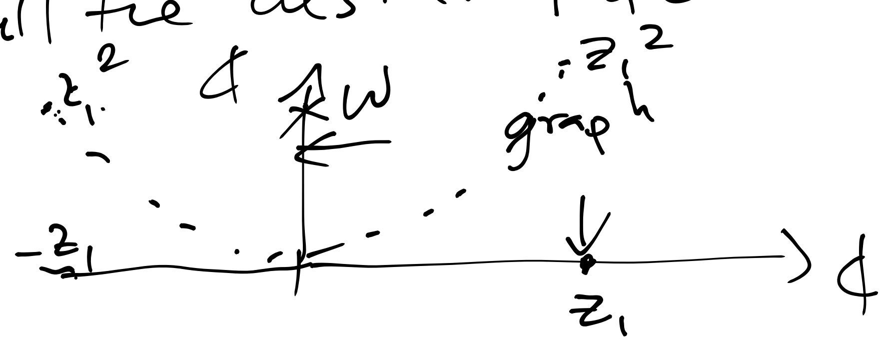


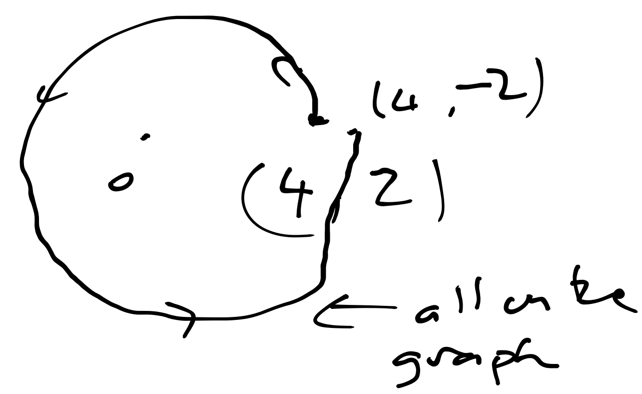
$\sqrt{z}$ is then a single valued function on the surface.

← topologically a covering space of $\mathbb{C}-\{0\}$ (away from 0)

done in $\mathbb{R}^3$. Same for $\log z$ with infinitely many sheets. There is a simple remedy though for this scissors and paste construction.

For $\sqrt{z}$ just consider $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = z_1^2\}$, the graph of the function z^2. This "curve" has all the desired properties.





One could have done this for $f(x) = x^2$ $f: \mathbb{R} \to \mathbb{R}$ and defined $\pm\sqrt{y}$ as a single valued function on the graph. But why?

$y^2 = x^3 + x$ → a subset of $\mathbb{C}^2$.

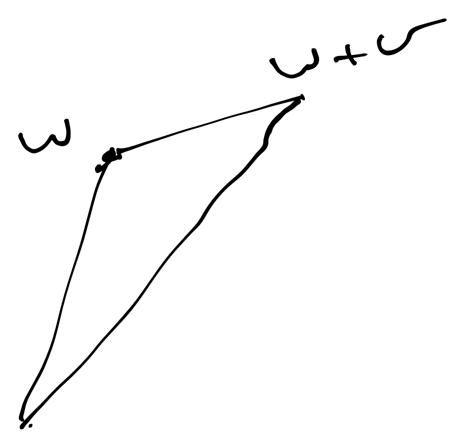
$= \{(z_1, z_2) : z_1^2 = z_2^3 + z_2\}$ Riemann surface.

Theorem Let Ω be a convex open subset of $\mathbb{C}$ and f be holomorphic on Ω. Then

$$\int_C f(z)dz = 0 \quad \forall \text{ closed curves in } \Omega.$$

Proof It suffices to show that f has a primitive.

Choose $z_0 \in \Omega$ and define $g(w) = \int_\gamma f(z)dz$ where $\gamma(t) = (1-t)z_0 + tw$ for $0 \le t \le 1$. The claim is that $\frac{dg}{dz} = f$ in Ω. Increment w by v



clearly $g(w+v) - g(w)$

$$= \int_\nu f(z)dz$$

where $\nu(t) = w + tv$ $0 \le t \le 1$ - a path of length $|v|$. Now by continuity of f (implied by holomorphic), for $\varepsilon > 0$ if we choose $0 < |v| < \delta$ we have $|f(w+tv) - f(w)| < \varepsilon$, $0 \le t \le 1$

