Chapter 11 Classification of positive energy representations of $LSU(2)$

Definition A positive energy (irreducible) representation of $LSU(N)$ at level ℓ will be an irreducible projective unitary representation of $LSU(N) \rtimes \mathrm{Rot}\, S^1$ occurring as a direct summand of $\mathcal{F}_p$ where $p: L^2(S^1, \mathbb{C}^{N\ell}) \to H^2(S^1, \mathbb{C}^{N\ell})$ is Hardy space projection and $LSU(N)$ is embedded diagonally into $U(\overbrace{\mathbb{C}^N \oplus \mathbb{C}^N \oplus \cdots \oplus \mathbb{C}^N}^{\ell \text{ copies}})$.

Let $\mathfrak{g} = su(N)$ (= antihermitian matrices of trace 0)

Theorem Let π on $\mathcal{H}$ be a +ve energy irrep of $LSU(N)$ at level ℓ, ~~Then there~~ and $\mathcal{H}^0$ the dense subspace of finite energy vectors (linear combinations of eigenvectors for $\mathrm{Rot}(S^1)$). Then there is a projective representation of $L^0\mathfrak{g} \rtimes \mathbb{R}$ trigonometric polynomials with values in $\mathfrak{g}$) on $\mathcal{H}^0$ satisfying

1) $[D, X(n)] = -nX(n) \quad (X(n) \in L^0\mathfrak{g} \otimes \mathbb{C},\ X(n) = z^n X)$

2) $D^* = D$

3) $X(n)^* = -X(-n)$

4) $[X(m), Y(n)] = [X, Y](n+m) + im\,\delta_{m,-n}\, \ell\, \mathrm{tr}(XY)$

5) Each $X(m)$ is essentially skew adjoint on $\mathcal{H}^0$

6) ~~$\exp(X(n) + X(-n))$~~ $\exp(t \cos m\theta X)$ and $\exp(t \sin m\theta X)$ agree with $\pi(e^{t\cos m\theta X})$ and $\pi(e^{t \sin m\theta X})$ (up to scalars)

~~7) The action of $L^0\mathfrak{g} \times \mathbb{R}$ on $\mathcal{H}^0$ is algebraically irreducible.~~

Proof The first job is to show that the $X(n)$'s previously defined on F_p^0, actually preserve $\mathcal{H}^0$. But if $u_t = e^{t\cos m\theta X}$ on F_p, then for $\xi \in F_p^0$, $\xi \in \mathrm{Dom}(\cos m\theta X)$ so $\frac{d}{dt} u_t \xi \big|_{t=0} = \cos m\theta X \xi \in \mathcal{H}$, hence by the same argument for $\sin m\theta$, $X(n)\xi \in \mathcal{H}_0$. This establishes 1) $\to$ 4) by restriction of the $X(n)$ on F_p^0 to $\mathcal{H}$.

If we denote by T the generator of the one parameter unitary group $u_t|_{\mathcal{H}}$, then $\mathrm{Dom}(T) \supset \mathcal{H}^0$. But by transplanting bosons as before, $\cos m\theta X|_{\mathcal{H}^0}$ is essentially self adjoint so its closure is equal to T and its exponential agrees with $u_t|_{\mathcal{H}}$. This establishes 5 and 6. QED.

Proposition The subgroups $e^{t\cos m\theta X}$ and $e^{t\sin m\theta X}$, for $X \in \mathfrak{g}$, generate a dense subgroup of $LU(N)$ (in the Sobolev $\frac{1}{2}$-norm topology).

Proof We have seen in the lecture on irrelevance of points that the groups $e^{tf(\theta)X}$ generate a dense subgroup of $LU(N)$ (because if t is small, there is a diffeo $\Theta: \mathbb{R}^{\dim \mathfrak{g}} \to \mathbb{R}^{\dim \mathfrak{g}}$ such that $e^{t\sum c_i X_i} = \prod e^{t\Theta(c_i)X_i}$, for X_i a basis of $\mathfrak{g}$, by the implicit function theorem). Thus it suffices to show that one may approximate $e^{tf(\theta)X}$, for $f \in C^\infty(S^1, \mathbb{R})$ by $\prod e^{ta_m \cos m\theta X} e^{tb_m \sin m\theta X}$. But this is clear by Fourier series and the continuity of exp shown also in the lecture on irrelevance of points. QED

<u>Corollary</u> With π, $\mathcal{H}$ as above, the representation of $L^0\mathfrak{g} \rtimes \mathbb{R}$ on $\mathcal{H}^0$ is algebraically irreducible.

