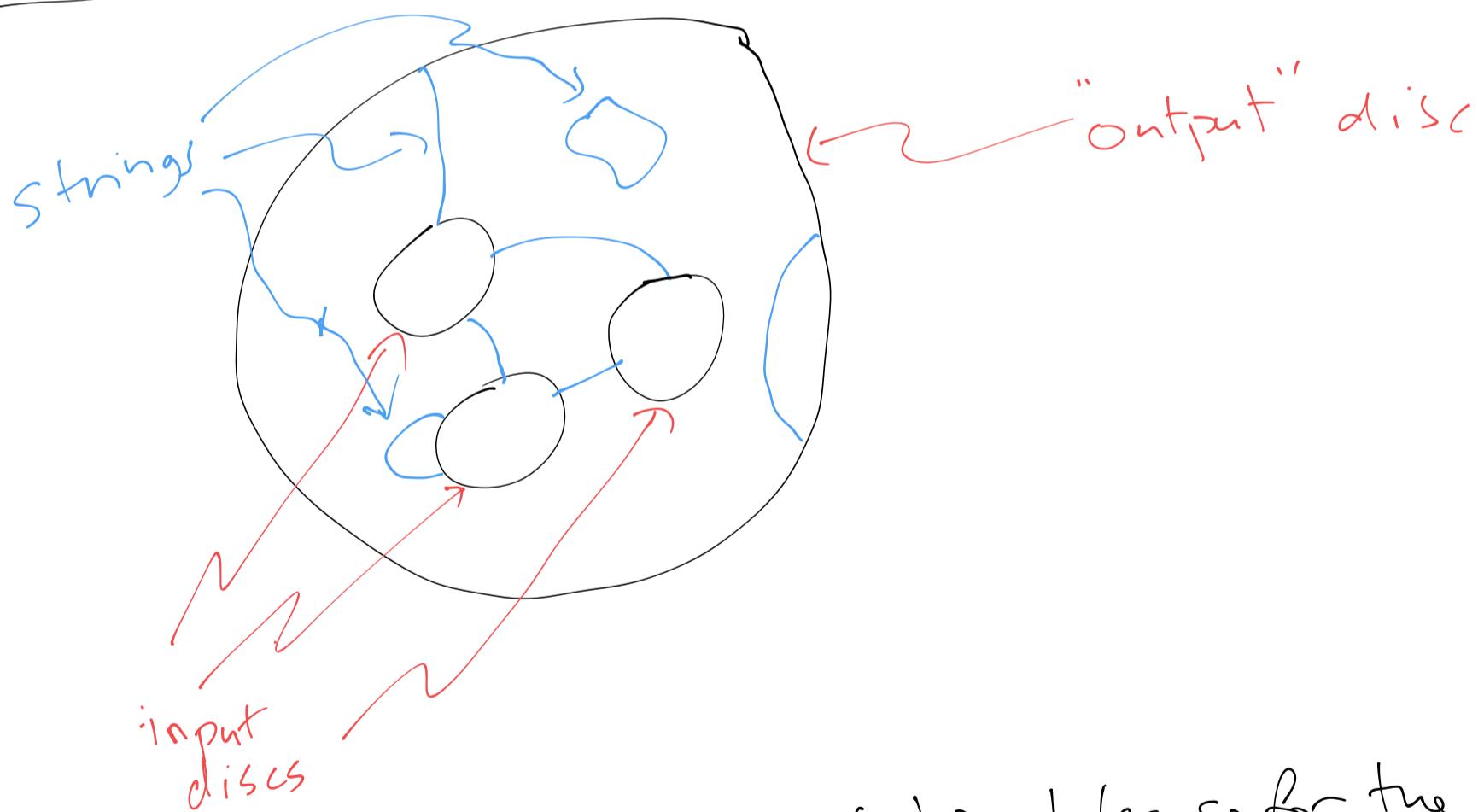


A planar algebra (PA) is a way of combining objects of a certain kind in a planar way to get object of the same kind. The operations are indexed by

