Fragments of residuated lattices axiomatized by simple equations and decidability

Gavin St. John

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Residuated Lattices

A (commutative) **residuated lattice** is an algebraic structure $\mathbf{R} = (R, \lor, \land, \cdot, \backslash, /, 1)$, such that

- (R, \lor, \land) is a lattice
- $(R, \cdot, 1)$ is a (commutative) monoid
- For all $x, y, z \in R$

$$x \cdot y \leq z \iff y \leq x \setminus z \iff x \leq z/y,$$

where \leq is the induced lattice order.

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- (C)RL denotes the **variety** of (commutative) residuated lattices.
- multiplication is order preserving:

$$x \leq y \implies uxv \leq uyv$$

• multiplication distributes of join:

$$x(y \lor z) = xy \lor xz \quad \& \quad (y \lor z)x = yx \lor zx$$

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Residuated structures are the algebraic semantics of substructural logics (i.e., axiomatic extension of the **Full Lambek Calculus**) **FL**.

$$\begin{array}{ccc} \frac{\Gamma \Rightarrow \alpha & \Delta_{1}, \alpha, \Delta_{2} \Rightarrow \Pi}{\Delta_{1}, \Gamma, \Delta_{2} \Rightarrow \Pi} & (\mathrm{cut}) & \overline{\alpha \Rightarrow \alpha} & (\mathrm{init}) & \overline{\Rightarrow 1} & (1r) \\ \\ \frac{\Gamma_{1}, \alpha, \beta, \Gamma_{2} \Rightarrow \Pi}{\Gamma_{1}, \alpha \cdot \beta, \Gamma_{2} \Rightarrow \Pi} & (\cdot l) & \frac{\Gamma \Rightarrow \alpha & \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta} & (\cdot r) & \frac{\Gamma_{1}, \Gamma_{2} \Rightarrow \Pi}{\Gamma_{1}, 1, \Gamma_{2} \Rightarrow \Pi} & (1l) \\ \\ \frac{\Gamma \Rightarrow \alpha & \Delta_{1}, \beta, \Delta_{2} \Rightarrow \Pi}{\Delta_{1}, \Gamma, \alpha \setminus \beta, \Delta_{2} \Rightarrow \Pi} & (\backslash l) & \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \setminus \beta} & (\backslash r) & \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0} & (0r) \\ \\ \frac{\Gamma \Rightarrow \alpha & \Delta_{1}, \beta, \Delta_{2} \Rightarrow \Pi}{\Delta_{1}, \beta / \alpha, \Gamma, \Delta_{2} \Rightarrow \Pi} & (/l) & \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \setminus \beta} & (/r) & \frac{\sigma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha} & (0l) \\ \\ \frac{\Gamma_{1}, \alpha, \Gamma_{2} \Rightarrow \Pi}{\Gamma_{1}, \alpha \vee \beta, \Gamma_{2} \Rightarrow \Pi} & (\lor l) & \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} & (\lor r_{1}) & \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} & (\lor r_{2}) \\ \\ \frac{\Gamma_{1}, \beta, \Gamma_{2} \Rightarrow \Pi}{\Gamma_{1}, \alpha \wedge \beta, \Gamma_{2} \Rightarrow \Pi} & (\land l_{1}) & \frac{\Gamma_{1}, \alpha, \Gamma_{2} \Rightarrow \Pi}{\Gamma_{1}, \alpha \wedge \beta, \Gamma_{2} \Rightarrow \Pi} & (\land l_{2}) & \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \wedge \beta} & (\land r_{2}) \end{array}$$

Structural rules have an algebraic meaning.

$$\begin{array}{ll} \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Pi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \Pi} \ (e) & \Leftrightarrow & xy \leq yx \\ \\ \frac{\Gamma, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi} \ (w) & \Leftrightarrow & x \leq 1 \\ \\ \frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi} \ (c) & \Leftrightarrow & x \leq x^2 \end{array}$$

We can use algebraic methods to answer questions about the logics.

(Quasi-) Equational Theory

A **quasi-equation** ξ is a universally-quantified formula

$$s_1 = t_1 \& \cdots \& s_n = t_n \implies s_0 = t_0,$$

where $s_0, t_0, s_1, t_1, \ldots, s_n, t_n \in T(X)$ are terms.

