Chapter 2 Connes-Sauvageot tensor product

2.1) Quick review of Tomita Takesaki theory & Connes Classification.

If M is a von Neumann algebra a faithful normal semifinite weight φ on M is a map $\varphi: M_+ \to [0,\infty]$ with the properties $\varphi(x+y)=\varphi(x)+\varphi(y)$, $\varphi(\lambda x)=\lambda\varphi(x)$, $\lambda\geq 0$, $\varphi(x)=0 \Rightarrow x=0$, $\varphi(\vee x_i)=\vee\varphi(x_i)$ for $x_i\uparrow$ an increasing net, bounded above, and $P_\varphi=\{x\in M_+ \mid \varphi(x)<\infty\}$ generates M. We will be most concerned with the special case of a state where φ extends to a linear function on M (and semifiniteness is automatic) with $\varphi(1)=1$

- Given a f.n sf weight φ one may form two Hilbert spaces

a) let $\mathcal{N}_\varphi=\{x \mid \varphi(x^*x)<\infty\}$ and $\mathcal{M}_\varphi=\{\sum y_i^*x_i \mid x_i,y_i\in\mathcal{N}_\varphi\}$. Then $\mathcal{N}_\varphi$ is a left ideal, $\mathcal{M}_\varphi$ is a subalgebra of M and φ extends to a linear functional on $\mathcal{M}_\varphi$. We then define the inner product $\langle x,y\rangle=\varphi(y^*x)$ on $\mathcal{N}_\varphi$. The inequality $(ax)^*ax\leq\|a\|^2x^*x$ shows that left multiplication on $\mathcal{N}_\varphi$ extends to a bounded linear operator $\pi_\varphi(a)$ on the completion $\mathcal{H}_\varphi$ ($L^2(M,\varphi)$) of $\mathcal{N}_\varphi$, $\pi_\varphi(M)$ is a von Neuman algebra on $\mathcal{H}_\varphi$

π_φ is faithful and $\pi_\varphi(M)$ is a

b) $_\varphi\mathcal{N}$ exactly the same except $\varphi(xx^*)$. Gets $[\varphi, M]$, a right M-module

If we define the map $S:\mathcal{N}_\varphi\cap\mathcal{N}_\varphi^*\to\mathcal{N}_\varphi\cap\mathcal{N}_\varphi^*$ by $S(x)=x^*$

Then S is a pre-closed conjugate linear involution on $L^2(M,\varphi)$ (identifying $\mathcal{N}_\varphi\cap\mathcal{N}_\varphi^*$ with a subspace of $L^2(M,\varphi)$). If $S=J\Delta^{\frac12}$ is ($J^2=1$) the polar decomposition of S then the TT theorem states that (i) $J\pi_\varphi(M)J=\pi_\varphi(M)'$ (ii) $\Delta^{it}\pi_\varphi(M)\Delta^{-it}=\pi_\varphi(M)$. Note that if $x\in\mathcal{N}_\varphi\cap\mathcal{N}_\varphi^*$ satisfies $\varphi(xy)=\varphi(yx)$ $\forall y\in M$ then $x\in\mathrm{Dom}(S^*S)$ and $\Delta x=x$, $\&$ $Jx=x^*$.

Thus

(i) $L^2(M,\varphi)$ becomes a bimodule (or correspondence) via

$$x\xi y = xJy^*J\xi$$

(ii) There is a 1-parameter group of automorphisms σ_t^φ on M defined by $\pi_\varphi(\sigma_t^\varphi(x)) = \Delta_\varphi^{it}\pi_\varphi(x)\Delta_\varphi^{-it}$, called the modular group.

Connes showed that σ_t^φ is independent of φ up to inner perturbations, i.e. if ψ is another n.f.s.f. weight then $\exists$ a map $t \to v_t$ from $\mathbb{R}$ to $\mathcal{U}(M)$ such that

$$\sigma_t^\psi(x) = v_t\sigma_t^\varphi(x)v_t^* \qquad (\text{and } v_s\sigma_s(v_t) = v_{s+t})$$

He also classified type III factors according to $\bigcap_\varphi \text{spectrum}(\Delta_\varphi)$ which automatically contains 0 in the type III case, so leave 0 out in following:

$\bigcap_\varphi \text{spectrum}\,\Delta_\varphi = \{1\}$ type III_0

$\bigcap_\varphi \text{spectrum}\,\Delta_\varphi = \{\lambda^n \mid n \in \mathbb{Z}\}$ type III_λ $(0<\lambda<1)$

$\bigcap_\varphi \text{spectrum}\,\Delta_\varphi = \mathbb{R}^+$ III_1.