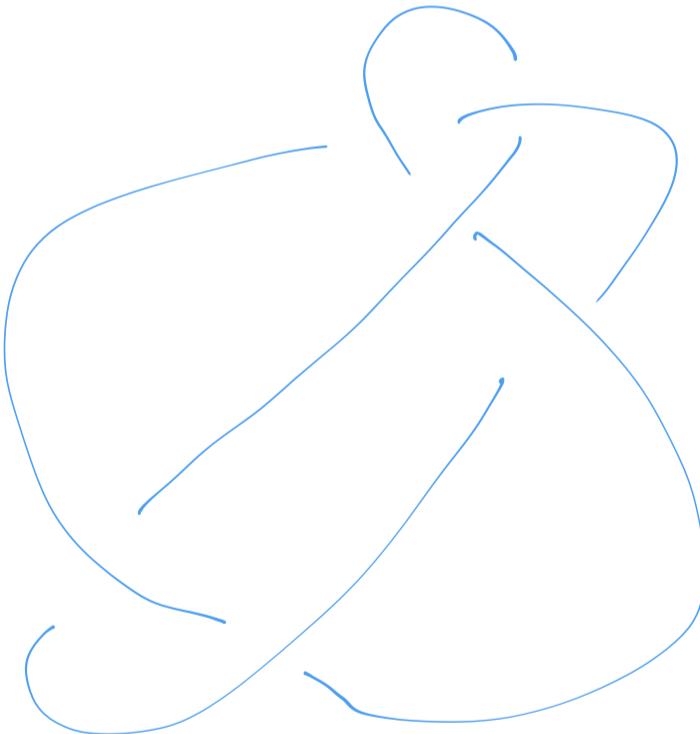
PLANAR TANGLES



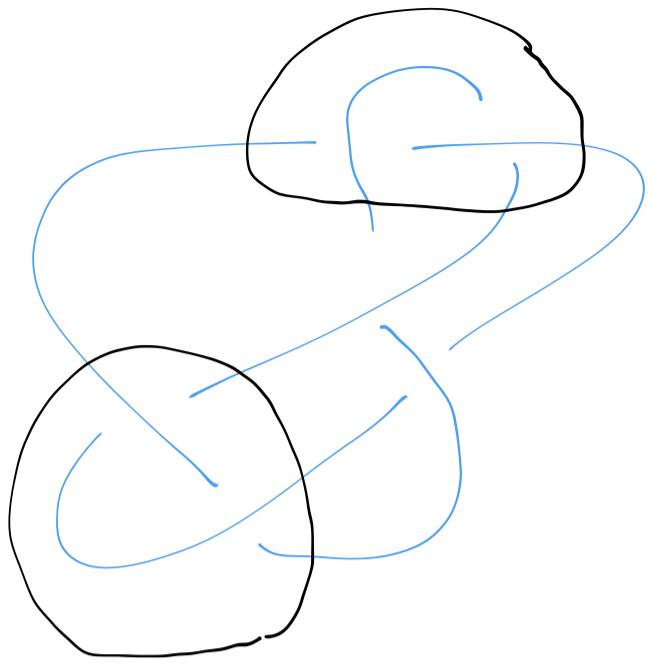
We'll make the formal definition later so for the moment let's let this picture serve. We see that the "discs" meet the strings in a certain number of boundary points so there are discs of type n for all $n \in \mathbb{N} \cup \{0\}$.

The objects on which the tangle operates thus also come with a grading by $\mathbb{N} \cup 0$. The above example operation would take as inputs objects of types 2, 3 and 4 and produce an object of type 3. An object of type n will be called an "n box". I had 4 different but related motivations for introducing planar algebras. The most obvious and elementary was knot theory.

