Some unitary representations of Thompson's groups — a category theoretic approach

Introduction Last year I gave a course in which I constructed unitary representations of R.Thompson's groups F and T using planar algebras. The construction was clumsy and would have to be repeated for other Thompson groups and it was not even clear how to do it for Thompson's group V. But while giving the course I understood how to make the construction much more elegant, general and easy to understand. The cost was that the new method uses categories which some may find distasteful. However the categories appear as a mathematical structure rather than a universal language and our use of them will be quite limited. All one needs to know is the basic properties of objects, morphisms (their sources and targets) and functors.

The new construction will allow lots of representations having nothing to do with planar algebras, which will appear as a more complicated framework. I will work my way there via simpler examples

Lecture 1 The (R.) Thompson groups.

Definition 1 Thompson's group F is the group of all homeomorphisms of $[0,1]$ with the following proerties.

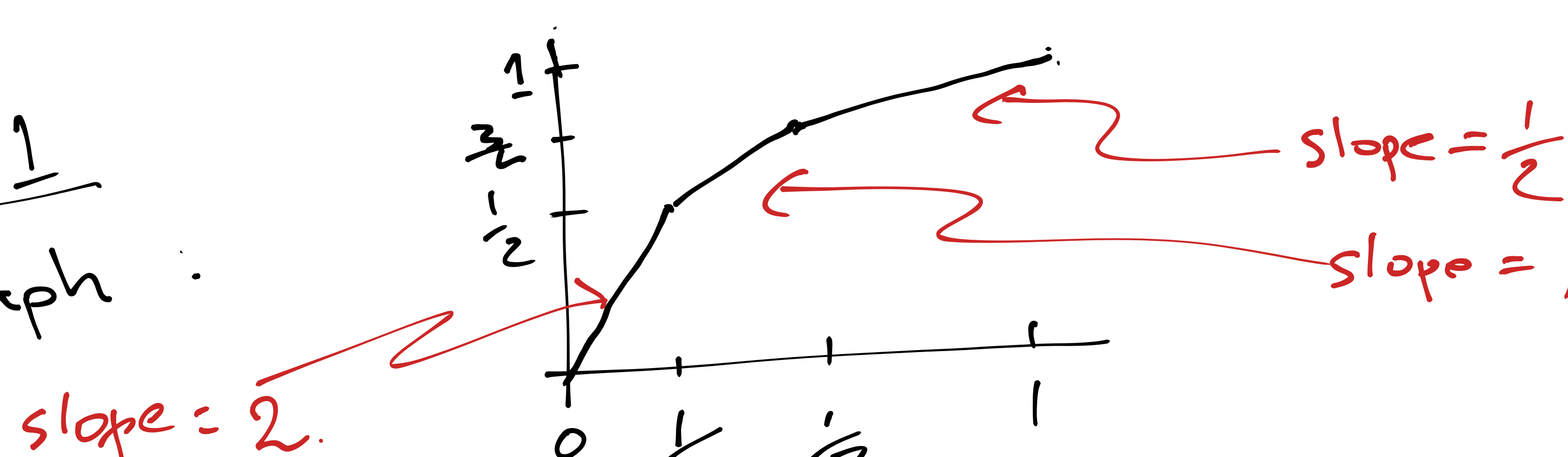
(i) Elements of F are piecewise linear (affine)

(ii) If $f \in F$ then f is smooth except at a finite set of dyadic rationals, i.e numbers of the form $\frac{a}{2^n}$ where a and n are integers.

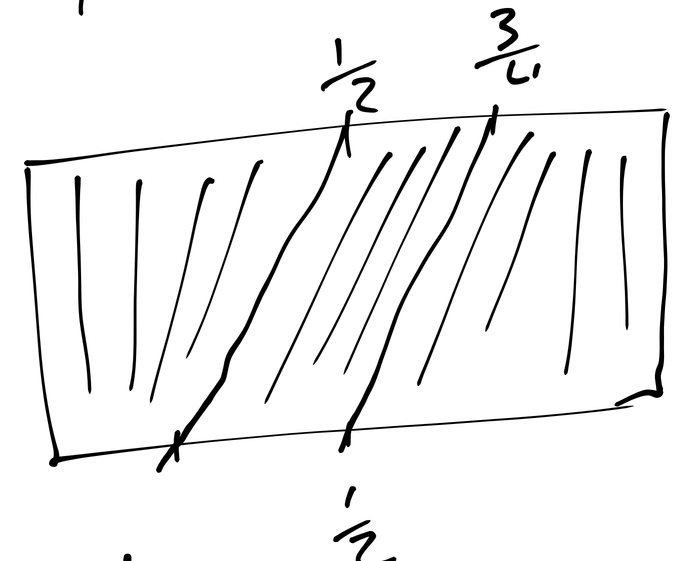
(iii) The slope of $f \in F$, when defined, is 2^k for some $k \in \mathbb{Z}$.

Example 1

a) graph :



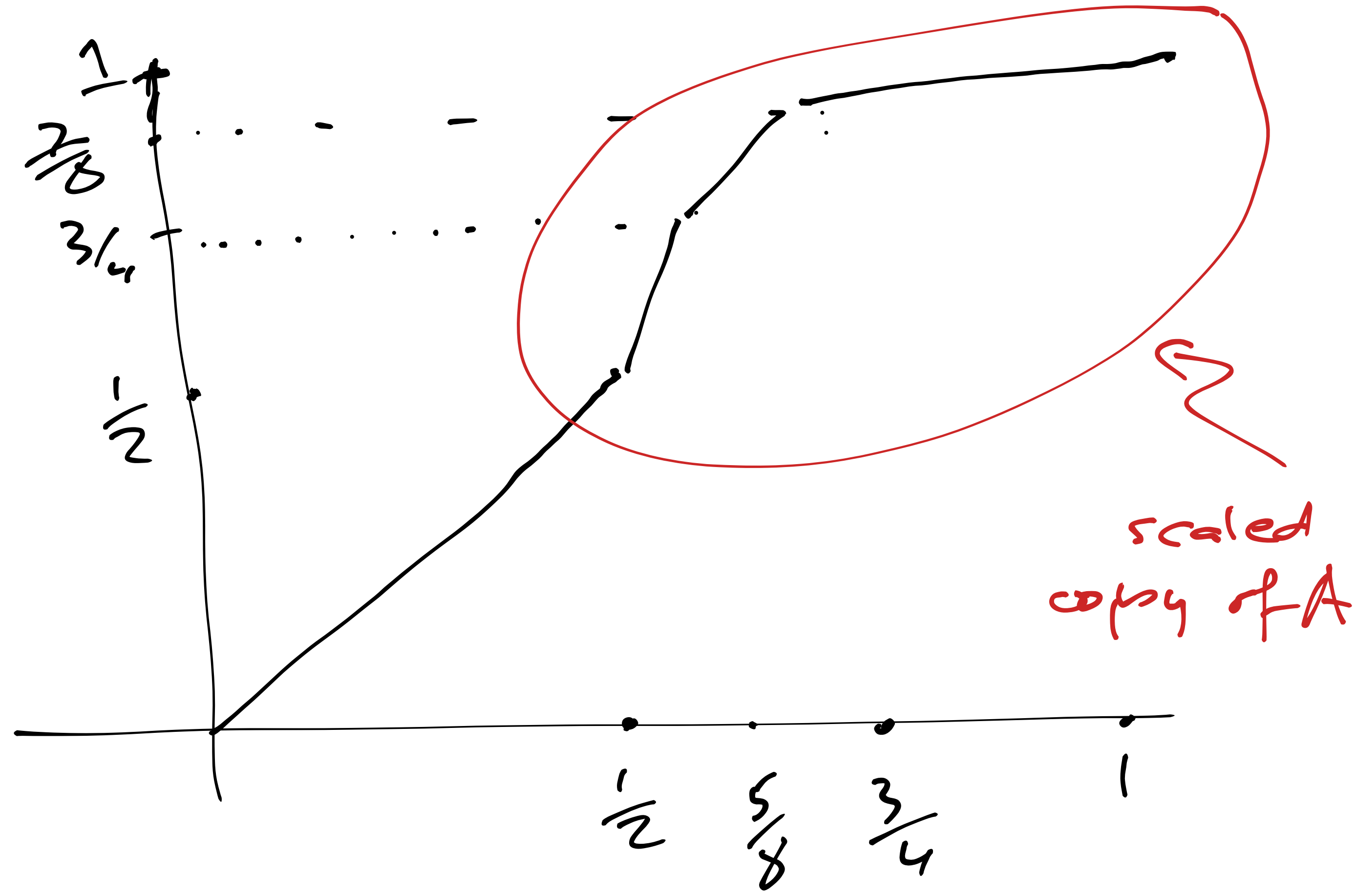
b) "Foliation" picture



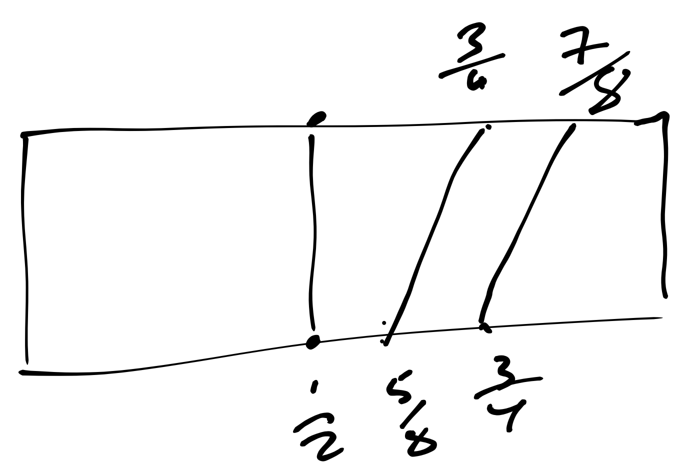
This element is known as A.

Example 2

(i) Graph



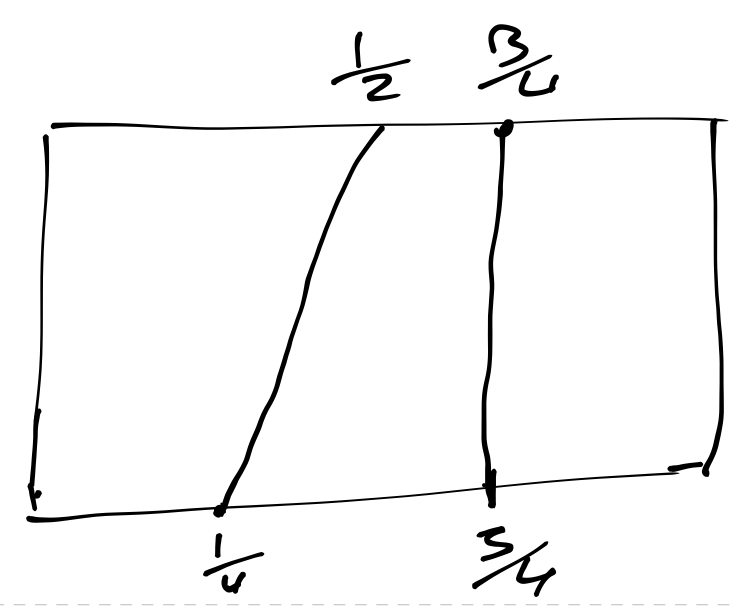
(ii) Foliation



This element of F is called B and it is known that A and B generate F. For this and much more information on F, T and V, consult the notes of Cannon, Floyd and Parry easily obtained on line.