<u>Proof</u> If $V \subseteq \mathcal{H}^0$ is an invariant subspace and $\xi \in V$, then $\xi \in$ a finite dimensional subspace invariant under D. Since e^{itD} is a polynomial in D on this subspace, V is invariant under $\mathrm{Rot}\, S^1$, hence the direct sum of its eigenspaces for $\mathrm{Rot}\, S^1$ and hence ~~the~~ V is the space of finite energy vectors of its closure. But the closure of V is invariant under a dense subgroup of $LG \rtimes \mathrm{Rot}\, S^1$ by the previous two results. Hence $V = \mathcal{H}^0$. QED

<u>Theorem</u> With π, $\mathcal{H}$ as above, then ~~...~~ after tensoring by a character of $\mathrm{Rot}\, S^1$ we may assume the lowest energy eigenspace of $\mathcal{H}$ is $\mathcal{H}(0)$. Then

1) ~~π_1 and π_2 ...~~ $\mathcal{H}(0)$ is irreducible as a G-module ($G = SU(N)$)

2) ~~...~~ $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are unitarily equivalent iff $\mathcal{H}_1(0) \cong \mathcal{H}_2(0)$ as $SU(N)$-modules

<u>Proof</u> 1) If $\xi, \eta \in \mathcal{H}(0)$ ~~the~~ and $(\xi, \eta) = 0$, then for $m, n < 0$, $\langle X(m)\xi, Y(n)\eta\rangle = 0$ for all $X, Y \in \mathfrak{g}$. ~~...~~ But for any $\xi \in \mathcal{H}(0)$, $[X(n)\xi \mid n < 0]$ is a $L^0\mathfrak{g} \rtimes \mathbb{R}$-module. Hence $\mathcal{H}(0)$ must be irreducible as a G-module.

2) The argument of 1) shows that all inner products between a spanning set for each $\mathcal{H}^0_i$ are determined

by the inner product on $\mathcal{H}(0)$ and the Lie algebra commutation relations. So the map $U(\pi_1(X(n))\xi) = \pi_2(X(n))\xi$ (where we have identified the isomorphic $\mathfrak{g}$-modules $H_1(0)$ and $H_2(0)$) ~~extends to an~~ defines a linear isometry of $\mathcal{H}_1(0)$ onto $\mathcal{H}_2(0)$. It obviously intertwines the actions of $L^0\mathfrak{g}$ (and $Rot(S^1)$). The isometry also sends a core for ~~the~~ $\pi_1(\cos m\theta X)$ and $\pi_1(\sin m\theta X)$ to a core for $\pi_2(\cos m\theta X)$ and $\pi_2(\sin m\theta X)$ so intertwines a dense subgroup of $LSU(N)$, hence all of $SU(N)$ (and of course $Rot S^1$). QED.

The classification of irreducible positive energy reps ~~can~~ will be complete when we know what G-modules can occur as $\mathcal{H}(0)$.

Irreducible reps of $SU(2)$

For each "spin" $j \in \frac{1}{2}\mathbb{N} \cup \{0\}$ there is a unique (unitary) irreducible representation of $SU(2)$. It may be realised on the space of Homogeneous polynomials of degree $2j$, aka the $2j$ th symmetric tensor power of the defining representation of ~~$\mathfrak{g}$~~ $SU(2)$ on $\mathbb{C}^2$. The (complexified) Lie algebra of $SU(2)$ has basis $H = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $E = \begin{pmatrix}0&1\\0&0\end{pmatrix}$, $F = \begin{pmatrix}0&0\\1&0\end{pmatrix}$ with $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$. On V_j H has spectrum $\{-2j, -2j+2, \ldots, 2j-2, 2j\}$.

} called V_j, $\dim V_j = 2j+1$

There is a unique non-zero vector $v \in V_j$ (up to scalars) with $Fv = 0$ and a basis for V_j is given by $\{v, Ev, E^2v, \ldots, E^{2j}v\}$, $HE^kv = 2(k-j)v$.

<u>Theorem</u> a) For each $j \in \frac{1}{2}\mathbb{N} \cup \{0\}$, $0 \le 2j \le \ell$ there is a (unique) irreducible positive energy representation of level ℓ of $LSU(2)$ on $\mathcal{H}$ with $\mathcal{H}(0) \cong V_j$.

b) If $\mathcal{H}$ is an irrep, positive energy of $LSU(2) \rtimes \mathrm{Rot}\,S^1$ (with $\mathrm{Rot}\,S^1$ action normalised so the the lowest energy is 0) of level ℓ, then $\mathcal{H}(0) \cong V_j$ for some j, $2j \le \ell$.

<u>Proof</u> a) Is easy. $\mathbb{C}^{2\ell} = \bigoplus^{\ell} \mathbb{C}^2$ and the corresponding "$\mathcal{F}_p$" is $\bigotimes_{i=1}^{\ell} \mathcal{F}_p(\mathbb{C}^2)$ as an $LSU(2)$ module, in particular as an $SU(2)$ module, so all irreps of $SU(2)$ contained in $\mathbb{C}^2$ occur as $\mathcal{H}(0)$ by our results. In particular V_j occurs, as the symmetric $2j$th power of $\mathbb{C}^2$.

b) Suppose $V_j = \mathcal{H}(0)$ and choose a $v \in V_j$, $v \ne 0$, $Hv = -2j\,v$.

Then consider $E(1)$, $F(-1)$ and $H(0) + \ell\,id$. They define a copy of $su(2)$ with $*$-properties as if they came from a unitary representation.

since $[E(+1), F(-1)] = H(0) + \underbrace{\ell}_{Tr(EF) \text{ on } \mathbb{C}^2} id$, $[H(0) + \ell\,id, E(1)] = 2E(1)$ etc.

But $E(1)v = 0$ so v must be an eigenvector of $H(0) + \ell\,id$ with non-negative eigenvalue. So $-2j + \ell \ge 0$ or $2j \le \ell$.

QED.