It is known that if σ_t^ψ acts ergodically, i.e. $\sigma_t^\psi(x) = x \ \forall t$ $\Rightarrow x \in \mathbb{C}\,\mathrm{id}$, then M is of type III_1

2.2 Spatial Theory

If M acts on $\mathcal{H}$, a fundamental idea (due to Connes) is to model $\mathcal{H}$ on (the closure of) the space of M-linear maps from $L^2(M,\psi)$ to $\mathcal{H}$. To this end we say that a vector $\xi \in \mathcal{H}$ is <u>bounded</u>, if the map $x \mapsto x\xi$ extends to a bounded linear map R_ξ from $L^2(M,\psi)$ to $\mathcal{H}$. This is equivalent to the existence of a $C > 0$ with $\|x\xi\|^2 < C\psi(x^*x)$. R_ξ if it exists is clearly M-linear. Similarly if $\mathcal{H}$ is a right M-module (or a left module over M^{opp}), we say a vector $\eta \in \mathcal{H}$ is bounded if the map $Jx^* \mapsto \eta x$ (defined for all x with $\psi(xx^*) < \infty$) extends to a bounded map L_η from $L^2(M,\psi)$ to $\mathcal{H}$. L_η is then right M-linear since the right action of M on $L^2(M,\psi)$ is $\xi x = Jx^*J\xi$ and for $a \in M$

$$
\begin{aligned}
L_\eta(Jx^* \cdot a) &= L_\eta(Ja^*JJx^*) \\
&= L_\eta(Ja^*x^*) \\
&= \eta x a \\
&= L_\eta(Jx^*)a
\end{aligned}
$$

$\left(\begin{array}{c}\text{note that} \\ L_{\xi\Omega_\psi^{1/2}}(b) = L_\xi b\end{array}\right)$

It is true that every M-linear map from $L^2(M,\psi)$ to $\mathcal{H}$ is of the form R_ξ. This is very easy to see if ψ is a state for then $1 \in L^2(M,\psi)$ and given $t: L^2(M,\psi) \to \mathcal{H}$, M-linear, let $\xi = t(1)$. Then for $a \in M$, $t(a) = t(a \cdot 1) = at(1) = R_{t(1)}(a)$. Similarly for a right M module and a map t, let $\eta = t(1)$. Then $t(Ja^*) = t(J1a^*) = t(Ja^*) \cdot 1) = t(Ja^*)t(1)$.

Moreover the inner product in $\mathcal{H}$ can be recovered from R_ξ, L_η as follows:

$R_{\xi_1}^* R_{\xi_2}$ is M-linear from $L^2(M,\psi)$ to itself, so

$J R_{\xi_1}^* R_{\xi_2} J \in M$ so we may form $\psi(J R_{\xi_1}^* R_{\xi_2} J)$

$= \langle J R_{\xi_1}^* R_{\xi_2} J \cdot 1, 1\rangle = \langle R_{\xi_1}^* R_{\xi_2} \cdot 1, 1\rangle = \langle \xi_2, \xi_1\rangle$

Similarly ~~if~~ $L_{\eta_1}^* L_{\eta_2} \in M$ and $\psi(L_{\eta_1}^* L_{\eta_2}) = \langle L_{\eta_1}^* L_{\eta_2} 1, 1\rangle$
$= \langle \eta_2, \eta_1\rangle$.

This leads to the following idea, due originally to Connes but reformulated following Takesaki vol II of the Connes spatial derivative

Suppose we are given M on $\mathcal{H}$ with ~~nfs~~ weights φ on M and ~~..~~ ψ on M'. Form $L^2(M',\psi)$ as a left M' module and suppose $\xi, \eta \in \mathcal{H}$ are bounded for the M'-actions. Then $R_\xi R_\eta^*$ is M'-linear from $\mathcal{H}$ to $\mathcal{H}$, hence an element of M. Provided it is in the domain of φ (automatic if φ is a state) we may evaluate φ on it to obtain the positive definite sesquilinear form

$\{\xi, \eta\} = \varphi(R_\xi R_\eta^*)$ on a dense subspace of $\mathcal{H}$. There is a corresponding positive self-adjoint (unbounded in general) operator $\frac{d\varphi}{d\psi}$ whose square root is essentially self-adjoint on the domain specified above.