Example 3

Foliation

$E=$ 

Note that there are only 2 break points.
This is the smallest possible.

<u>Exercise</u> Write the above element as a word on A and B (and their inverses).

<u>Facts</u> 0) F is torsion free, countable.

1) There are homomorphisms $\log_2 f'(0)$, $\log_2 f'(1)$ from F onto $\mathbb{Z}$ and thus a surjection

$$F \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}.$$

This is known to induce an isomorphism

$$F/[F,F] \cong \mathbb{Z} \oplus \mathbb{Z}.$$

2) Moreover the ~~kernel~~ $[F,F]$ is simple.

3) Any noncommutative subgroup of F contains a copy of the free abelian group on infinitely many generators. Hence F does not contain free groups.

4) The relations $[AB^{-1}, A^{-1}BA] = 1$
and $[AB^{-1}, A^{-2}BA^{2}] = 1$

hold in F and give a presentation of F.

5) The group $\langle X_0, X_1, X_2, \ldots \mid X_i X_j X_i^{-1} = X_{j+1}$ for $0 \le i < j \rangle$

is isomorphic to F $(X_0 \to A,\ X_1 \to B)$.

6) It is a famous/notorious open question as to whether F is amenable. False proofs of both amenability and non-amenability have appeared.

Amenability of a group G was defined by von Neumann. It is the existence on $\ell^\infty(G)$ of an invariant mean, i.e. a linear function $m : \ell^\infty(G) \to \mathbb{C}$ satisfying $m(1) = 1$, $m(f) \geq 0$ for $f \geq 0$, $m(f \circ \lambda_g) = m(f)$, λ_g being left translation on G. Equivalently a <u>finitely additive</u> <u>G invariant</u> <u>measure</u> μ: subsets of $G \to [0,1]$. Note that singletons would have to have measure 0.

Thus defined amenability is "transcendental" in that the existence of such means uses weak limits. But Følner gave a more combinatorial proof:

$$\forall g_1, \ldots, g_k \in G, \ \forall \varepsilon > 0 \ \exists S \underset{\text{finite}}{\subset\subset} G$$

such that
$$\frac{\#(g_i S \,\Delta\, S)}{\#(S)} < \varepsilon, \text{ all } i.$$

It is known that amenability of a group generated by a and b is equivalent to $\| u_a + u_b + u_a^{-1} + u_b^{-1} \| = 4$ where

$u_a : \ell^2(G) \to \ell^2(G)$ is the left regular representation

$$(u_a f)(g) = f(a^{-1}g)$$

This norm is easily shown to be $\lim_{k\to\infty} (N_k)^{\frac{1}{2k}}$

where N_k is the number of ways of writing id as a word of length $2k$ on a, b, a^{-1}, b^{-1}!

Numerical evidence (see Haagerup, Haagerup, Ramirez-Solano) is inconclusive!

The amenability question accounts for much of the current interest in the Thompson group though my own interest is of completely different genesis.

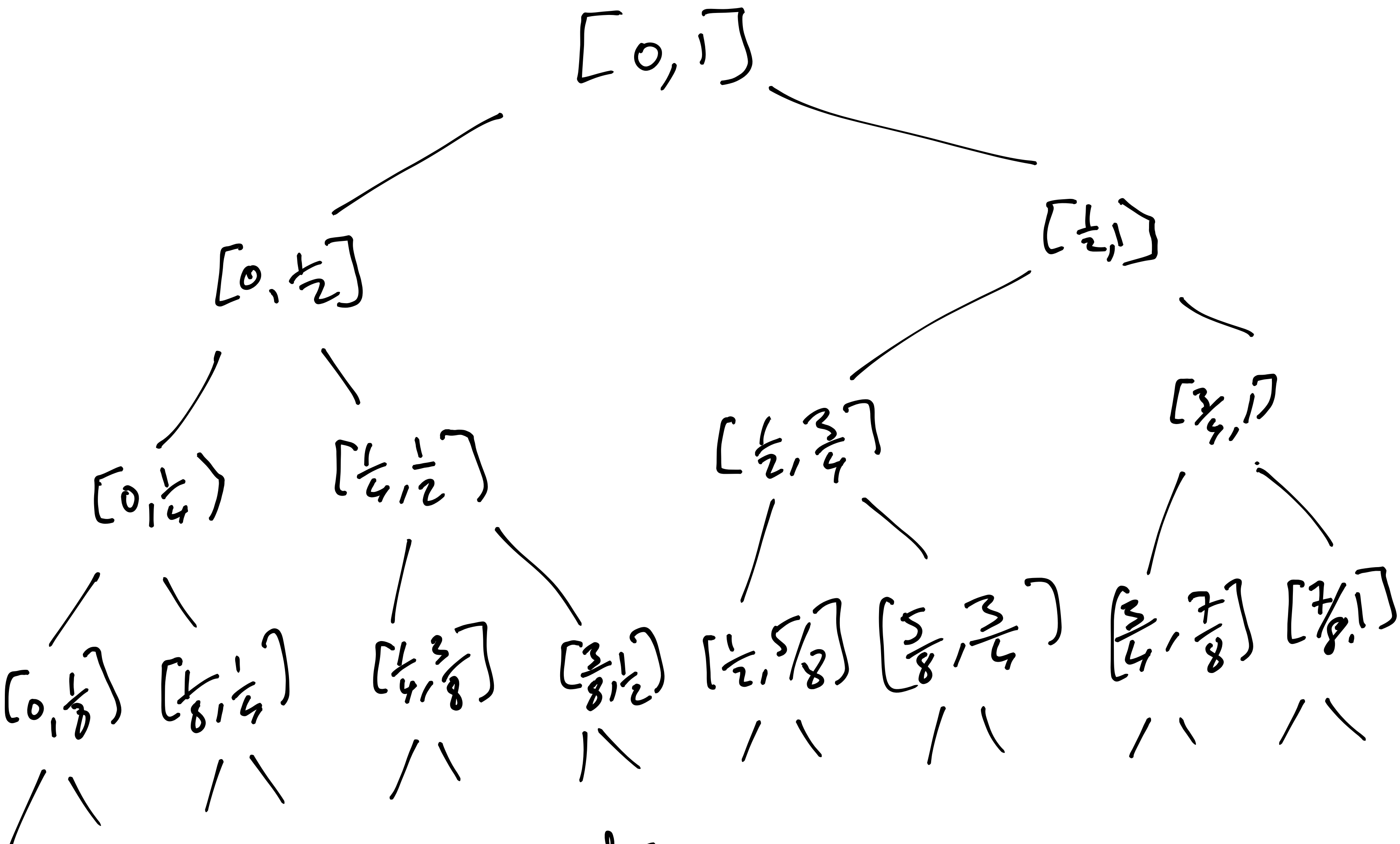
7) (Ghys Sergiescu) Inside Homeo([0,1]), F is conjugate to a subgroup of Diff([0,1]), C^∞ diffeomorphisms. This gives rise to another set of questions .....

8) F has exponential growth.

The tree of <u>standard dyadic</u> intervals.

<u>Definition 2</u> An interval $[x,y]$ is called standard dyadic if there are integers a and $m\ (>0)$ such that $x = \frac{a}{2^m}$, $y = \frac{a+1}{2^m}$.

The standard dyadic intervals in $[0,1]$ form a *tree*:

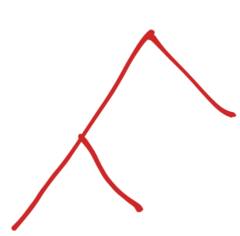


etc.

Thus *partitions* of $[0,1]$ into dyadic intervals are exactly (full) subtrees of the above complete binary planar rooted tree.

For instance

$[0,\frac{1}{4}]$, $[\frac{1}{4},\frac{1}{2}]$, $[\frac{1}{2},1]$ is the subtree:



or $[0,\frac{1}{4}]$, $[\frac{1}{4},\frac{3}{8}]$, $[\frac{3}{8},\frac{1}{2}]$ $[\frac{1}{2},\frac{3}{4}]$, $[\frac{3}{4},1]$ is

The following lemma is key to understanding F in a simple combinatorial way:

Lemma For $f \in F$ there is a partition of $[0,1]$ into standard dyadic intervals $I_1, I_2, \ldots, I_n$ such that

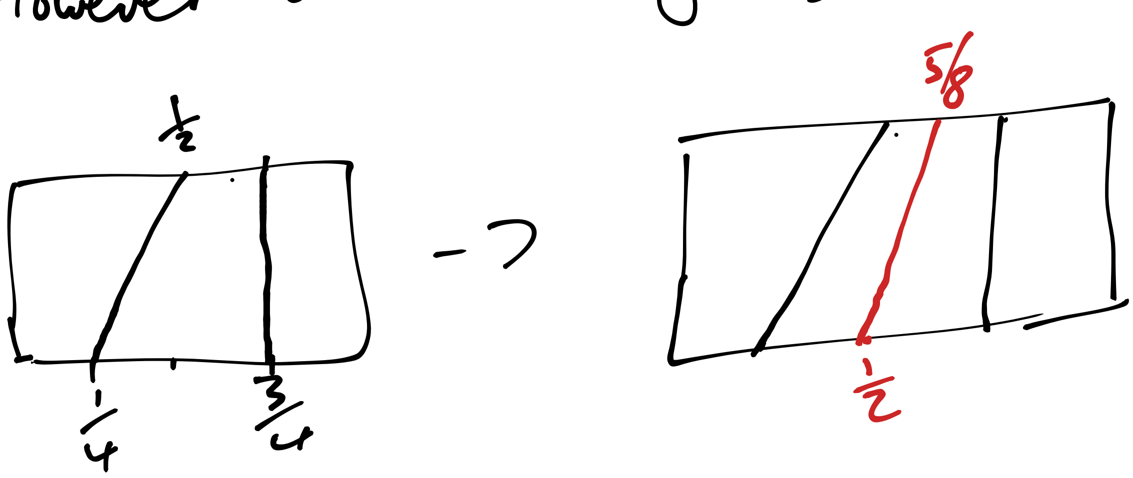
(i) f is smooth on each I_k

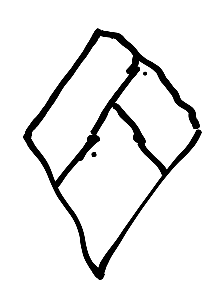
(ii) $f(I_k)$ is standard dyadic.

Proof Subdivide until it's true. QED.

Note: For the element we have called E the intervals $I_1, \ldots, I_k$ are not immediately obvious. (The maximal intervals on which E is smooth are not standard dyadic!

However on subdividing they become standard:





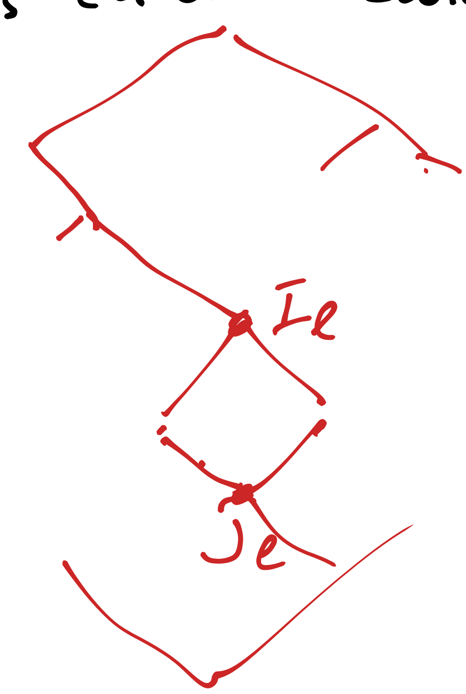
This lemma shows that each $f \in F$ is defined by a pair of binary trees represented as pairs of standard dyadic partitions $\{I_1, \ldots I_k\}, \{J_1, J_2, \ldots J_k\}$

for which $f(I_\ell) = J_\ell$ and f is affine on I_ℓ. Moreover given any pair of such trees the unique affine maps from I_ℓ to J_ℓ patch together to give an element of F.