Here is a link

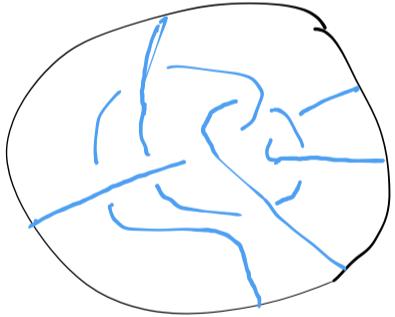


A "divide and conquer" strategy would be to collect together certain parts of the link or below



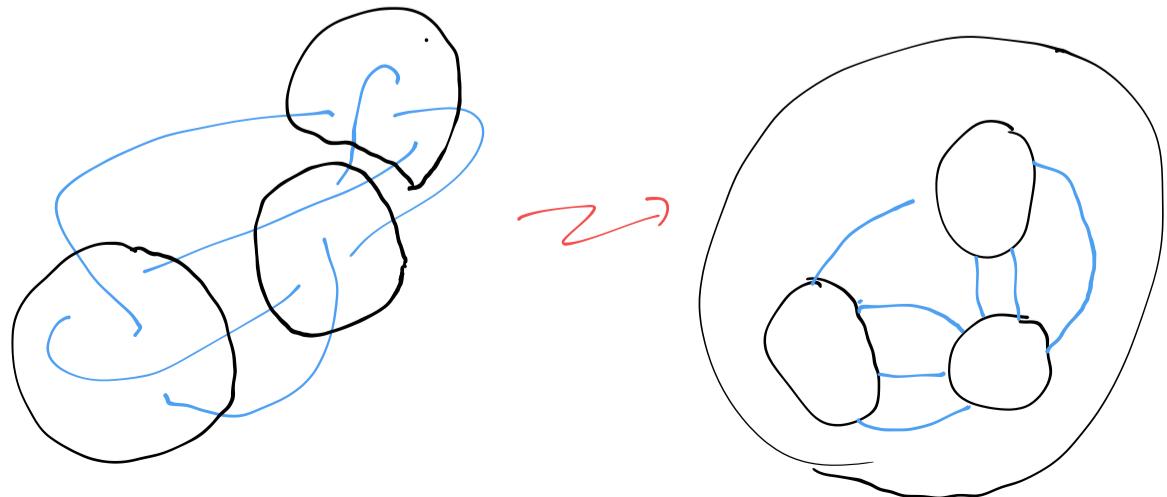
We are thus led to the idea of a "link with boundary":

Conway tangles
picture like



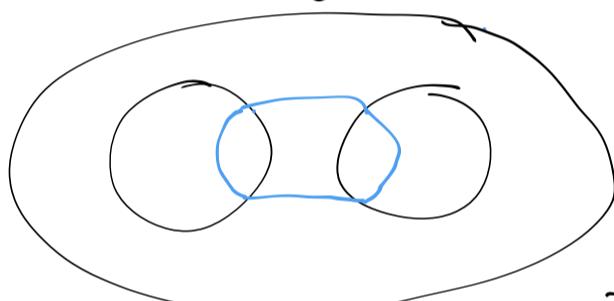
A (Conway) tangle is a
(modulo the Reidemeister
moves $\mathcal{R} = /$
 $\mathcal{T} = \text{)(}$
 $\mathcal{L} = -\text{)(}-\text{)(}-$)

Carrying the divide and conquer strategy to
its ultimate conclusion we will enclose all
the crossings and be left with a planar
tangle