The operator was defined by Connes and is called the spatial derivative of φ w.r.t. ψ. It satisfies nice properties, especially

$$\left(\frac{d\varphi}{d\psi}\right)^{it} x \left(\frac{d\varphi}{d\psi}\right)^{-it} = \sigma_t^{\varphi}(x), \quad x \in M$$

$$\left(\frac{d\varphi}{d\psi}\right)^{-it} y \left(\frac{d\varphi}{d\psi}\right)^{it} = \sigma_t^{\psi}(y) \quad y \in M'.$$

Let us do a simple exercise where domain questions vanish, $M = \mathbb{C}$, $\mathfrak{H} = \mathbb{C}^n$. Then $M' = M_n(\mathbb{C})$ and a state on M' is given by $\psi(x) = \mathrm{Trace}(hx)$ where h is positive definite, $\mathrm{Trace}(h) = 1$. There is no choice for the state φ ..... The main thing is to determine R_v^* for $v \in \mathbb{C}^n$. Let us use the notation $|v\rangle\langle w|$ for the operator $|v\rangle\langle w|(u) = \langle u, w\rangle v$ for vectors v & $w \in \mathbb{C}^n$.

<u>Proposition</u> $R_v^*(w) = |w\rangle\langle v| h^{-1}$

<u>Proof</u> $\langle R_v^*(w), x\rangle = \langle w, R_v x\rangle = \langle w, xv\rangle = \langle x^* w, v\rangle$

But also
$$\langle |w\rangle\langle v| h^{-1}, x\rangle = \varphi(x^* |w\rangle\langle v| h^{-1})$$
$$= \mathrm{Trace}(h x^* |w\rangle\langle v| h^{-1})$$
$$= \mathrm{Trace}(x^* |w\rangle\langle v|)$$
$$= \langle x^* w, v\rangle$$
(complete v to a basis, normalise)

It is now easy that $\frac{d\varphi}{d\psi} = h^{-1}$:

$R_u R_v^* w = R_u(|w\rangle\langle v| h^{-1}) = |w\rangle\langle v| h^{-1}(u) = \langle h^{-1}u, v\rangle w$

so that $R_u R_v^* = \langle h^{-1}u, v\rangle\, \mathrm{id}$ and $\varphi(R_u R_v^*) = \langle h^{-1}u, v\rangle$. Note the special case φ = normalized trace, $h = \frac{1}{n}\,\mathrm{id}$ and $\frac{d\varphi}{d\psi} = n = \dim_{\mathbb{C}} \mathbb{C}^n$.

2.3 The relative tensor product.

If $\mathcal{H}_M$ is a right M-module and ${}_M\mathcal{K}$ is a left one, and ψ is a n.f.s. weight on M we consider the vector spaces $\mathcal{H}^0$ and $\mathcal{K}^0$ of bounded vectors. On the (purely algebraic) tensor product we define the sesquilinear form

form $$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle L_{\xi_2}^* L_{\xi_1} \eta_1, \eta_2 \rangle$$

if ψ is a state this is just $\langle R_{\eta_2}^* \underbrace{L_{\xi_2}^* L_{\xi_1}}_{\in M} R_{\eta_1}(1), 1\rangle$

$$= \langle L_{\xi_2}^* L_{\xi_1} \underbrace{R_{\eta_2}^* R_{\eta_1}}_{\in JMJ}(1), 1\rangle$$

$$= \langle R_{\eta_2}^* R_{\eta_1} \xi_1, \xi_2 \rangle$$

Or if we had dispensed with bounded vectors and just considered intertwiners (bounded)

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_2^* x_1\, y_2^* y_1 1, 1\rangle .$$

Note though that the ~~inner product~~ sesquilinear form obviously extends to $\mathcal{H}^0 \otimes \mathcal{K}$ and $\mathcal{H} \otimes \mathcal{K}^0$. To see that $\langle , \rangle$ is positive semidefinite, we have to show

$$\langle \sum_{i=1}^n x_i \otimes y_i, \sum x_i \otimes y_i \rangle = \sum_{\substack{i=1\\ j=1}}^n \langle x_j^* x_i\, y_j^* y_i 1, 1\rangle \geq 0 .$$ But observe that $x_j^* x_i$ is positive as an element of $M \otimes M_n(\mathbb{C})$ and so is $y_j^* y_i$ in $M_n(M')$. So we can write $x_j^* x_i = \sum_p a_{pj}^* a_{pi}$, $y_j^* y_i = \sum_q b_{qj}^* b_{qi}$ for a's in M and b's in M'. So what we have to calculate is

$$\sum_{ijpq} \langle a_{pj}^* a_{pi} b_{qj}^* b_{qi} 1, 1\rangle = \sum_{ijpq} \langle a_{pi} b_{qi} 1, a_{jp} b_{qj} 1\rangle = \sum_{p,q} \| \sum_i a_{pi} b_{qi} 1 \|^2 \geq 0.$$