How unique is the pair of trees defined by $f \in F$?

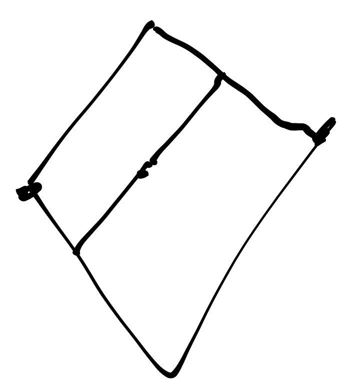
Suppose I_ℓ and J_ℓ as above are $[x,y]$ and $[z,t]$. Then replacing I_ℓ by $[x, \frac{x+y}{2}], [\frac{x+y}{2}, y]$ and J_ℓ by $[z, \frac{z+t}{2}], [\frac{z+t}{2}, t]$ produces a pair of trees defining the same element of F.

This process is called "cancelling carets":

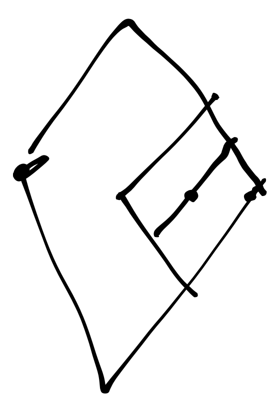


and can obviously be reversed. It is not too hard to see (Cannon Floyd Parry) that this is the whole story: given any pair of binary trees with the same number of leaves, keep cancelling carets until there are no more do cancel. The resulting pair of trees is the unique pair with no cancelling carets, representing f. (Uniqueness is easy to show - exercise)

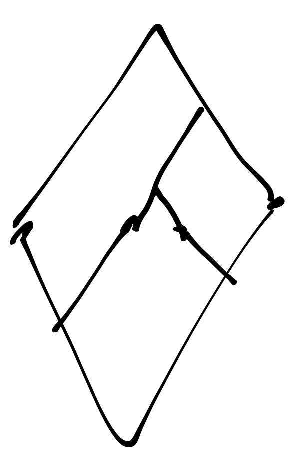
Notation: we will always write the pair of trees one on top, one on the bottom, with all leaves on the same horizontal line; the top identified with the bottom. Thus



is A



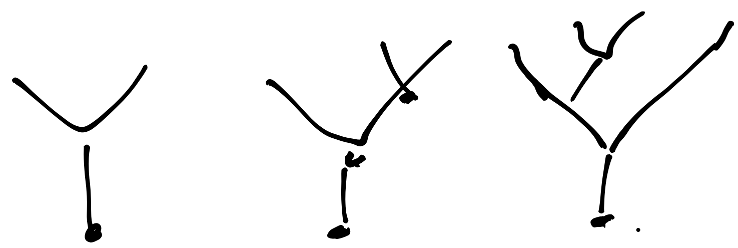
is B and



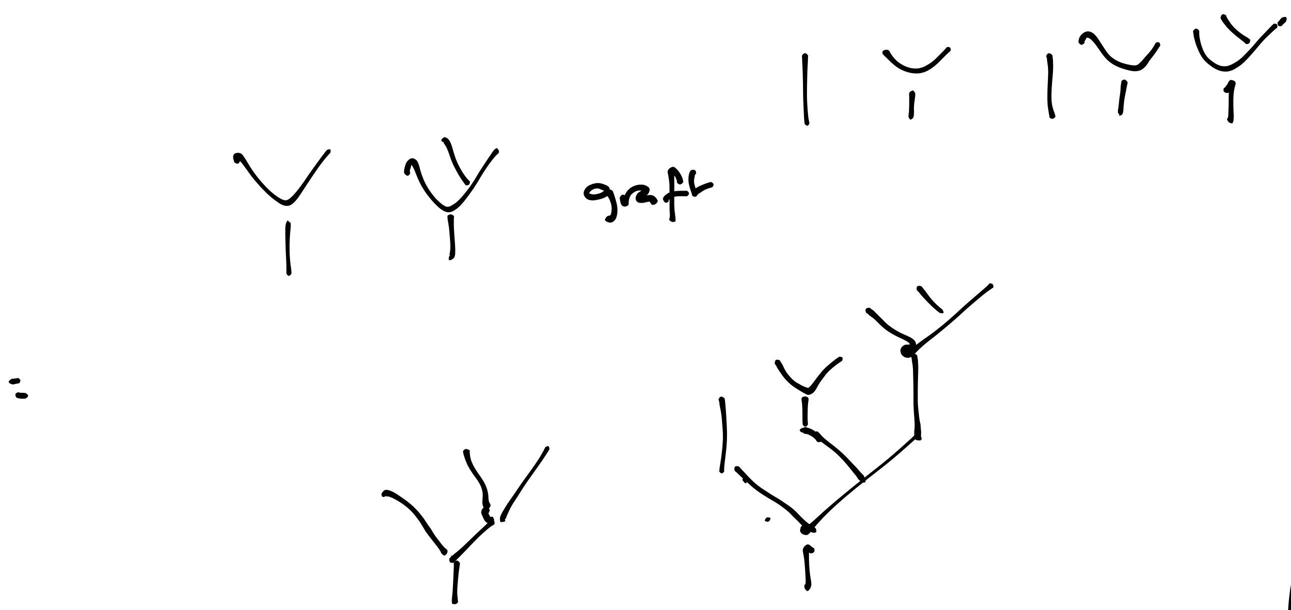
is E.

Remark. Binary trees are the same as partitions of a certain kind and as such are partially ordered by refinement. $T_1 \leq T_2$.

In fact $T_1 \leq T_2$ is expressed in terms of the trees alone by grafting of forests.
A binary planar rooted forest is a gadget like



and under the right circumstances these forests may be grafted:



We see that forests are a category $\mathcal{F}$ under grafting with objects $\{n \in \mathbb{Z} \mid n \geq 1\}$,
and $T_1 \leq T_2 \iff \exists\, f \in$ Forests such that

$$T_2 = f T_1 .$$

Refinement allows us to understand group composition.

Observe that any tree can be refined to the full tree so that given two pairs (S_1, T_1) and (S_2, T_2) one may always refine T_1 and S_2 so that they become the same. i.e.

$$\forall\, T_1, S_2 \quad \exists \text{ forests } t_1 \text{ and } s_1 \in F \text{ so that}$$

$$t_1 T_1 = s_2 S_2 .$$

But if $(S_1, T_1)(S_2, T_2)$ define Thompson group elements with $T_1 = S_2$ then their compositio is, tautologically, given by the pair of trees (S_1, T_2).

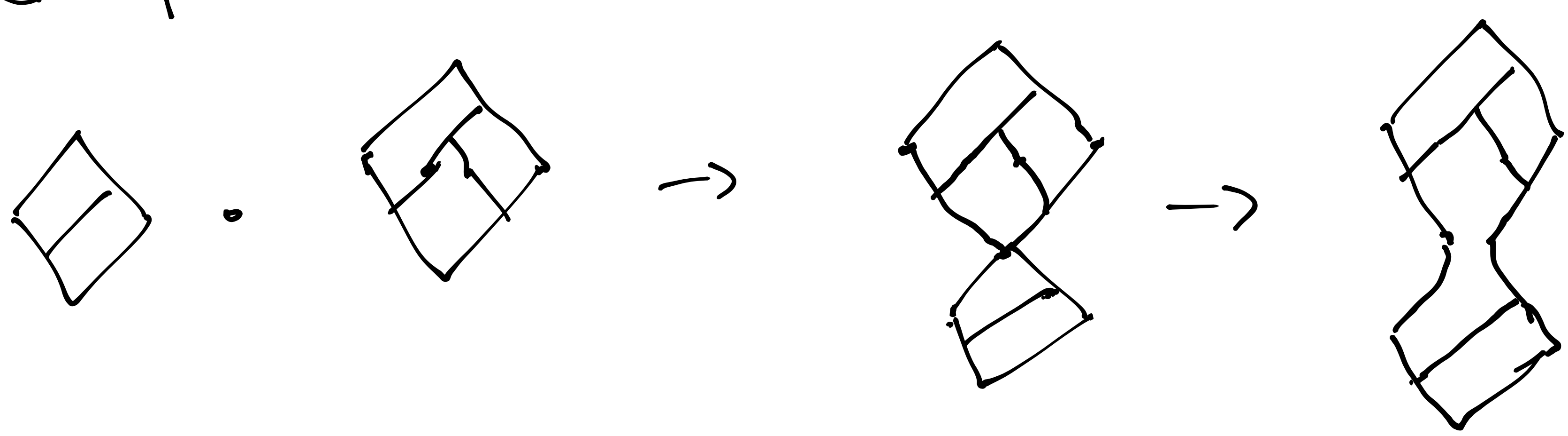
Moreover replacing (S, T) by (fS, fT) is just cancelling carets so they defie the same elements of F.

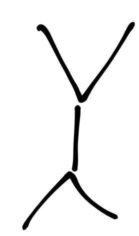
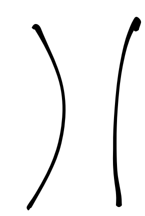
Thus to sum up,

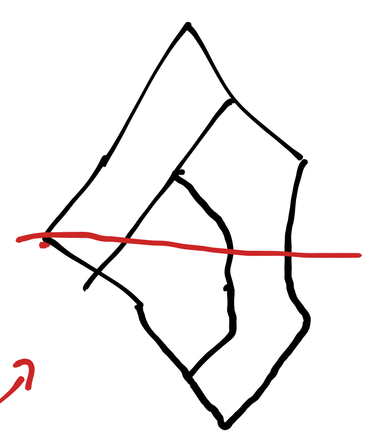
$$\boxed{(S_1, T_1)\circ(T_1, T_2) = (S_1, T_2)}$$

defies the group operation on F.

Aside For manual calculations this way of composing elements of F can be painful. There is another much more efficient way due to Belk and Sapir-Gupta. Let me illustrate by example:



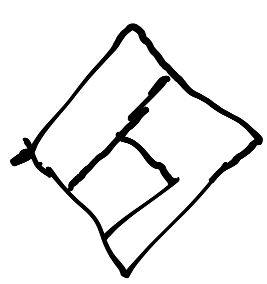
now cancel carets and replace  by 

So $\rightarrow$ 

when no more cancellation is possible there is a pair of trees:

Y's below

Y's above

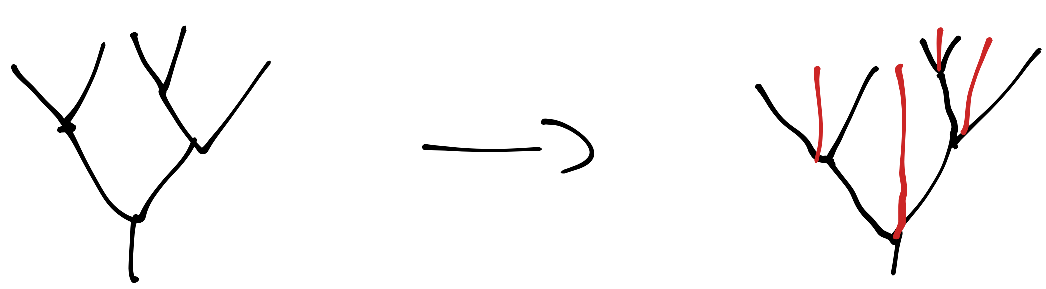


Exercise: Check by refinement.