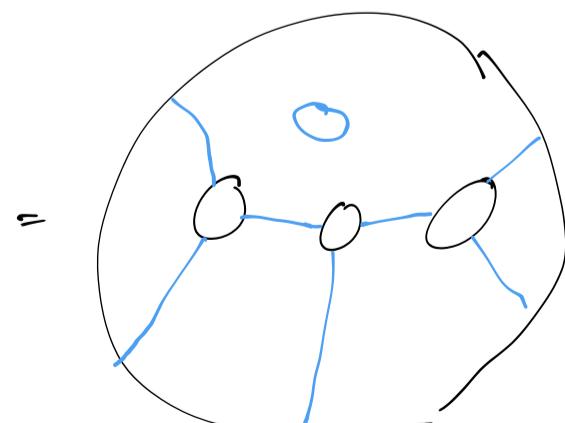
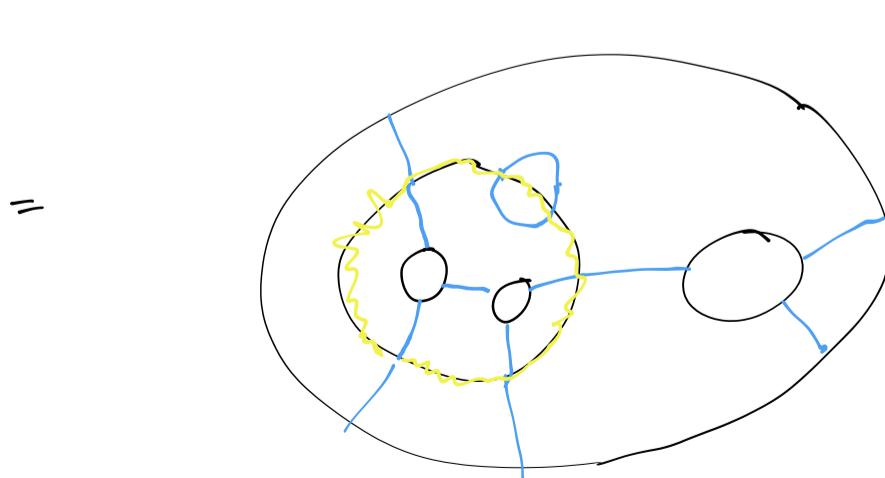
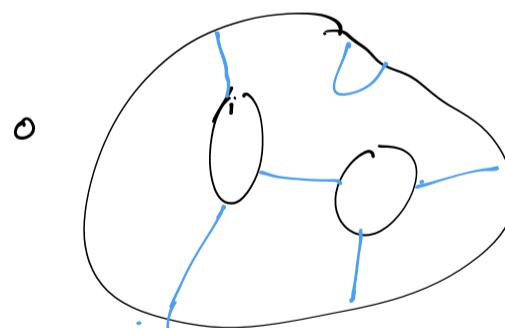
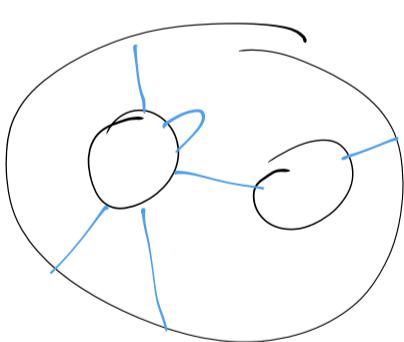


Several remarks are in order:

- 1) The tangle of a link has a somewhat artificial output disc with no boundary points, but clearly the calculus of these tangles will use boundary discs.
- 2) Conway tangles form a planar algebra
- 3) The number of boundary points is always even ("what goes in must come out")
- 4) Closed loops  may be formed when combining tangles even if they're not present initially. e.g.



- 5) What has emerged is a "gluing" operation on planar tangles



Thus planar tangles themselves form a planar algebra.

Typically the objects of a planar algebra will be vector spaces. In Conway's case one simply takes formal linear combinations of Conway tangles. The interesting thing that Conway did was to introduce certain " skein relations" and take the quotient of tangles by these relations. The most famous is the following:

$$\text{Diagram 1} - \text{Diagram 2} = \left(\sqrt{t} - \frac{1}{\sqrt{t}} \right) \text{Diagram 3}$$

And in fact the Reidemeister moves themselves can be thought of as skein relations.

(The result of applying the above relation is that any knot/link diagram can be reduced to a multiple of a single unknotted circle, the multiple being the Alexander polynomial Δ of the link - exercise calculate $\Delta(\text{circle})$.)

Observe that we were required to add extra structure to our tangles, namely an orientation of the strings

Second motivation:

Subfactors

If $N \subseteq M$ are II_1 factors (or Neumann algebras with trivial centre and a trace $\text{tr} : M \rightarrow \mathbb{C}$ with $\text{tr}(1) = 1$, $\text{tr}(ab) = \text{tr}(ba)$) Then one may

form a tower $N \subset M \subset \langle M, e_N \rangle \subset \dots \langle M_i, e_i, e_{i+1} \rangle \subset \dots$

$$\vdots \quad \vdots \quad \vdots \\ M_0 \quad M_1 \quad M_2 \subset \dots \subset M_n$$

$M_{i+1} = \langle M_i, e_{i+1} \rangle$ "basic construction"

where each e_k is orthogonal projection from M_k to M_{k-1} . The e_i 's satisfy the

relations

- a) $e_i^* = e_i^2 = e_i$ (projection) (i a number, $i = [0: N]$)
- b) $e_i e_i + e_i = e_i$.
- c) $e_i e_j = e_j e_i \quad |i - j| \geq 2$

(Historically the Jones polynomial was discovered by representing the braid group via $\sigma_i : 1 \mapsto t e_i - (1 - e_i)$, $2 + t + t^{-1} = \overline{t}^{-1}$ and taking the trace of the braid.)

