Fragments of residuated lattices axiomatized by simple equations and decidability

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A (commutative) **residuated lattice** is an algebraic structure $\mathbf{R} = (R, \vee, \wedge, \cdot, \backslash, /, 1)$, such that

- $(R, \vee, \wedge)$ is a lattice
- $(R, \cdot, 1)$ is a (commutative) monoid
- For all $x, y, z \in R$

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y,$$

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Residuated Lattices

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- (C)RL denotes the **variety** of (commutative) residuated lattices.
- multiplication is order preserving:

$$x \leq y \implies uxv \leq uyv$$

- multiplication distributes of join:

$$x(y \vee z) = xy \vee xz \quad \& \quad (y \vee z)x = yx \vee zx$$

Residuated structures are the algebraic semantics of substructural logics (i.e., axiomatic extension of the **Full Lambek Calculus**) $\mathbf{FL}$.

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \alpha, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \Pi}\ (\mathbf{cut}) \qquad \frac{}{\alpha \Rightarrow \alpha}\ (\mathbf{init}) \qquad \frac{}{\Rightarrow 1}\ (1r)$$

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \cdot \beta, \Gamma_2 \Rightarrow \Pi}\ (\cdot l) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot r) \qquad \frac{\Gamma_1, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Pi}\ (1l)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \beta, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \alpha\backslash\beta, \Delta_2 \Rightarrow \Pi}\ (\backslash l) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\backslash r) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0r)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \beta, \Delta_2 \Rightarrow \Pi}{\Delta_1, \beta/\alpha, \Gamma, \Delta_2 \Rightarrow \Pi}\ (/l) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (/r) \qquad \frac{}{0 \Rightarrow}\ (0l)$$

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$$\frac{\Gamma_1, \beta, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Pi}\ (\wedge l_1) \qquad \frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Pi}\ (\wedge l_2) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r)$$

Structural rules have an algebraic meaning.

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Pi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \Pi}\ (\mathsf{e}) \quad \Leftrightarrow \quad xy \leq yx$$

$$\frac{\Gamma, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi}\ (\mathsf{w}) \quad \Leftrightarrow \quad x \leq 1$$

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi}\ (\mathsf{c}) \quad \Leftrightarrow \quad x \leq x^2$$

We can use algebraic methods to answer questions about the logics.

(Quasi-) Equational Theory

A **quasi-equation** ξ is a universally-quantified formula

$$s_1 = t_1 \;\&\; \cdots \;\&\; s_n = t_n \implies s_0 = t_0,$$

where $s_0, t_0, s_1, t_1, \ldots, s_n, t_n \in T(X)$ are terms.

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The *equational theory* for $\mathcal{V}$ is the set of equations that it satisfies

$$\{s = t : \mathcal{V} \models s = t\}$$

The *Word Problem*

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We say $\mathcal{V}$ has an **undecidable word problem** if there exists a finite presentation $\langle X, E\rangle$ such that there is no algorithm deciding whether the q.e. ($\&E \implies s = t$) holds in $\mathcal{V}$ having $s, t \in T(X)$ as inputs.

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Or equivalently, there is a finitely presented algebra $\mathbf{A} \in \mathcal{V}$ generated by X such that the following set is undecidable:

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- undecidable word problem $\Rightarrow$ undecidable q.e. theory.

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$$\begin{array}{lcl} \{\leq, \cdot, 1\} & : & \text{Ordered Monoid Fragment} \\ \{\vee, \cdot, 1\} & : & \text{Idempotent Semiring (ISR) Fragment.} \end{array}$$

Overview of Decidability Results

$\mathcal{V}$	Eq. Th.	WP
RL	FMP	Und. $\{\leq, \cdot, 1\}$
$\mathsf{RL} + x \leq x^2$	Und.	Und. $\{\leq, \cdot, 1\}$
$\mathsf{RL} + x \leq x^2 \vee 1$	?	Und. $\{\leq, \cdot, 1\}$
$\mathsf{RL} + xy \leq yx \vee xyx$	?	Und. $\{\leq, \cdot, 1\}$
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CRL	FMP	Und. $\{\vee, \cdot, 1\}$
CRL $+ x^m \leq x^n$	FMP	FEP
CRL $+ x \leq x^2 \vee x^3$	Und.	Und. $\{\vee, \cdot, 1\}$
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Encoding an Undecidable Problem

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k-Counter Machine

A k-CM is a tuple $\mathtt{M} = (\mathtt{R}_k, \mathtt{Q}, \mathtt{P}, q_f)$ where,

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As we will see, they can be encoded using a some *string rewriting system.*

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Theorem (Minsky)

There exists a 2-CM whose set of accepted configurations is undecidable.

Example: $\mathtt{M}_{\text{even}}$

$\mathtt{M}_{\text{even}} = (\mathtt{R}_2, \mathtt{Q}_{\text{even}}, \mathtt{P}_{\text{even}}, q_f)$ where

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p_f : If in state q_2 and register $\mathtt{r}_2$ is empty, transition to the final state q_f.