Other Thompson groups

The most obvious thing to vary is 2. And there is indeed a Thompson group F_n for each integer $n \geq 2$ (not to be confused with the free group!) The abelianization is a bit more complicated for $n > 2$ but mostly the groups behave in the same way. They are all embedded in each other which makes the amenability question the same.

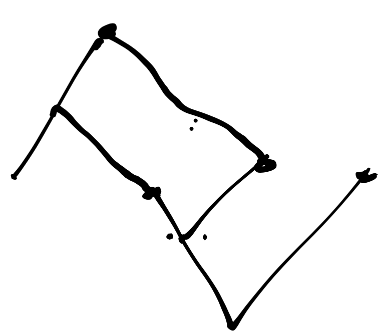
Example To embedd F_2 in F_3 one many work on the level of trees:



This clearly defines a map $F_2 \to F_3$ which is a group homorphism.

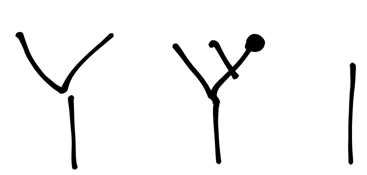
Thompson's group V

By definition V is the group of all PL homeomorphisms of $\mathbb{R}/\mathbb{Z} = S^1$ with dyadic break points and slopes all powers of 2. As for F one can show that elements are given by pairs of standard dyadic partitions of $[0,1]$, only the leftmost interval (leaf of tree) of the first is not necessarily mapped to the leftmost interval of the second. We amend our pair of trees notation thus:

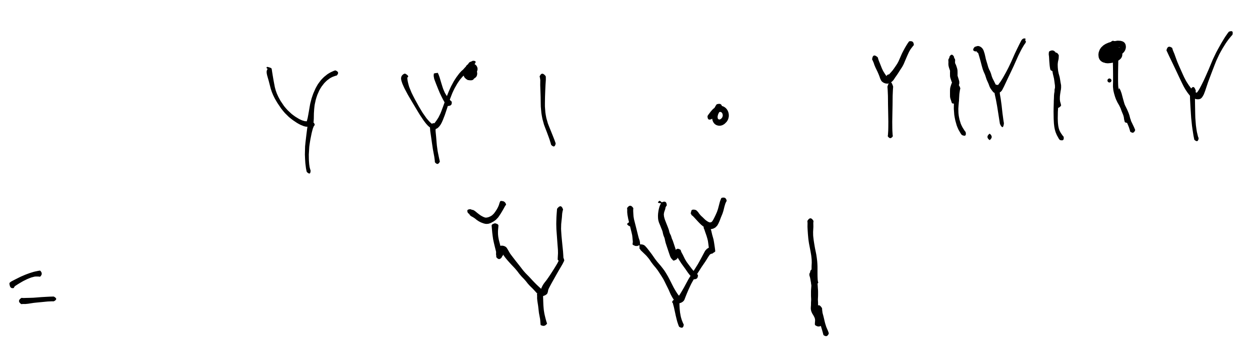


← this is the homeo of $\mathbb{R}/\mathbb{Z}$ sending $[0,\frac{1}{4}]$ to $[\frac{1}{4},\frac{1}{2}]$, $[\frac{1}{4},\frac{1}{2}]$ to $[\frac{1}{2},1]$ and $[\frac{1}{2},1]$ to $[0,\frac{1}{4}]$

As before we can define forests with marked leaves:



which form a category under composition:



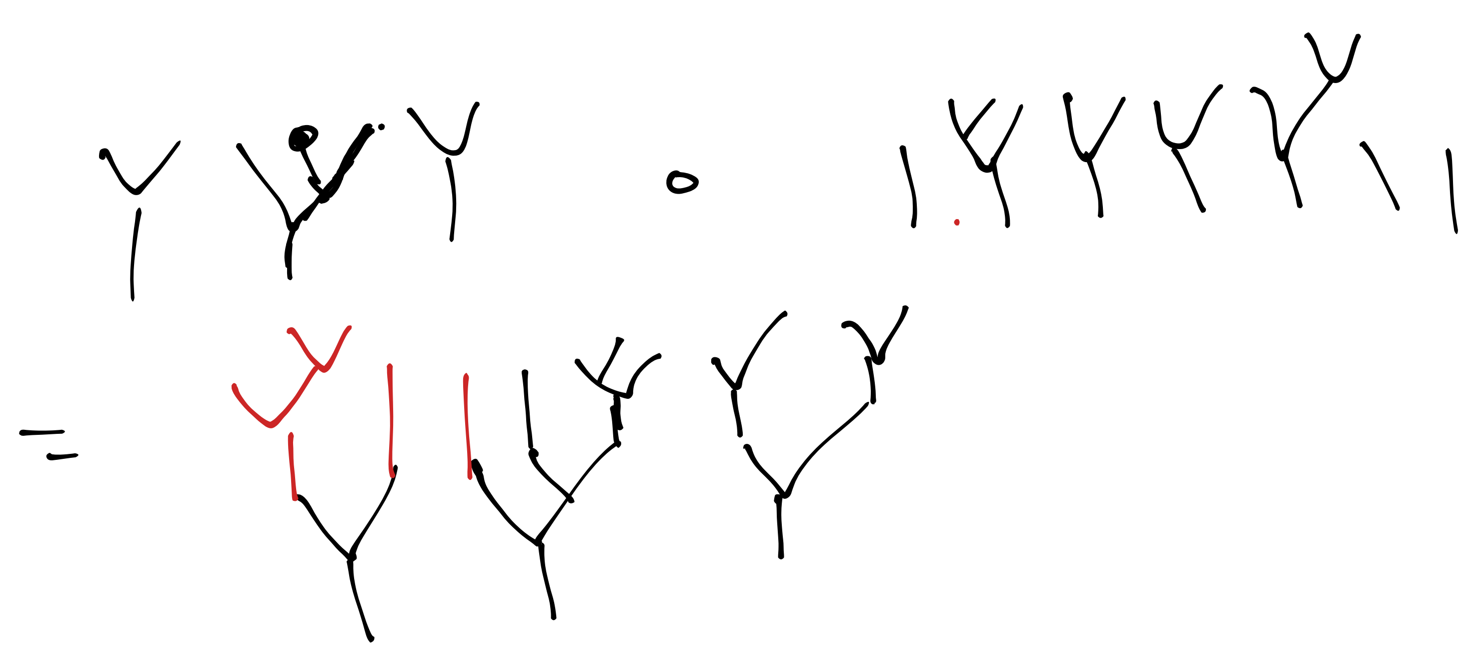
Trees in these decorated forests form a directed set $t_1 \le t_2 \iff \exists f$ s.t. $t_2 = ft_1$

and elements of V are pairs of trees (t_1, t_2) which define the same element of V if both are changed by composing with f

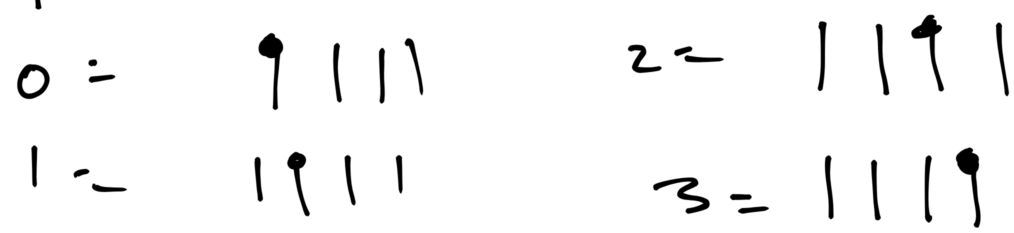
$$(t_1, t_2) \sim (ft_1, f\circ t_2)$$

Given t, there is always an f such that ft has its marked leaf at the extreme left. elements of V are given by pairs of trees as above.

Begin day 2 with another example of a category, i.e. dotted planar binary forests: $\dot{\mathcal{F}}$

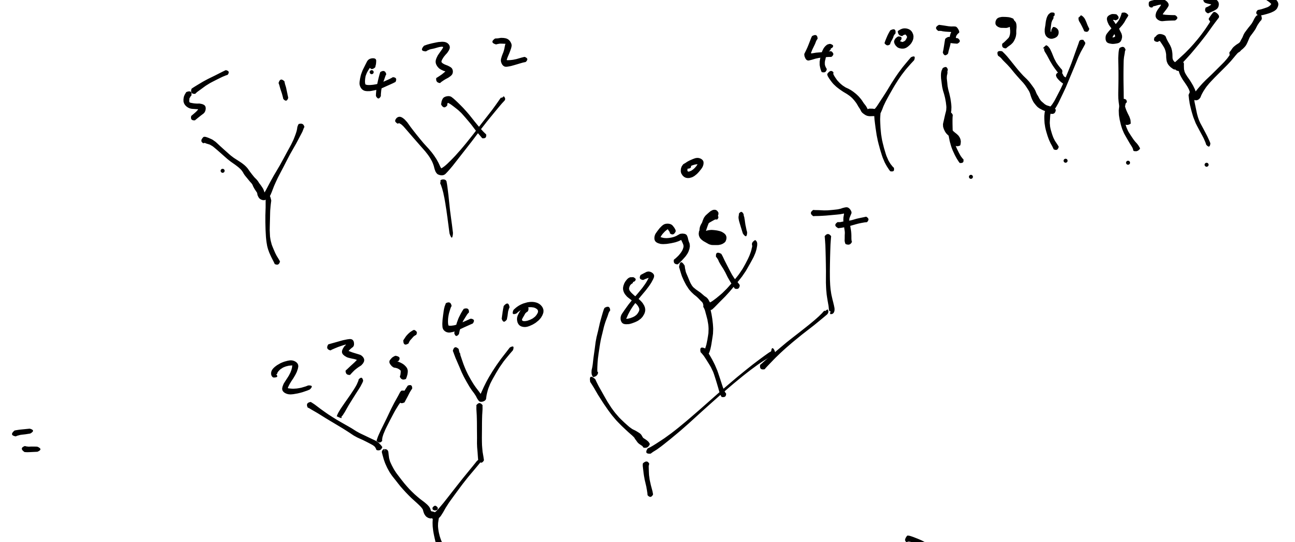


It is clear that this category satisfies the axioms – the cyclic groups $\mathbb{Z}/n\mathbb{Z}$ are groups of morphisms from n to n:

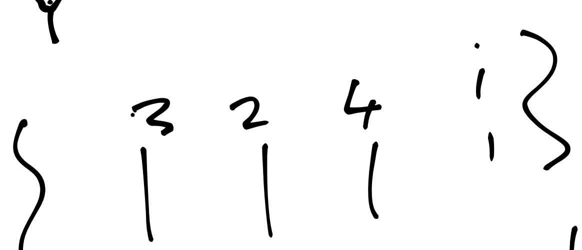


So for instance to prove stabilisation, first prove it for the undotted trees (done), then move between the dots with the cyclic group.

Another example $\mathcal{F}^{\Sigma}$



clearly



under categorical composition form the symmetric group so we can check axioms as before.