Kauffman introduced a pictorial representation of these e_i 's as follows (multiplication = vertical stacking)

$$E_i = \frac{e_i}{\delta} = \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$\delta = \sqrt{\epsilon}$$

$$E_i^2 = \delta E_i : \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} = \delta \begin{array}{c} \diagup \\ \diagdown \end{array} \text{ if } \delta = 0$$

"loop parameter"

$$E_i E_j - E_j E_i :$$

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$E_i E_j - E_j E_i, |i-j| \geq 2 : \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array}$$

Kauffman's diagrammatic was extremely revealing and successful and I was intent on seeing to what extent the diagrammatics could be extended to the whole tower

M_i , especially the "higher relative commutants"

$$M_0 \cap M_i = \{ s \in M_i \mid x_m = m \in H_m \cap M_0 \}$$

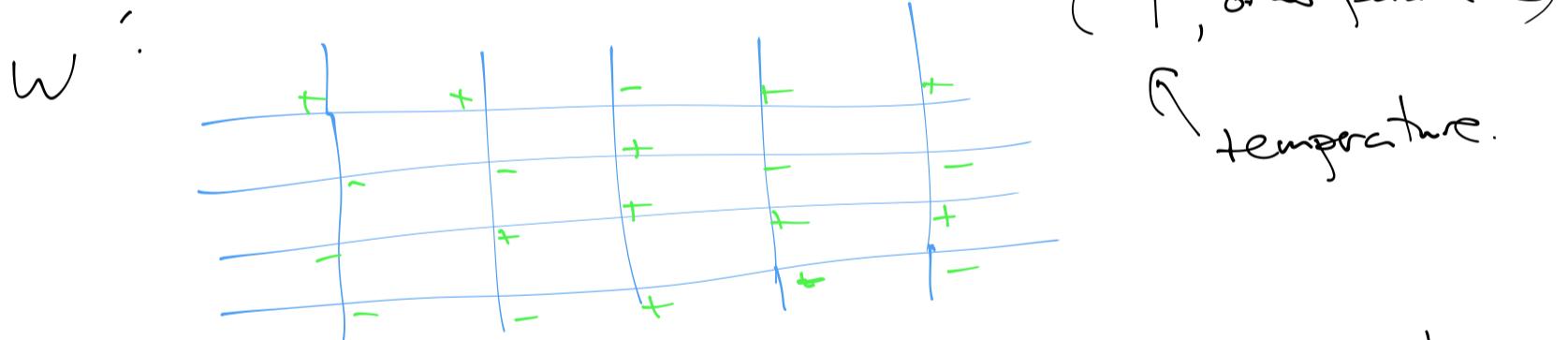
"centraliser tower"

Next motivation

Statistical mechanical
models in 2d

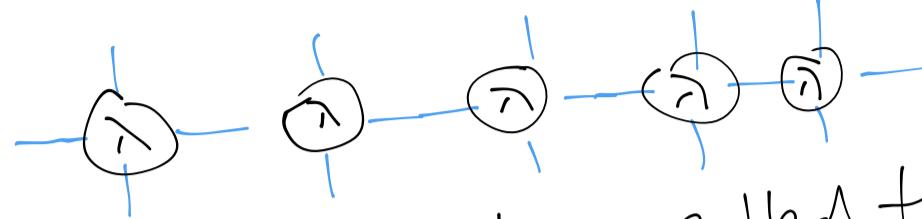
The relations among the $\phi_i <$, as well as the Kauffman diagrammatics, are both known today as "Temperley Lieb algebra", going back to a paper of Temperley and Lieb where they were used to show a mathematical equivalence between two types of (classical) stat mech lattice models in 2d, dimensions (Lieb's ice-type model and the Potts model).