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The equational theory for \mathcal{V} is the set of equations that it satisfies

$$\{s = t : \mathcal{V} \models s = t\}$$

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Or equivalently, there is a finitely presented algebra $\mathbf{A} \in \mathcal{V}$ generated by X such that the following set is undecidable:

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• undecidable word problem \Rightarrow undecidable q.e. theory.

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 - $\begin{array}{lll} \{\leq,\cdot,1\} &: & \mbox{Ordered Monoid Fragment} \\ \{\vee,\cdot,1\} &: & \mbox{Idempotent Semiring (ISR) Fragment.} \end{array}$

Overview of Decidability Results

| \mathcal{V} | Eq. Th. | WP |
|---------------------------|---------|---------------------------|
| RL | FMP | Und. $\{\leq, \cdot, 1\}$ |
| $RL + x \le x^2$ | Und. | Und. $\{\leq, \cdot, 1\}$ |
| $RL + x \leq x^2 \vee 1$ | ? | Und. $\{\leq, \cdot, 1\}$ |
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| $RL + yx \leq xyx \lor y$ | ? | ? |

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| $RL + yx \le xyx \lor y$ | ? | ? |
| CRL | FMP | Und. $\{\lor, \cdot, 1\}$ |
| $CRL + x^m \le x^n$ | FMP | FEP |
| $CRL + x \leq x^2 \vee x^3$ | Und. | Und. $\{\lor, \cdot, 1\}$ |
| $CRL + x \leq x^2 \vee x^3 \vee 1$ | ? | Und. $\{\lor, \cdot, 1\}$ |
| $CRL + x \le x^2 \lor 1$ | ? | ? |

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k-Counter Machine

A $k\text{-}\mathsf{C}\mathsf{M}$ is a tuple $\mathtt{M}=(\mathtt{R}_k,\mathtt{Q},\mathtt{P},q_f)$ where,

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As we will see, they can be encoded using a some *string rewriting system*.

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Theorem (Minsky)

There exists a 2-CM whose set of accepted configurations is undecidable.

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$\mathsf{Example:}\ \mathtt{M}_{\mathrm{even}}$

$$M_{\text{even}} = (R_2, Q_{\text{even}}, P_{\text{even}}, q_f)$$
 where

$$Q_{\text{even}} = \{q_0, q_1, q_2, q_f\}$$
 & $P_{\text{even}} = \{p_0, p_1, p_2, p_f\}$

Example: M_{even}

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- p_f : If in state q_2 and register r_2 is empty, transition to the final state q_f .

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How should we do this?

The relation \leq_{M}

Let $\operatorname{Stp}_k = \{S_0, S_1, ..., S_k\}$

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 \blacktriangleright Increment r_i

$$p: q \mathbb{S}_i \leq q' \mathbf{r}_i \mathbb{S}_i$$

Let
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lncrement r_i

$$p: q \mathbb{S}_i \leq q' \mathbf{r}_i \mathbb{S}_i$$

Decrement r_i

 $p:q\mathbf{r}_i\mathbf{S}_i \leq q'\mathbf{S}_i$

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Zero-test r_i

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Let $\leq_{\mathbb{M}}$ be the least compatible preorder generated by P and the finite sets $\{qx \leq xq : q \in \mathbb{Q} \& x \in \mathbb{R}_k \cup \operatorname{Stp}_k\}$ and $\{xq \leq qx : q \in \mathbb{Q} \& x \in \mathbb{R}_k \cup \operatorname{Stp}_k\}$.

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$$\frac{s \leq^p t}{usv \leq^p utv}$$

The M_{even} machine

Let
$$M_{even} = (R_2, Q_{even}, P_{even}, q_f)$$
 be a 2-CM where
 $\mathbf{Q}_{even} = \{q_0, q_1, q_2, q_f\}$
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$$q_0 S_0 \mathbf{r}_1^n S_1 \mathbf{r}_2^m S_2 \in Acc(\mathbb{M}_{even})$$
 iff n is even and $m = 0$.