Day 2

The categorical construction

Let $\mathcal{C}$ be a small category with the following 3 properties.

i) $\exists\ 1$ st. $\mathrm{mor}(1,x) \neq \emptyset \quad \forall x \in \mathrm{Obj}(\mathcal{C})$

Let $\mathcal{T} = \coprod_{x \in \mathrm{Obj}(\mathcal{C})} \mathrm{mor}(1,x)$.

(ii) $\forall\ t_1, t_2 \in \mathcal{T}\ \exists\ f_1, f_2$ st. $f_1 t_1 = f_2 t_2$.
(stabilization)

(iii) If $f_1 t = f_2 t$ for $t \in \mathcal{T}$ then $f_1 = f_2$
(cancellation)

<u>Proposition</u> $\mathcal{T}$ forms a <u>directed set</u> under $s \leq t \iff \exists f$ st. $fs = ft$.

<u>Proof</u> If $s, t \in \mathcal{T}$ then by stabilisation $\exists\ f, g$ with $fs = gt$ so $s \lesssim fs = gt \gtrsim t$
Transitivity of $\lesssim$ is composition of morphisms. QED

<u>Proposition</u> Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a small functor then the map $A_\Phi : t \to \Phi(\mathrm{target}(t))$ a <u>direct system</u> with $\eta_s^t = \Phi(f)$, where $t = fs$.

Proof By definiton a direct system on a directed set $(S, \lesssim)$ is a map $s \longmapsto \Phi(s)$, a set, together with "inclusion maps" $\iota_s^t : \Phi(s) \to \Phi(t)$

Satisfying $\iota_t^r \circ \iota_s^t = \iota_s^r$ if $s \lesssim t$, $t \lesssim r$.

These properties are obvious from composition of morphisms. Note that we have used cancellation to define ι_s^t. QED.

So we may define the direct limit

$\varinjlim A_\Phi$ which is the set

$$\frac{\coprod_{s \in S'} A_\Phi(s)}{\sim} \quad := \quad \frac{\coprod_{s \in S} \Phi(\mathrm{target}(s))}{\sim}$$

where $(s,x) \sim (t,y) \iff \exists f, g$ s.t. $(fs, \iota_s^{fs}(x)) = (gt, \iota_t^{gt}(x))$

(i.e. $\exists\, r \geq s,t$ with $\iota_s^r(x) = \iota_t^r(y)$)

Exercise i) Check that this is an equivalence relation

ii) In the tree example $\lesssim$ is a partial order but not in the dotted tree example.

Identifying the direct limit may be tricky but if all the ι_s^t's are injections we have:

<u>Proposition</u> For each $s \in \mathcal{S}$ the map $x \to [(s,x)]$ is an injection.

<u>Proof</u> If $(s,x) \sim (s,y)$ then $\exists$ $t \geq s$ with $\iota_s^t(x) = \iota_s^t(y) \Rightarrow x = y$.
QED.

<u>The group of fractions of $\mathcal{C}$.</u>

(this can be interpreted as a $\varinjlim_{\mathcal{C}} A_x$ but it's confusing)

Form
$$\boxed{\mathcal{P} = \frac{\coprod_{x \in \mathrm{Ob}(\mathcal{C})} \{ (s,t) \mid s,t \in \mathrm{Mor}(1,x) \}}{\sim}}$$
where $(s_1,t_1) \sim (s_2,t_2) \iff \exists f,g$ s.t. $(fs_1, ft_1) = (gs_2, gt_2)$

$\frac{s}{t}$

Given (s_1,t_1) and (s_2,t_2) in $\mathcal{P}$ we want to define a composition. By stabilisation $\exists$ $f,g \in \mathcal{C}$ with $ft_1 = st_2$
so we define $(s_1,t_1) \circ_{f,g} (s_2,t_2) = (fs_1, gs_2)$

<u>Theorem</u> The map $\mathcal{P} \times \mathcal{P} \to \mathcal{P}/\sim$ defined by $(s_1,t_1) \circ_{f,g} (s_2,t_2)$ depends neither on the choice of f and g, nor on the choice of (s_1,t_1) and (s_2,t_2) in their equivalence classes. It makes $P/\sim$ into a group.

Proof. The well definedness of the operation follows from a judicious use of stabilisation and cancellation.

If we choose $\bar{f}, \bar{g}$ with $\bar{f}t_1 = \bar{g}s_2$

then there are p and q (stabilisation) with

$p\bar{f}t_1 = qft_1$ so by cancellation $\boxed{p\bar{f} = qf}$ $\boxed{ft_1 = gs_2}$

further, $qft_1 = qgs_2$ and $p\bar{f}t_1 = p\bar{g}s_2$

so $qgs_2 = p\bar{g}s_2 \Rightarrow \boxed{qg = p\bar{g}}$

$$\begin{aligned} \text{so} \quad (fs_1, gt_2) &\sim (qfs_1, qgt_2) \\ &= (p\bar{f}s_1, p\bar{g}t_2) \\ &\sim (\bar{f}s_1, \bar{g}t_2). \end{aligned}$$

So composition is well defined on classes

This makes associativity easy:

$$\left([(s_1, t_1)][(s_2, t_2)] \right) \left([s_3, t_3] \right)$$

choose a, b s.t. $at_1 = bs_2$

and this $= \left([as_1, at_1][(bs_2, bt_2)] \right) \left([s_3, t_3] \right)$

now choose c, d s.t. $cbt_2 = ds_3$.

ans $= [cas_1, ds_3]$, but

$\left([(s_2, t_2)][s_3, t_3] \right) = (b$

$$[(s_2,t_2)][(s_3,t_3)] = [(bs_2, bt_2)(s_3 t_3)]$$
$$= [(cbs_2, cbt_2)(ds_3, dt_3)]$$
$$= [(cbs_2, dt_3)]$$

and $(s_1, t_1) = (cas_1\ cat_1) = (cas_1, cbs_2)$

so other order get $[(cas_1, dt_3)]$.

So it's an associative operation on classes.

Identity : $[(1,1)]$ $\quad : [(1,1)][(s,t)]$
$= [(s,s)]$ $\quad = [(s,s)][(s,t)] = [(s,t)]$

Inverse $\quad ((s,t)(t,s)) = [(s,s)]$.

& it's a group. $\quad G_{\mathcal{C}}$

Group of fractions of $(\mathcal{C}, 1)$

Examples (i) $\mathcal{C} = \mathcal{F} \to G_{\mathcal{C}} = F$ (Thompson)

(ii) $\mathcal{C} = \dot{\mathcal{F}} \to G_{\mathcal{C}} = T$ (Thompson on circle)

(iii) $\mathcal{C} = \mathcal{F}^{\Sigma} \to G_{\mathcal{C}} = V$ (F + permute intervals)

(iv) $\mathcal{C}$ = fundamental groupoid of a connected nice space X, $1 = 1$ base point
$G_{\mathcal{C}} = \pi_1(X, 1)$

(v) $\mathcal{C}$ = a group Γ category with 1 object.

$G_{\mathcal{C}} = \Gamma$

(vi) $\mathcal{C} = (\mathbb{N} \cup \{0\}, +)$, $1 = 0$.

$G_{\mathcal{C}} = \mathbb{Z}$.

(vii) $\mathcal{C} = (\mathbb{N}, \cdot)$ $1 = 1$, $G_{\mathcal{C}} = (\mathbb{Q}^+, \cdot)$

(viii) $\mathcal{C}$ = {positive braids on n strings} - semigroup
- category with 1 object

$G_{\mathcal{C}} = B_n$ (braid group on n strings)

exercise. check axioms.

Note Garside theory

If Δ is a $\frac{1}{2}$ twist then any element of B^n is $\Delta^k \times$ a positive braid for some $k \leq 0$

(ix) $\mathcal{C}$ = n-ary planar forests F_n, 1.

$G_{\mathcal{C}}$ = Thompson group F_n.

$\dot{F}_n$. F_n^{Σ} obviously.

So what?

Let $(\mathcal{C}, 1)$ be a category satisfying the axioms and $\Phi : \mathcal{C} \to \mathcal{D}$ be a functor.

Theorem $[(s,t)]([t,x]) = [s,x]$ $\quad (x \in \Phi(\text{target}(t)))$

defines an action of $G_{\mathcal{C}}$ on $\varinjlim A_\Phi$.

Proof It is formally a subset of the proof that $G_{\mathcal{C}}$ is a group. QED

Note: Action will preserve structure in $\mathcal{D}$

Examples

(i) $\mathcal{C} = (\mathbb{N} \cup \{0\}, +)$

a functor from $\mathcal{C}$ to Vect (finite dimensional vector spaces over some field)

is just a linear map $: V \to V$.

each such map gives a vector space $\varinjlim A_\Phi$, and an action of $\mathbb{Z}$ on it.

The case of an injection is standard — make an endomorphism invertible

(ii) Similarly, $(\mathbb{N}, \cdot)$. A functor to Vect is just a linear map for each prime.

(iii) For the fundamental groupoid, if X is a manifold and $\mathcal{V}$ is a vector bundle with flat connection, fibre V then parallel transport gives a functor and the ensuing rep of π_1 is the holonomy representation.

(iv) The braid group.

Recall that B_n is $\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \ \ |i-j|\geq 2, \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$

and positive braids are just words on $\sigma_1, \ldots, \sigma_{n-1}$.

So every time one has $n-1$ matrices, <u>invertible or not</u>, our construction gives a representation of the braid group.

Here is a construction which achieves that. Presumably it is what the category group of fractions method gives:

Δ^2 is the full twist $(= (\sigma_1\sigma_2\ldots\sigma_{n-1})^n)$ which is in the centre of B_n, and is a positive braid. So if the matrices are $s_1 \ldots s_{n-1}$, let Δ^2 be the matrix $(s_1 \ldots s_{n-1})^n$. Let W be the subspace of V (V = vector space on which s_i's act) spanned by the non-zero eigenspaces of Δ^2. Then on W, each s_i is invertible. ✓

Not so clear in infinite dimensions!

If the s_i's are just self maps of any set X, $s_i : X \to X$, one can define $X_0 = \bigcap_{i=1}^{\infty} \Delta^{2i}(X)$

This is invariant under Δ^2 and Δ^2 is invertible on it. So since $\Delta^2 = s_i P_i$ (P_i positive)
$= Q_i s_i$ (Q_i positive)
for any i, s_i is invertible on X_0 for any i.

<u>Another example</u> $\mathcal{C}$ = homotopy groupoid, $x = 1$, X a manifold. Take a vector bundle $\mathcal{V}$ with fibre V over X, and a flat connection on $\mathcal{V}$.
Parallel transport yields a functor $\Phi : \mathcal{C} \to \text{vect}$
($\Phi(x) = V$, $\Phi(\text{path})$ = parallel transport along path)

The representation of $G_{\mathcal{C}} = \pi_1(X, x)$ is obviously the Holonomy rep of the connection.

Let us come to the main example, Thompson's group F.

The group of fractions method gives an action of F whenever one has a functor from $\mathcal{F}$ to some category $\mathcal{D}$.

<u>Claim</u> $\mathcal{F}$ is generated by $Y_i =$ 

and the only relation is 

this relation is $\boxed{\Phi(Y_j)\,\Phi(Y_i) = \Phi(Y_i)\,\Phi(Y_{j-1})}$ for $\boxed{i < j-1}$

<u>Proof</u> Clear enough.

So any time you have a category with this structure you are in business.