Stat mech models associate a "Boltzmann weight" to spin configurations on a lattice



Obviously we need to consider boundary conditions and some kind of locality. These are perhaps best expressed by saying that the model gives a planar algebra which frees the model from any particular lattice structure.

For instance the whole lattice is built up from a single "transfer matrix"

$T(\lambda) =$  where λ is a 4-box called the "spectral parameter"

"Solving" the model is then a question of diagonalising $T(\lambda)$. This is the divide and conquer strategy at work.

The famous Yang-Baxter equation can then be thought of as a "skew relation" in the planar algebra: $\forall \lambda, \mu \exists \sigma \text{ s.t.}$

$$\begin{array}{ccc} \text{Diagram with } r \text{ at top, } \lambda \text{ and } m \text{ below it} & = & \text{Diagram with } m \text{ and } r \text{ swapped, } \sigma \text{ below them} \end{array}$$

and the "commuting transfer matrix" argument becomes a pure calculation in planar algebras (with attention to the boundary conditions) $T(\lambda)T(\mu) = T(\mu)T(\lambda)$

My fourth and final (last but not least!!)
 motivation was an apparently technical thing
 internal to the theory of subfactors. Namely
 the calculation of the centraliser tower
 for a commuting square.

A commuting square is a quadrilateral

$$\begin{matrix} \mathcal{B}_0 & \subset & \mathcal{B}_1 \\ \downarrow & & \downarrow \\ \mathcal{A}_0 & \subset & \mathcal{A}_1 \end{matrix}$$

of finite dimensional unital
 semisimple \ast -algebras over
 (direct sums of matrix algebras)

together with a trace $\text{tr} : \mathcal{B}_1 \rightarrow \mathbb{C}$ which is
 positive definite ($\text{tr}(a^* a) > 0$ if $a \neq 0$) so
 that the algebras become Hilbert spaces under
 $\langle a, b \rangle = \text{tr}(b^* a)$. The commuting square
 condition is that $\boxed{\mathcal{B}_0 \perp \mathcal{A}_1 \text{ mod } \mathcal{A}_0}$
 which has many alternative formulations.

One may then perform the basic construction
 horizontally for $\mathcal{B}_0 \subset \mathcal{B}_1$ to obtain

$$\begin{array}{c} \mathcal{B}_0 \subset \mathcal{B}_1 \subset \langle \mathcal{B}_1, e_{\mathcal{B}_1} \rangle \\ \cup \quad \cup \quad \cup \\ \mathcal{A}_0 \subset \mathcal{A}_1 \subset \{ \mathcal{A}_1, e_{\mathcal{B}_0} \}'' \end{array}$$

algebra generated by
A and $e_{\mathcal{B}_0}$

A simple nondegeneracy condition on the
commuting square guarantees that

$$1) \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \{ \mathcal{A}_1, e_{\mathcal{B}_0} \}'' \cong \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \langle \mathcal{A}_1, e_{\mathcal{A}_0} \rangle$$

under $e_{\mathcal{A}_0} \rightarrow e_{\mathcal{B}_0}$

$$\begin{array}{ccc} 2) & \mathcal{B}_1 \subseteq \langle \mathcal{B}_1, e_{\mathcal{B}_1} \rangle & \text{is also a commuting} \\ & \cup \quad \cup & \text{square, and the} \\ & \mathcal{A}_1 \subseteq \langle \mathcal{A}_1, e_{\mathcal{A}_0} \rangle & \text{trace extends} \end{array}$$

One may thus iterate to obtain:

$$\begin{array}{ccccccccc} \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \dots & \subseteq & \mathcal{B}_\infty \\ \cup \quad \cup \quad \cup \quad \cup & & \cup \\ \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3 \subseteq \dots & \subseteq & \mathcal{A}_\infty \end{array}$$

The traces are coherent so $\mathcal{A}_\infty \subseteq \mathcal{B}_\infty$ may be
made into a pair of $\overline{\mathbb{II}}_1$ factors NCM

whose index is easy to calculate from the data of the commuting square.