One may thus form the Hilbert space completion

$$\mathcal{H} \otimes_M \mathcal{K} \quad \text{of the quotient of } \overset{\circ}{\mathcal{H}} \otimes \mathcal{K}^\circ \text{ by}$$

the kernel of $\langle\, ,\rangle$. This is called the relative tensor product, Connes tensor product, Connes-Sauvageot tensor product or Connes fusion of $\mathcal{H}$ and $\mathcal{K}$.

Some properties are more or less immediate.

a) $\mathcal{H} \otimes_M \mathcal{K}$ is a $\mathcal{L}_{-M}(\mathcal{H}) - \mathcal{L}_{M-}(\mathcal{K})$ bimodule.

(exercise)

b) $\mathcal{H} \otimes_M L^2(M,\psi) \cong \mathcal{H}$ as a $\mathcal{L}_{-M}$ -M bimodule

(use the map $L_\xi \overset{R}{\otimes} Jm^* \Rightarrow \xi m$)

$L^2(M,\psi) \otimes_M \mathcal{H} \cong \mathcal{H}$ (use $L_x \otimes \xi \mapsto x\xi$)

c) If $\mathcal{H}_M$, ${}_M\mathcal{K}_N$ and ${}_N\mathcal{L}$ with weights(...)

$(\mathcal{H}_M \otimes_M \mathcal{K}) \otimes_N \mathcal{L} \cong \mathcal{H}_M \otimes_M (\mathcal{K} \otimes_N \mathcal{L})$. (check isometry property of algebraic t.p.)

d) The notion $\mathcal{H} \otimes_M \mathcal{K}$ does not depend on the weight ψ.
(this is the most difficult)

However, note that $\otimes_M$ is <u>not</u> middle M-linear. In fact $\xi a \otimes \eta = \xi \otimes \sigma^\psi_{-i/2}(a)\eta$ for appropriate ξ, η and a. (we ~~denote~~ mean by $\xi \otimes \eta$ its image in the completed quotient)

Example. The basic construction for subfactors, II_1 case

If $N \subset M$ is a subfactor of the II_1 factor M there is a canonical trace-preserving conditional expectation $E_N : M \to N$ which extends to the projection $e_N : L^2(M, tr) \to L^2(N, tr)$. The basic construction is the von Neumann algebra $\langle M, e_N \rangle$ on $L^2(M, tr)$ generated by M and e_N. There is a tracial normal faithful semifinite weight $Tr : \langle M, e_N \rangle$ with the subalgebra $M e_N M$ in its domain (note that $e_N x e_N = E_N(x) e_N$ for $x \in M$) satisfying

$$Tr(x e_N y) = tr(xy)$$

(to say that $[M:N]$ is finite, equal to r is to say $Tr(1) = r$).

Now $L^2(M)$ is an N-N bimodule.

Claim $\quad L^2(\langle M, e_N \rangle, Tr) = L^2(M) \otimes_N L^2(M)$.

Proof We define a map from the dense subspace $M e_N M$ to $L^2(M, tr) \otimes_N L^2(M, tr)$ by $x e_N y \mapsto x \otimes y$ (note that x and y are bounded vectors). For this it suffices to show that the inner products on the left and right are the same, thus

$$\langle x_1 e_N y_1, x_2 e_N y_2 \rangle = Tr(y_2^* e_N x_2^* x_1 e_N y_1)$$
$$= tr(y_2^* E_N(x_2^* x_1) y_1)$$
$$= tr(E_N(y_1 y_2^*) E_N(x_2^* x_1))$$

and $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = tr(L_{x_2}^* L_{x_1} R_{y_2}^* R_{y_1} 1, 1) = tr(\quad E_N(x_2^* x_1))$

so we need to calculate $L_x^* : L^2(M) \to L^2(N)$ claim $L_x^*(m) = E_N(x^* m)$

because $\langle L_x^*(m), n \rangle = \cancel{tr(n^* L_x^*(m))}$ $\langle m, xn \rangle = tr(n^* x^* m)$

and $\langle E_N(x^* m), n \rangle = tr(n^* E_N(x^* m)) = tr(E_N(n^* x^* m))$
$= tr(n^* x^* m)$ etc.