so

$$\left| \frac{g(w+v) - g(w)}{v} - f(w) \right|$$

$$= \left| \frac{1}{v}\int_0^1 f(w+tv) \cdot v\,dt - \int_0^1 f(w)dt \right|$$

$$= \left| \int_0^1 f(w+tv) - f(w)\, dt \right|$$

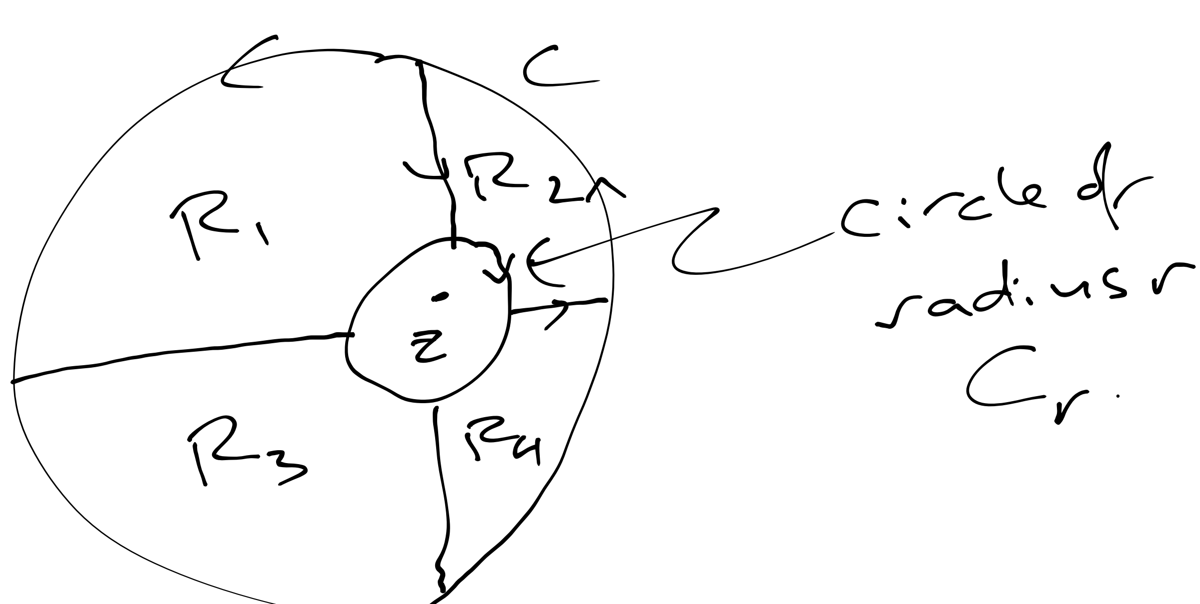
$$< \varepsilon.$$

Cauchy's formula for a circle

If C is a counterclockwise / positive oriented circle which, along with its interior, is in Ω open subset of $\mathbb{C}$, and f is holomorphic on Ω, then $\forall z \in \underline{\text{Interior}}(C)$,

$$\boxed{f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta}$$

Proof (Sarason's notes)



observe that

$$\Rightarrow 0 = \sum_{i=1}^{4} \int_{\partial R_i} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{C_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

All R_i contained in convex regions in $\Omega - \{z\}$ positive orientation by continuity of f, for r small and $\varepsilon > 0$ given we have

$$\left| \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{C_r} \frac{f(z)}{\zeta - z} \right| < \varepsilon \left| \int_{C^r} \frac{1}{\zeta - z} dz \right|$$

$$= \varepsilon \, 2\pi$$

so

$$\int_C \frac{f(\zeta)}{\zeta - z} dz = f(z) \int_{C_r} \frac{1}{\zeta - z} dz$$

$$= 2\pi i f(z)$$

QED.

Observe mean value property

$$f(z) = \frac{1}{2\pi} \int f(z + re^{i\theta}) \, d\theta \qquad z \in \Omega, \; r \text{ small}$$

Cauchy integral (defn) C a piecewise C^1 curve. ϕ cts on

$$C \text{ Integral} = \int_C \frac{\phi(\zeta)}{\zeta - z} d\zeta = f(z)$$

(e.g. f holomorphic, inside the circle. a circle, $CI = 2\pi i f(z)$)

Thm. The series

$$\sum_{n=0}^{\infty} \left(\int_C \frac{\phi(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n$$

converges to $f(z)$ for $|z - z_0| <$ distance$(z_0, C) = R$

Proof Write $\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)}$ and observe that $|\zeta - z_0| \geq R$, so for $|z - z_0| < R$ $\sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n$ converges uniformly in some ball around ζ which guarantees $\left|\frac{z - z_0}{\zeta - z_0}\right| \leq r < R$. Covering C with such balls and using compactness of C gives the uniform convergence of the sum so that

$$\int_C \frac{\phi(\zeta)}{\zeta - z} d\zeta = \sum_{n=0}^{\infty} \left(\int \frac{\phi(\zeta) d\zeta}{(\zeta - z_0)^{n+1}} \right) (z - z_0)^n$$

Thus $\int_C \frac{\phi(\zeta)}{\zeta - z} d\zeta$ is infinitely (complex) differentiable and $f^{(n)}(z_0) = n! \int_C \frac{\phi(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$. The radius of convergence of the series is at least $d(z_0, C)$. QED

Corollary If f is holomorphic in Ω then $f^{(n)}(z)$ exists at all $z \in \Omega$ and is given by $\frac{n!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$ where $z \in C_r$, C_r is a circle centred at z_0, radius $< d(z_0, \mathbb{C} \setminus \Omega)$ and f is given by its Taylor series at z_0, $\forall z$ with $|z - z_0| < d(z_0, \mathbb{C} \setminus \Omega)$