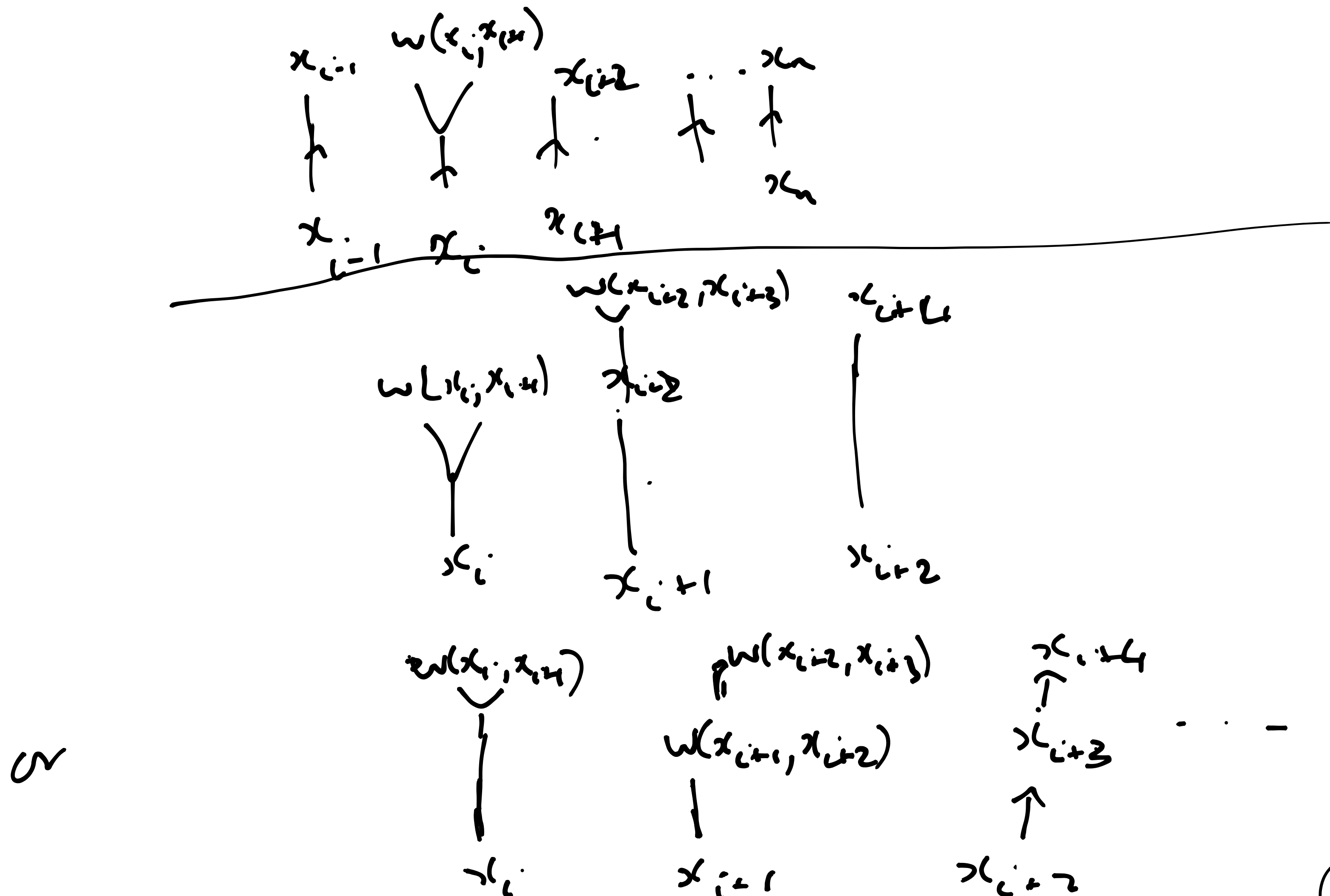
<u>Observation</u> If the spaces in $\mathcal{D}$ are Hilbert spaces and the $\Phi(\,||Y|||\,)$ are <u>isometries</u> ($u^*u = 1$), then $\varinjlim A_\Phi$ has a positive definite scalar product invariant under the action of F (to be checked) so we may <u>complete</u> it to obtain a <u>unitary</u> representation of F on a Hilbert space

Before going on let's do a "simpler" example: $\mathcal{D}$ = category, objects are $\mathbb{F}_n$ = free group on n generators $(x_1, \dots, x_n)$

Choose any element w of F_2 $w(a,b)$

and define $\Phi(Y_i)(x_j) = \begin{cases} x_j & j<i \\ w(x_i, x_{i+1}) & j=i \\ x_{j+1} & j>i. \end{cases}$

(extended to Free groups by universal property)



Rem

Get an action of F on some group which is a direct limit of free groups

Remark on cofinal sequences ⊂ full binary trees

Check that F preserves inner product.

First, existence of $\langle , \rangle$ on $\varinjlim A_{\Phi}$: This is general and directed set, any direct system with isometry

The only thing is to show that it is well-defined, sesquilinearity and positive definiteness will then follow from each $\Phi(t)$.

So if $v \in \Phi(t)$, $w \in \Phi(s)$ and $s, t \lesssim r$ then $\langle [v], [w] \rangle$ is to be defined by $\langle \iota_t^r(v), \iota_s^r(w) \rangle$

By composition this is independent of r.

But $\sim$ in the definition of the direct limit is generated by $(s, v) \sim (fs, \Phi_f(v))$

Choose a sufficiently large r and we're done.

That action of F preserves $\langle , \rangle$

given $[v]$ & $[w]$ in $\varinjlim$, can assume $v, w \in \Phi(t)$ for some t, and if $p, q \in \mathcal{T}$

$$\left\langle \frac{p}{q}(t, v), \frac{p}{q}(t, w) \right\rangle = \langle (p, v), (t, w) \rangle = \langle [v], [w] \rangle$$

provided $t = q$, which can be supposed

QED

<u>Sum example</u> $\Phi(n) = \bigoplus_{i=1}^{n} \mathcal{H}$, $\mathcal{H}$ fixed

$$|| \ \underset{i}{Y} \ || \ (\xi_1, \xi_2 \ldots \xi_i \cdot \xi_{i+1} \ldots \xi_n) = (\xi_1, \ldots, W\xi_i, \xi_{i+1} \ldots \xi_n)$$

$\in \bigoplus^{n+1} \mathcal{H}$.

where W is <u>any</u> isometry from $\mathcal{H}$ to $\mathcal{H} \oplus \mathcal{H}$.

<u>Formally</u> $\Phi(Y_i)(\xi_1, \ldots, \xi_n) = (\eta_1, \ldots, \eta_{n+1})$

where
$$\eta_j := \begin{cases} \xi_j & j < i \\ A\xi_i & j = i \\ B\xi_i & j = i+1 \\ \xi_{j+1} & j > i+1 \end{cases}$$

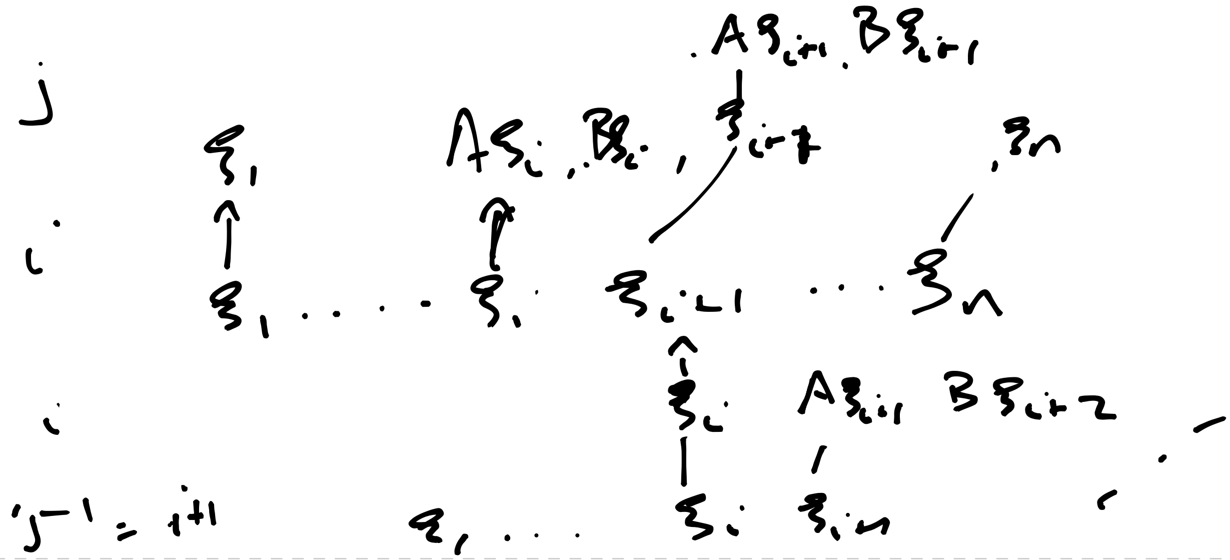
where $W(\xi) = (A\xi, B\xi)$.

<u>note</u> the condition for W (hence the $\Phi(Y_i)$) to be isometries is

$$\boxed{A^*A + B^*B = 1}$$

($|A|^2 + |B|^2 = 1$, Pythagoras.)

check that $\Phi(Y_j)\Phi(Y_i) = \Phi(i)\Phi(Y_{j-1})$ for $i < j-1$



$j = i+2$

ok

So every time you have a representation of $|A|^2 + |B|^2 = 1$ you get a unitary rep of F (and T).

<u>Aside</u> Unitary representations and coefficients $g \mapsto u_g$

A <u>coefficient</u> is a function $g \mapsto \langle u_g \xi, \eta \rangle$.

If you know $g \to \langle u_g \xi, \xi \rangle$ for some ξ, you know the representation on $\overline{[G\xi]}$.

$\to$ take finitely supported functions on Γ. $= \mathbb{C}\Gamma$,

$$\text{“}\langle \sum c_g u_g \xi, \sum d_h u_h \xi \rangle\text{”} = \sum_{g,h} c_g \overline{d_h} \langle u_g \xi, u_h \xi \rangle$$
$$= \sum_{g,h} c_g \overline{c_h} \langle u_h^{-1} g \xi, \xi \rangle.$$

so $$\langle c, d \rangle = \sum_{g,h} c_g \overline{d_h} \underset{\text{coeff.}}{f(h^{-1}g)}$$

This will be a positive semidefinite ... with $\overline{[\Gamma \xi]}$ as quotient, action ok.

So we want to have a nice vector ξ and calculate its coefficients

Fourth day.

Do something with reps?