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How should we do this?

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Let $\leq_{\mathtt{M}}$ be the least compatible preorder generated by $\mathtt{P}$ and the finite sets $\{qx \leq xq : q \in \mathtt{Q} \,\&\, x \in \mathtt{R}_k \cup \mathtt{Stp}_k\}$ and $\{xq \leq qx : q \in \mathtt{Q} \,\&\, x \in \mathtt{R}_k \cup \mathtt{Stp}_k\}$.

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We often write $\leq^p$ to be compatible preorder generated by p:

$$\frac{s \leq^p t}{usv \leq^p utv}$$

The $\mathtt{M}_{\text{even}}$ machine

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- **Configuration**: $\mathtt{C} = uqv$, where $q \in \mathtt{Q}$ and $uv = \mathrm{S}_0\mathtt{r}_1^n\mathrm{S}_1\mathtt{r}_2^m\mathrm{S}_2$.

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CMs in ordered monoids: Motivating example

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CMs in ordered monoids: Motivating example

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CMs in ordered monoids: Motivating example

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Admissibility

We are interested in whether the set $\mathrm{Acc}(\mathtt{M})$ is *resilient* to certain inference rules corresponding to certain inequations.

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- $q_0\mathtt{r}_1^4, q_0\mathtt{r}_1^6 \in \mathrm{Acc}(\mathtt{M}_{\text{even}})$ but $q_0\mathtt{r}_1^3 \notin \mathrm{Acc}(\mathtt{M}_{\text{even}})$.

$$
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E.g., representing n-many tokens in a register by $\phi(n)$, for some (computable) function ϕ on $\mathbb{N}$.

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What about commutativity $xy \leq yx$?

- Impossible for the standard encoding. Commutative ordered monoids have a decidable word problem! [Mayr 1984]

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And-branching Counter Machines

An ACM is a machine $\mathtt{M} = (\mathtt{R}_k, \mathtt{Q}, \mathtt{P}, q_f)$ is a CM containing no zero-test instruction but allows **branching instructions**

$$q \leq q' \vee q''.$$

Acceptance and Quasi-equations

Let $\mathtt{M} = (\mathtt{R}_k, \mathtt{Q}, \mathtt{P}, q_f)$ be a counter machine.

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- For a monoid term u, we define the quasi-equation $\mathrm{acc}_{\mathtt{M}}(u)$ to be

$$\&\mathtt{P}_{\text{com}} \implies u \leq \mathtt{C}_f$$

where for CM's $\mathtt{C}_f = q_f \mathrm{S}_0 \mathrm{S}_1 \cdots \mathrm{S}_k$ and ACM's $\mathtt{C}_f = q_f$.

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Completeness is achieved by constructing a counter-model using the theory of **residuated frames**.

Residuated frames

Definition

A **residuated frame** is a structure $\mathbf{W} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/, 1)$, s.t.

- $(W, \circ, 1)$ is a monoid and W' is a set.
- $N \subseteq W \times W'$,
- $\backslash\!\!\backslash : W \times W' \to W'$ and $/\!\!/ : W' \times W \to W'$ such that
- N is **nuclear**, i.e. for all $u, v \in W$ and $w \in W'$,
  $$(u \circ v)\; N\; w \quad \text{iff} \quad u\; N\; (w /\!\!/ v) \quad \text{iff} \quad v\; N\; (u \backslash\!\!\backslash w).$$

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$$\wp(W) \underset{\lhd}{\overset{\rhd}{\rightleftarrows}} \wp(W') : \quad \begin{aligned} X^{\rhd} &= \{y \in W' : X \mathrel{N} y\} \\ Y^{\lhd} &= \{x \in W : x \mathrel{N} Y\} \end{aligned}$$

- $({}^{\rhd}, {}^{\lhd})$ is a Galois connection.
- The map $X \xmapsto{\gamma_N} X^{\rhd\lhd}$ is a closure operator on $\mathcal{P}(W)$.
- N is nuclear iff γ_N is a nucleus.

Residuated frames cont.

Theorem [Galatos & Jipsen 2013]

$\mathbf{W}^+ := (\gamma_N[\mathcal{P}(W)], \cup_{\gamma_N}, \cap, \circ_{\gamma_N}, \backslash, /, \gamma_N(\{1\}))$,

$$X \cup_{\gamma_N} Y = \gamma_N(X \cup Y) \text{ and } X \circ_{\gamma_N} Y = \gamma_N(X \circ Y),$$

is a residuated lattice.

Completeness of Encoding

Let $\mathtt{M} = (\mathtt{R}_k, \mathtt{Q}, \mathtt{P}, q_f)$ be a counter machine.