This has been the most successful way of constructing subfactors. $N \leq M$

Obviously we want to compute the centraliser tower. It turns out to be a "simple" computation in linear algebra, best explained in terms of planar algebras. Let us consider an example, perhaps the easiest; it will involve the construction of our second real live PA — the spin PA — which is in fact the PA underlying the Potts model.

Suppose we look for a commuting square of the form

$$\Delta \subset M_n(\mathbb{C})$$
$$\cup$$
$$\Delta \subset M_n(\mathbb{C})$$
$$\Delta \subseteq u \Delta u^{-1}$$

where Δ consists of the diagonal matrices. u a unitary U_{ij}

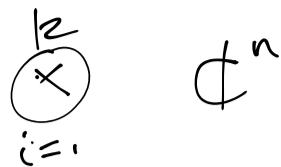
Then the commuting square condition (exercise) turns out to be exactly

$$|U_{ij}| = \frac{1}{\sqrt{n}} \quad \forall i, j = 1, 2, \dots, n$$

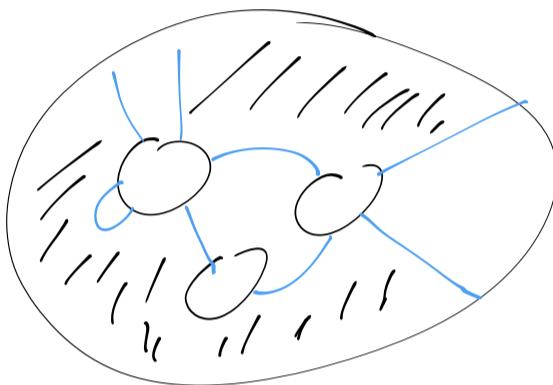
Complex Hadamard matrix

We now form the planar algebra whose

$2k$ -box space is (ignoring P_0) $\binom{2k+1 \text{ box space}}{\emptyset}$

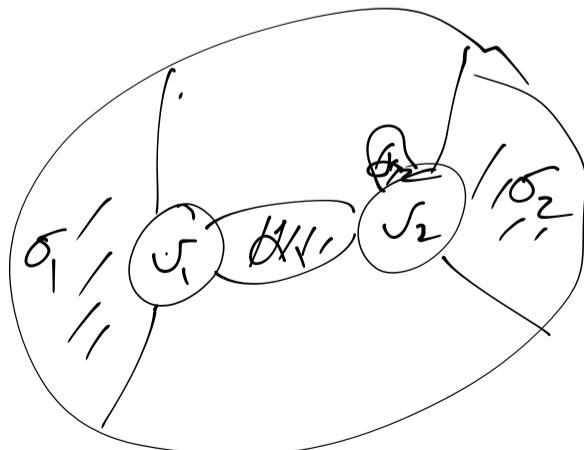


We consider only planar tangles with even numbers of boundary points, to which we add the extra data of a shading, so that a shaded planar tangle looks like:



The planar algebra structure is given as follows:

A "state" of a ^{shaded} _{planar} tangle is a function $\sigma: \text{shaded regions} \rightarrow \{1, 2, \dots, n\}$. A state induces a function $\partial\sigma: \text{shaded intervals on } \partial \text{ of each disc} \rightarrow \{1, 2, \dots, n\}$. Illustration of PA structure. Coefficient of (σ_1, σ_2) :



$v_1 \in P_4 = \text{functions from } \{\sigma_1, \sigma_2\} \rightarrow \mathbb{C}$
 $v_2 \in P_6 = \text{functions for } \{\sigma_1, \sigma_2, \sigma_3\} \rightarrow \mathbb{C}$.

$$= \sum_{\alpha_1, \alpha_2} v_1(\sigma_1, \alpha_1) v_2(\alpha_1, \alpha_2, \sigma_2)$$

(resolve cyclic ambiguity with '\$\circ\$')

Now given a U defining a commuting square,
let

$$\begin{array}{c} \sigma_1 \\ \diagup \quad \diagdown \\ \text{---} \\ R \\ \diagdown \quad \diagup \\ \sigma_2' \end{array} = U_{\sigma_1 \sigma_2}$$

Result: -The space $M'_0 \cap M_k$ is the
space of solutions of the equation

$$\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ X \\ \diagdown \quad \diagup \\ \text{---} \\ Y \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ Y \\ \diagdown \quad \diagup \\ \text{---} \\ X \end{array}$$

$$(X_{\sigma_1 \sigma_2 \sigma_3}, Y_{\sigma_1 \sigma_2 \sigma_3}) \rightarrow \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ Y \\ \diagdown \quad \diagup \\ \text{---} \\ X \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ R \\ \diagdown \quad \diagup \\ \text{---} \\ X \end{array}$$

Result: For any commuting square there is
a PA for which $M'_0 \cap M_k$ is the space
of solutions of these equations, which make
sense in any shaded PA.