Property T for a group Γ

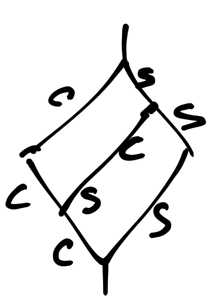
"Trivial rep is isolated in space of all unitary reps"

or $\boxed{\exists\ g_1 \dots g_n \in \Gamma,\ \varepsilon > 0 \text{ s.t. } \pi : \Gamma \to \mathcal{U}(\mathcal{H}),\ \xi \in \mathcal{H},\ \|\xi\| = 1,\ \|\pi(g_i)\xi - \xi\| < \varepsilon\ \forall i \Rightarrow \exists\ \xi' \neq 0,\ \pi(g)\xi' = \xi'\ \forall g.}$

Obvious for Γ finite (compact) but non trivial indeed for Γ infinite (Kazdan)

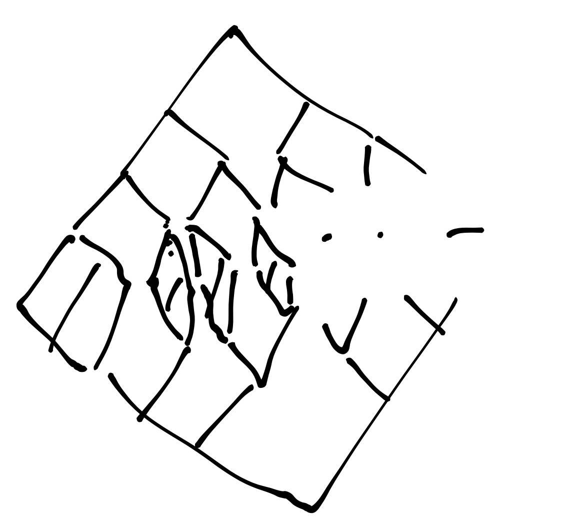
We can show that F, $[F,F]$ and T don't have T. (known but by less direct methods).

consider $A = \cos\theta$ $B = \sin\theta$ for W.

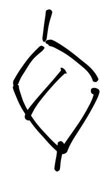


$= x = c^3 + s^3 + c^2 s^2 < 1$ for $\theta \neq 0, \pi/2$

let $g_n =$

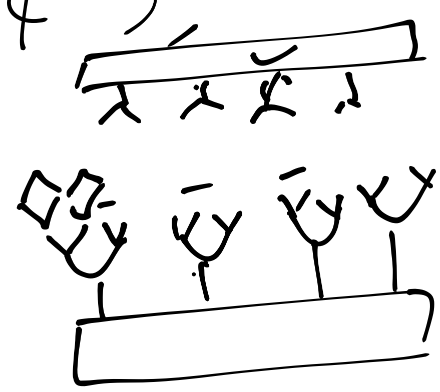


all paths from top to bottom must go through

 , and, grouping them together we get

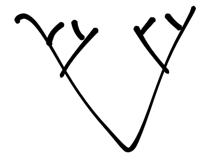
$\langle g_n \Omega, \Omega \rangle = x$

Moreover if  $\leftarrow 2^n$ is an arbitrary element of $\bigoplus^{2^n}(\mathbb{C}^2)$

$\langle g_n v, v \rangle$ =  $= x\langle v, v\rangle$,

n large

so $\langle g_n v, v\rangle \to x\langle v, v\rangle$

But $\{$  $\mid 2^n \}$ are cofinal in $\mathcal{H}$ so

$\langle g_n v, v\rangle \to x\langle v, v\rangle$ $\forall$ v in the prehilbert space. $\Rightarrow$ $\langle g_n v, v\rangle \to x\langle v, v\rangle$ $\forall v$ in Hilbert space

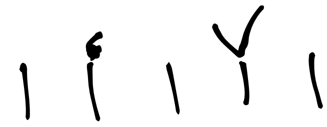
But if v is fixed, $g_n v = v$, a contradiction.

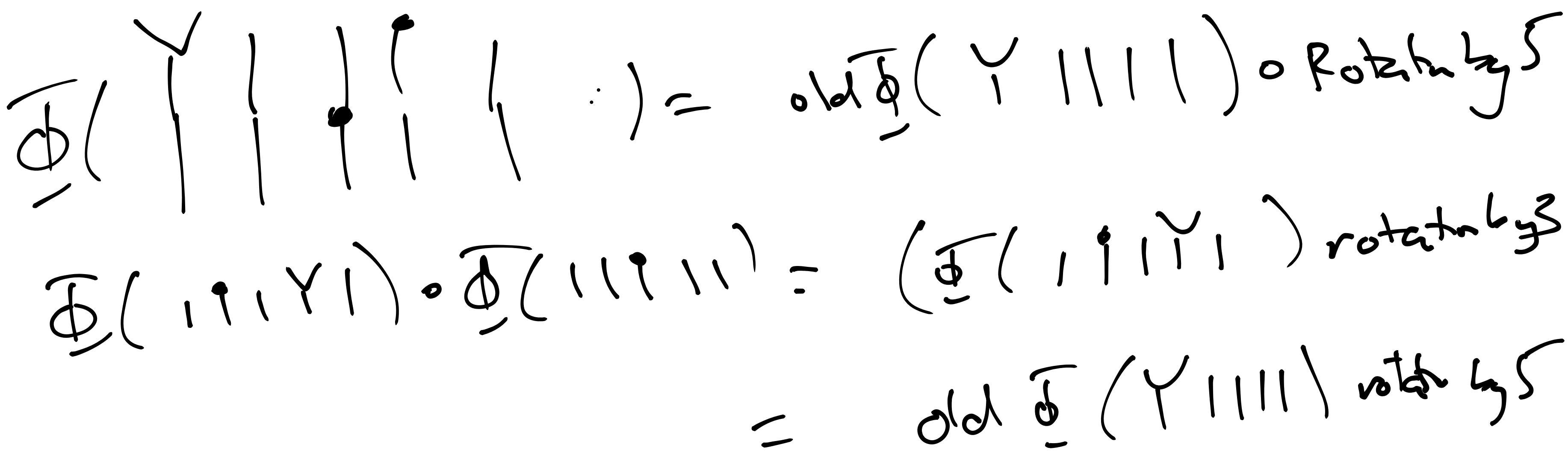
$\Rightarrow$ no fixed vectors, but as $\theta \to 0$, π tends to trivial. So F doesn't have T.
(? check)

Actually this is completely obvious since $F \twoheadrightarrow \mathbb{Z}\oplus\mathbb{Z}$ and $\mathbb{Z}\oplus\mathbb{Z}$ doesn't have T.

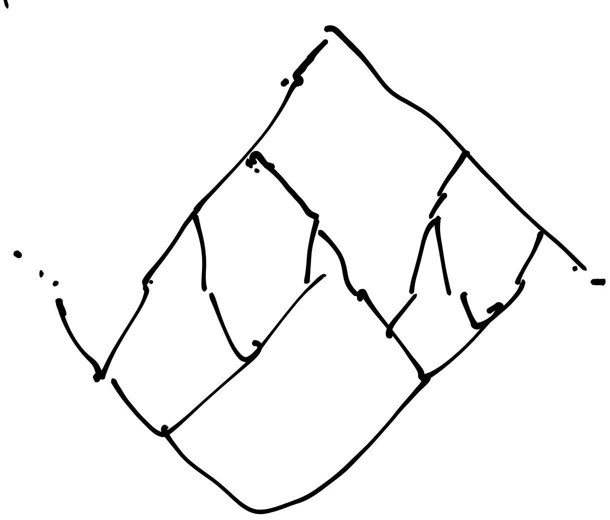
However we can easily change g_n to $\tilde{g}_n$, just at the end points, so that $\langle g_n v, v\rangle \sim \langle \tilde{g}_n v, v\rangle$ and $\tilde{g}_n \in [F,F]$. So $[F,F]$ doesn't have T.

Observation any Y that works for F will work for T.





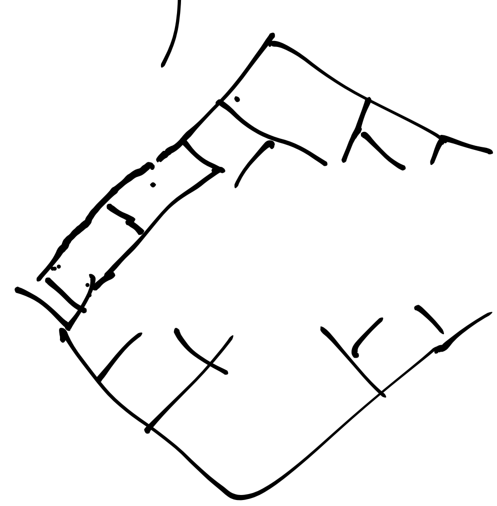
Add coefficients are obviously

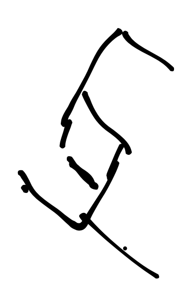


etc.

Unfortunately above method doesn't work for

T:

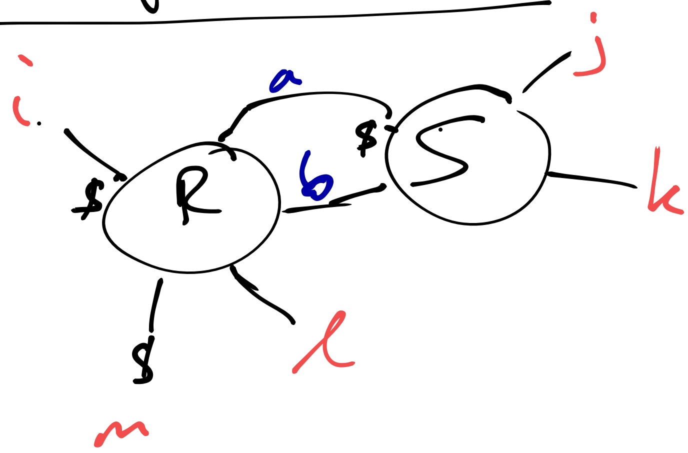




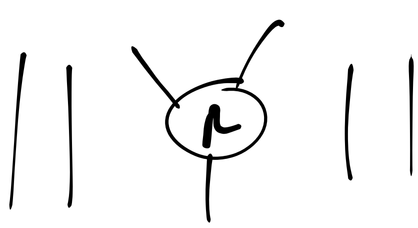
The tensor setup:

replace $\bigoplus_{i=1}^{n} \mathcal{H}$ by $\bigotimes_{i=1}^{n} \mathcal{H}$.

Notation for tensors



$$(i,j,k,l,m) = \sum_{a,b} R_{iablm}\, S_{ajkb}.$$

So  defines a linear map from

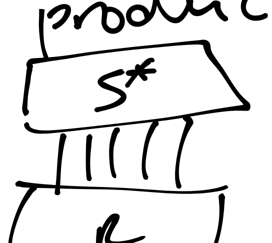
$$\bigotimes^{n} \mathcal{H} \longrightarrow \bigotimes^{n+1} (\mathcal{H})$$

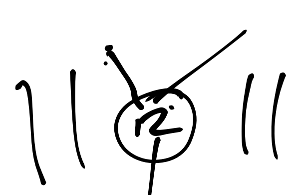
f.d.v.s

a tensor

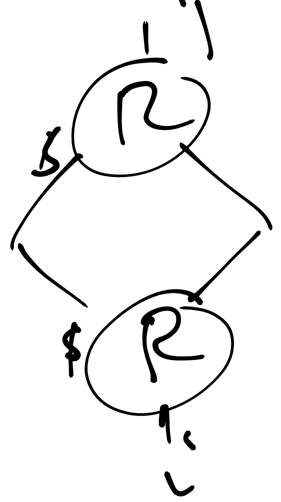
$\longrightarrow$  another tensor

Inner product on tensors

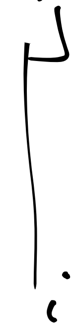


so when is  an

Bonely ?