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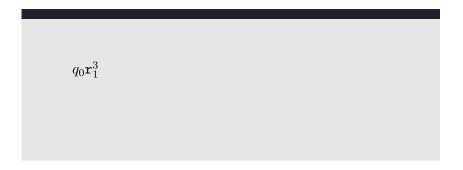
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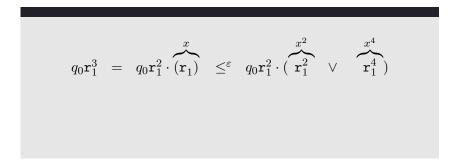
▶ $x \le x^2 \lor x^4$ is admissible in M if $ux^2v, ux^4v \in Acc(M) \implies uxv \in Acc(M).$

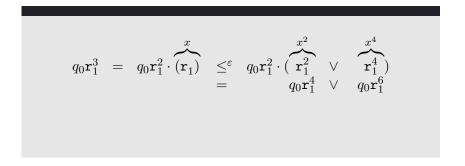
$\mathbb{M}_{\mathrm{even}}$ and $\varepsilon: x \leq x^2 \lor x^4$

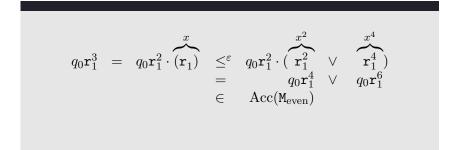


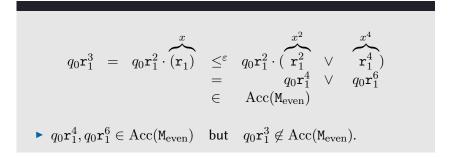
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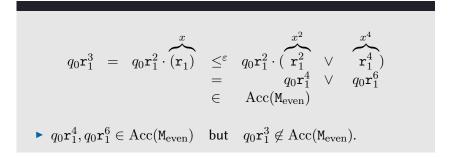
$$q_0 \mathbf{r}_1^3 = q_0 \mathbf{r}_1^2 \cdot \overbrace{(\mathbf{r}_1)}^x$$











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And-branching Counter Machines

An ACM is a machine $M = (R_k, Q, P, q_f)$ is a CM containing no zero-test instruction but allows **branching instructions**

$$q \le q' \lor q''.$$

Acceptance and Quasi-equations

Let $\mathtt{M} = (\mathtt{R}_k, \mathtt{Q}, \mathtt{P}, q_f)$ be a counter machine.

$$\blacktriangleright \text{ Let } P_{\text{com}} = P \cup \{qx = xq : q \in Q, x \in R_k \cup \text{Stp}_k\}$$

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- \blacktriangleright For a monoid term u, we define the quasi-equation $\operatorname{acc}_{\mathtt{M}}(u)$ to be

$$\& \mathsf{P}_{\mathrm{com}} \implies u \leq \mathsf{C}_f$$

where for CM's $C_f = q_f S_0 S_1 \cdots S_k$ and ACM's $C_f = q_f$.

Lemma

If u is accepted in M then $\mathsf{RL} \models \operatorname{acc}_{\mathsf{M}}(u)$

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Lemma

If u is accepted in M then $\mathsf{RL} \models \operatorname{acc}_{\mathsf{M}}(u)$

Completeness is achieved by constructing a counter-model using the theory of **residuated frames**.

Residuated frames

Definition

A residuated frame is a structure $\mathbf{W} = (W, W', N, \circ, \mathbb{N}, //, 1)$, s.t.

- $(W, \circ, 1)$ is a monoid and W' is a set.
- $\blacktriangleright N \subseteq W \times W',$
- $\blacktriangleright \ \|: W \times W' \to W' \text{ and } /\!\!/ : W' \times W \to W' \text{ such that}$
- ▶ N is **nuclear**, i.e. for all $u, v \in W$ and $w \in W'$, $(u \circ v) N w$ iff u N (w / v) iff $v N (u \ w)$.

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$$\wp(W) \underset{\triangleleft}{\stackrel{\triangleright}{\underset{\triangleleft}{\leftarrow}}} \wp(W'): \quad X^{\triangleright} = \{y \in W' : X N y\}$$
$$Y^{\triangleleft} = \{x \in W : x N Y\}$$

- $({}^{\triangleright},{}^{\triangleleft})$ is a Galois connection.
- The map $X \xrightarrow{\gamma_N} X^{\triangleright \triangleleft}$ is a closure operator on $\mathcal{P}(W)$.
- N is nuclear iff γ_N is a nucleus.

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Theorem [Galatos & Jipsen 2013]

 $\mathbf{W}^+ := (\gamma_N[\mathcal{P}(W)], \cup_{\gamma_N}, \cap, \circ_{\gamma_N}, \backslash, /, \gamma_N(\{1\})),$

$$X \cup_{\gamma_N} Y = \gamma_N(X \cup Y)$$
 and $X \circ_{\gamma_N} Y = \gamma_N(X \circ Y)$,

is a residuated lattice.