- $W_{\mathtt{M}} = (\mathtt{Q} \cup \mathtt{R}_k \cup \mathtt{Stp}_k)^*$
- $W'_{\mathtt{M}} = W \times W$
- $x \, N_{\mathtt{M}} \, (u, v) \iff uxv \in \mathrm{Acc}(\mathtt{M})$

Theorem

$\mathbf{W}_{\mathtt{M}}$ is a residuated frame.

Proof.

$$\begin{aligned} xy \, N_{\mathtt{M}} \, (u, v) &\iff uxyv \in \mathrm{Acc}(\mathtt{M}) \iff x \, N_{\mathtt{M}} \, (u, yv) \\ &\iff y \, N_{\mathtt{M}} \, (ux, v) \end{aligned} \qquad \square$$

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Theorem

For a variety $\mathcal{V} \subseteq \mathsf{RL}$, if $\mathbf{W}^+_{\mathtt{M}} \in \mathcal{V}$ then for all u

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Corollary

If $\mathbf{W}^+_{\mathtt{M}} \in \mathcal{V}$, then the word problem for $\mathcal{V}$ in the $\{\leq, \cdot, 1\}$-fragment is at least as hard as acceptance in $\mathtt{M}$. In particular, if $\mathtt{M}$ has an undecidable set of accepted configurations, then the word problem for $\mathcal{V}$ is undecidable.

Linearization and Simple Equations

In RL:

- Every equation $s = t$ over the signature $\{\vee, \cdot, 1\}$ can be written as the conjunction of *basic (in)equations* of the form

  $$w \leq v_1 \vee \cdots \vee v_k,$$

  where $w, v_1, ..., v_k$ are monoid terms over a set of variables X.

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- These basic equations can be **linearized** in a uniform way producing an equivalent *simple equation* of the form

$$[\mathrm{R}] : x_1 x_2 \cdots x_n \leq \bigvee_{r \in \mathrm{R}} r$$

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is equivalent to, via the substitution σ: $u \overset{\sigma}{\mapsto} x \vee y$ and $v \overset{\sigma}{\mapsto} z$,

$$(\forall x)(\forall y)(\forall z)\ xyz \leq x^3 \vee x^2 y \vee xy^2 \vee y^3 \vee xz \vee yz$$

Simple Equations and Simple Rules

Any simple equation $[\mathrm{R}]$ corresponds to a **simple structural rule** (R). For example

$$[\mathrm{R}] : xy \leq x^2 \vee y \iff \frac{\Delta_1, \Gamma, \Gamma, \Delta_2 \Rightarrow \Pi \quad \Delta_1, \Psi, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \Psi, \Delta_2 \Rightarrow \Pi} \ (\mathrm{R})$$

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In general,

$$[\mathrm{R}] : x_1 \cdots x_n \leq \bigvee_{r \in \mathrm{R}} r \iff \frac{\{\Delta_1, r^{\mathbf{FL}}(\Gamma_1, \ldots, \Gamma_n), \Delta_2 \Rightarrow \Pi\}_{r \in \mathrm{R}}}{\Delta_1, \Gamma_1, \ldots, \Gamma_n, \Delta_2 \Rightarrow \Pi} \ (\mathrm{R})$$

Theorem [Galatos & Jipsen 2013]

Extensions of $\mathbf{FL}$ by simple rules enjoy **cut-elimination**.

Simple equations and Residuated Frames

Lemma [Galatos & Jipsen 2013]

All simple equations ε are preserved by $(-)^{+}$:

$$\mathbf{W} \models (\varepsilon) \text{ iff } \mathbf{W}^{+} \models \varepsilon,$$

where for all $x_1, \ldots, x_n \in W$ and $w \in W'$,

$$\frac{r_1(x_1, ..., x_n)\, N\, w \quad \cdots \quad r_k(x_1, ..., x_n)\, N\, w}{x_1 \circ \cdots \circ x_n\, N\, w}\ (\varepsilon)$$

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Admissibility in $\mathbf{W}_{\mathtt{M}}$

$\mathbf{W}_{\mathtt{M}} \models (\varepsilon)$ is equivalently stated as

$$\frac{ur_1(x_1, ..., x_n)v \in \mathrm{Acc}(\mathtt{M}) \quad \cdots \quad ur_k(x_1, ..., x_n)v \in \mathrm{Acc}(\mathtt{M})}{u \cdot x_1 \cdots x_n \cdot v \in \mathrm{Acc}(\mathtt{M})}\ (\varepsilon)$$

Undecidable word problems

Theorem (Horčík 2015)

Let ε be a simple equation that "always contains a square as a subword" on its RHS. Then $\mathsf{RL} + \varepsilon$ has an undecidable word problem witnessed in its ordered monoid fragment.

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Theorem (Galatos and S.)

Let ε be a *spineless* simple equation. Then $\mathsf{RL} + \varepsilon$ has an undecidable word problem witnessed in its ordered monoid fragment and $\mathsf{CRL} + \varepsilon$ has an undecidable word problem witnessed in its idempotent semiring fragment.

Thank you!