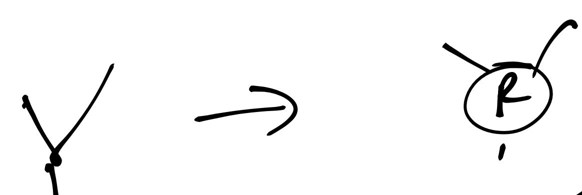
$=$



$= \delta_{ij}$

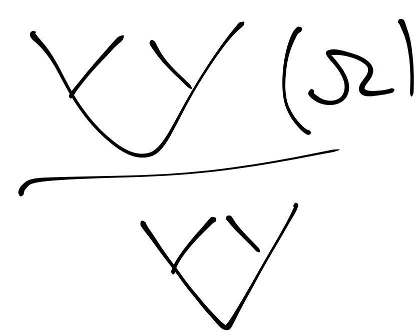
$$\sum_{ab} R_{abi} \overline{R}_{abj} = \delta_{ij}.$$

Given any such R you get a unitary rep of F.



Coefficients choose $\Omega \in \mathcal{H}$

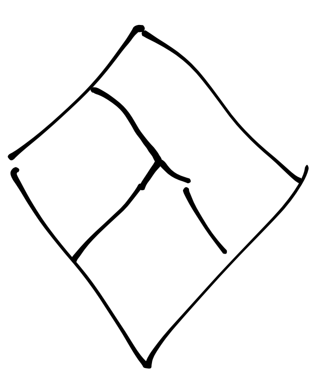
$\Omega = \frac{\Omega}{1} =$



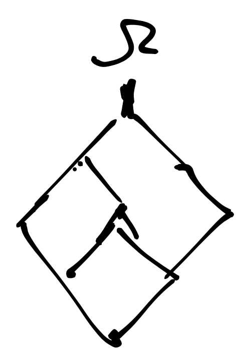
$= \sum R..R...$

so $\langle g\Omega, \Omega \rangle$

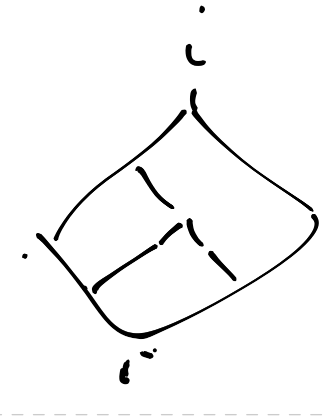
$g =$



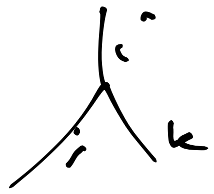
; ans =



if Ω_i a basis vector $=$



e.g. dim $H = 3$

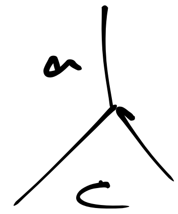


$= \begin{cases} 0 & \#\{i,j,k\} \neq 3 \\ 1 & i,j,k \text{ distinct.} \end{cases}$

"Wellknown fact" $\langle g\Omega, \Omega \rangle > 0 \ \forall g$

$\Longleftrightarrow$ four colour theorem.

Other example

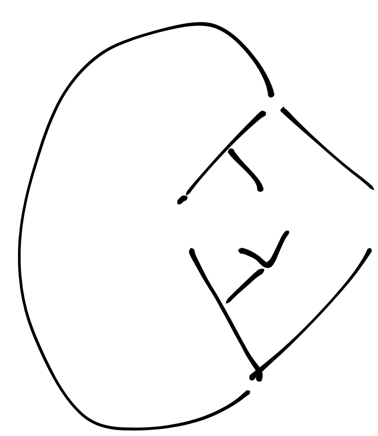


$= \delta_{abc}$

indizes in regions

In this case $\langle g\Omega, \Omega \rangle = \#$ of ways

of 3 colouring map



which is either possible or not

$\langle g\Omega, \Omega \rangle = \begin{cases} \text{"} 1 \text{"} \\ \text{"} 0 \text{"} \end{cases}$

$\Rightarrow$ rep induced.

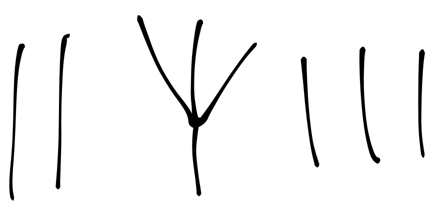
Yunxiang Ren - subgroup $\cong F_4'$ (!!)

Finally

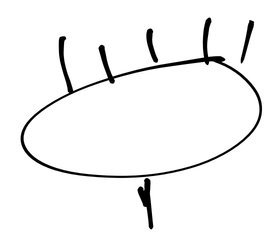
Knots & links

Easiest approach: Work with F_3:

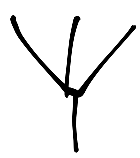
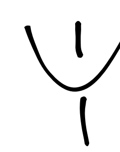
look for $\Phi(\Psi_i)$



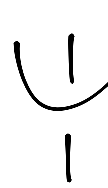
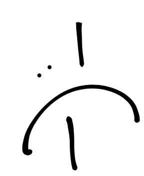
Conway's linear skein theory.

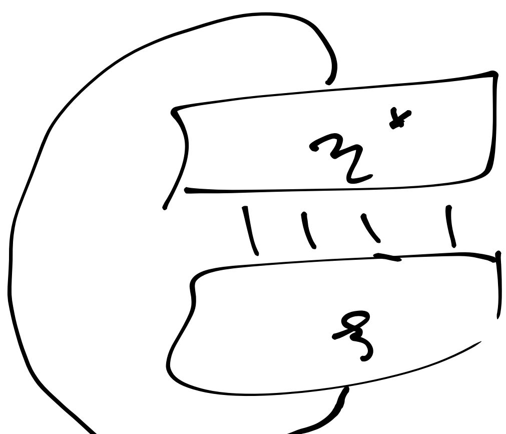
$\Phi(n)$ = linear combs of tangles 

(n odd) 0 n even.

Choose $\delta \neq 0$ ($\in \mathbb{R}^+$) & set  $= \frac{1}{\sqrt{\delta}}$ 

inner product with * structure

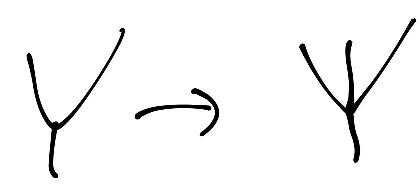
$\left(\frac{1}{\sqrt{\delta}} \right.$  $\left. \right)^* = \frac{1}{\sqrt{\delta}}$ 

$\langle \xi, \eta \rangle$ =  $\in$ linear combination of knots and links.

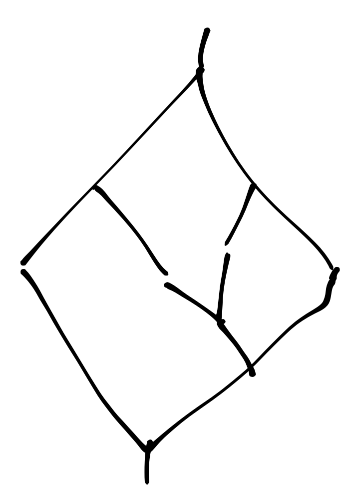
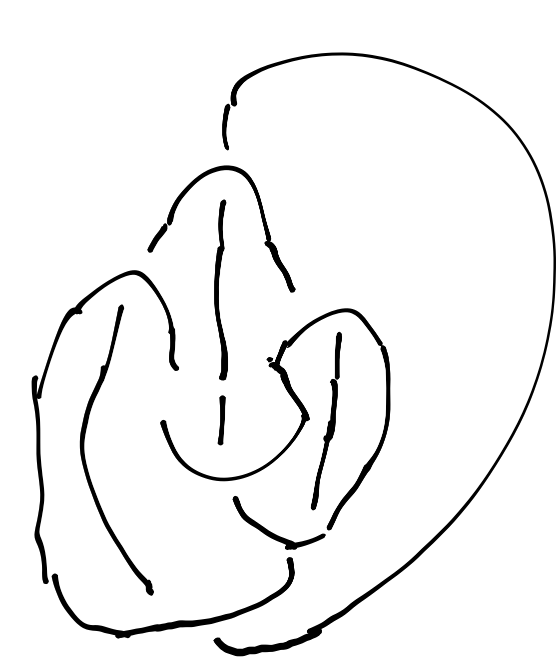
Apply TQFT (e.g. Jones polynomial/Kauffman Bracket) to get a number. If $\delta = 2\cos \pi/n$ the rep factors through a Hilbert space quotient so is unitary → $\langle g\Omega, \Omega \rangle$ = Jones poly of link, is positive definite on F_3

Ω = | .

<u>Obvious question</u>: Does every link arise as $\langle g\Omega, \Omega \rangle$ (up to δ's - distant unknots)

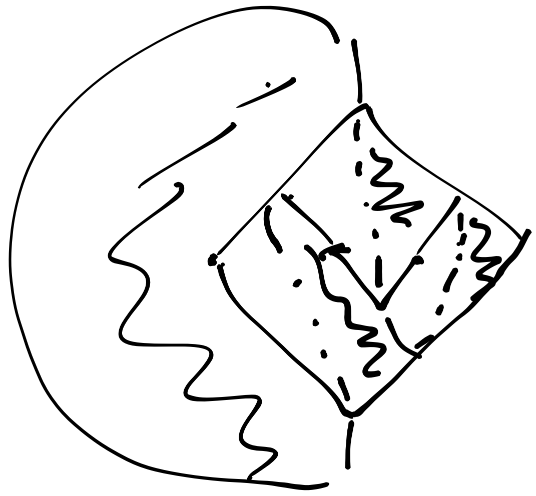
<u>Theorem</u> $\forall L \ \exists g \in Im(F_2) < F_3$ under homomorphism $F_2 \to F_3$ 
s.t. $\langle g\Omega, \Omega \rangle = L$

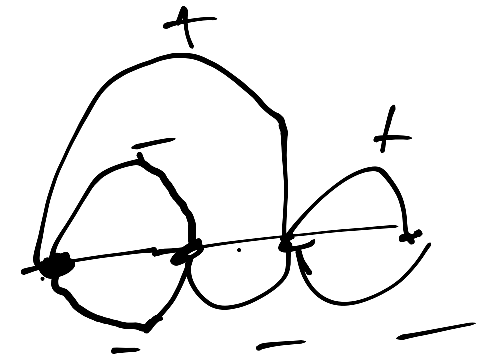
<u>Proof</u>.

ex. g:  → 

not clear how to get anything

idea. Shading

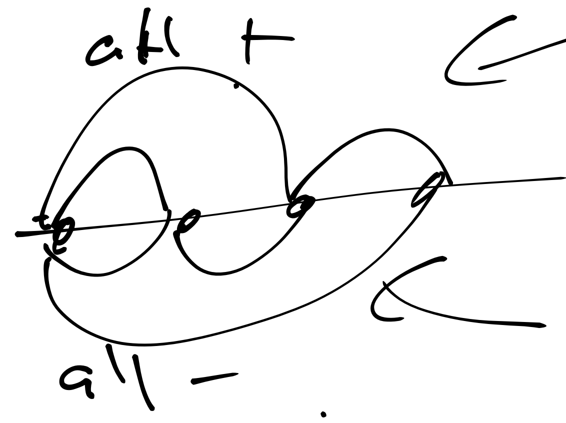




take the Tait picture of a link.

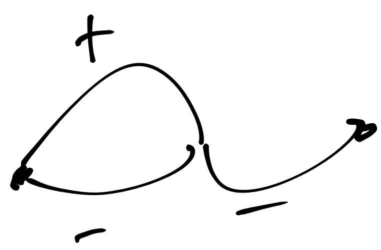
→ make it look like

planar graph



planar graph

need some moves - Reidemeister



→

etc. see published paper
(arxiv)