Relations with other structures

There is a general theory of structures with multiple inputs and an output. It is the theory of operads (\mathbb{C}) and the collection of planar tangles with the gluing operation is an example of a "coloured" operad where the colours are the elements of $N \cup \{\circ\}$ which grade everything. From the operad point of view a PA is nothing but an "algebra over the planar operad".

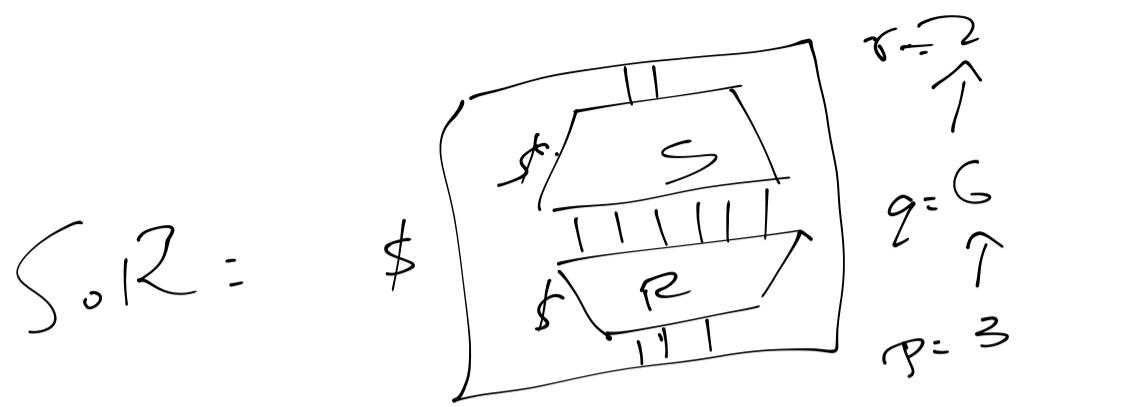
A planar algebra also gives a tensor category whose objects are $N \cup \{\circ\}$ and whose morphisms are the elements of the PA.
 $\text{Hom}(m, n) =$ The $(n+m)$ -box space of the PA.

Composition of morphisms

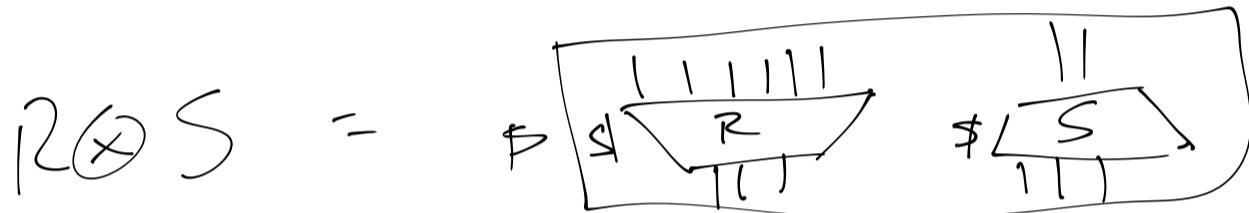
is given by stacking thus:

If $R: p \rightarrow q$ and $S: q \rightarrow r$ then

$S \circ R: p \rightarrow r$ is the element



and the tensor product of 2 morphisms
 $R^{r \rightarrow s}$ and $S^{r \rightarrow s} \rightsquigarrow R \otimes S : p+q \rightarrow r+s$



The rest of the PA structure, given basically
 by the Temperley-Lieb tangles, supplies a
 "pivotal structure" on the tensor category
 (see [7]).