Completeness of Encoding

Let $M = (R_k, Q, P, q_f)$ be a counter machine.

$$\blacktriangleright \ W_{\mathtt{M}} = (\mathtt{Q} \cup \mathtt{R}_k \cup \mathtt{Stp}_k)^*$$

$$\blacktriangleright W'_{\mathsf{M}} = W \times W$$

► $x N_{\mathsf{M}} (u, v) \iff uxv \in \operatorname{Acc}(\mathsf{M})$

Theorem

 $\mathbf{W}_{\mathtt{M}}$ is a residuated frame.

Proof.

$$\begin{array}{rcl} xy \, N_{\,\mathrm{M}} \ (u,v) & \Longleftrightarrow & uxyv \in \mathrm{Acc}(\mathrm{M}) \ \Longleftrightarrow & x \, N_{\,\mathrm{M}} \ (u,yv) \\ & \Longleftrightarrow & y \, N_{\,\mathrm{M}} \ (ux,v) \end{array}$$

Let $M = (R_k, Q, P, q_f)$ be a counter machine.

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Theorem

For a variety $\mathcal{V} \subseteq \mathsf{RL}$, if $\mathbf{W}^+_{\mathtt{M}} \in \mathcal{V}$ then for all u

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Corollary

If $\mathbf{W}_{\mathtt{M}}^{+} \in \mathcal{V}$, then the word problem for \mathcal{V} in the $\{\leq, \cdot, 1\}$ -fragment is at least as hard as acceptance in \mathtt{M} . In particular, if \mathtt{M} has an undecidable set of accepted configurations, then the word problem for \mathcal{V} is undecidable.

In RL:

► Every equation s = t over the signature {∨, ·, 1} can be written as the conjunction of *basic (in)equations* of the form

 $w \leq v_1 \vee \cdots \vee v_k,$

where $w, v_1, ..., v_k$ are monoid terms over a set of variables X.

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These basic equations can be **linearized** in a uniform way producing an equivalent *simple equation* of the form

$$[\mathbf{R}]: x_1 x_2 \cdots x_n \le \bigvee_{r \in \mathbf{R}} r$$

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is equivalent to, via the substitution $\sigma: u \xrightarrow{\sigma} x \lor y$ and $v \xrightarrow{\sigma} z$,

 $(\forall x)(\forall y)(\forall z) \; xyz \leq x^3 \lor x^2y \lor xy^2 \lor y^3 \lor xz \lor yz$

Simple Equations and Simple Rules

Any simple equation $[\mathrm{R}]$ corresponds to a simple structural rule $(\mathrm{R}).$ For example

$$[\mathbf{R}]: xy \le x^2 \lor y \iff \frac{\Delta_1, \Gamma, \Gamma, \Delta_2 \Rightarrow \Pi \quad \Delta_1, \Psi, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \Psi, \Delta_2 \Rightarrow \Pi} \ (\mathbf{R})$$

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In general,

$$[\mathbf{R}]: x_1 \cdots x_n \le \bigvee_{r \in \mathbf{R}} r \iff \frac{\{\Delta_1, r^{\mathbf{FL}}(\Gamma_1, \dots, \Gamma_n), \Delta_2 \Rightarrow \Pi\}_{r \in \mathbf{R}}}{\Delta_1, \Gamma_1, \dots, \Gamma_n, \Delta_2 \Rightarrow \Pi}$$
(R)

Theorem [Galatos & Jipsen 2013]

Extensions of \mathbf{FL} by simple rules enjoy **cut-elimination**.

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Fragments of residuated lattices axiomatized by simple equations and decidability 25 / 28

Simple equations and Residuated Frames

Lemma [Galatos & Jipsen 2013]

All simple equations ε are preserved by $(-)^+$:

$$\mathbf{W} \models (\varepsilon) \text{ iff } \mathbf{W}^+ \models \varepsilon,$$

where for all $x_1, \ldots, x_n \in W$ and $w \in W'$,

$$\frac{r_1(x_1,...,x_n) N w \cdots r_k(x_1,...,x_n) N w}{x_1 \circ \cdots \circ x_n N w} (\varepsilon)$$

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$$\frac{r_1(x_1,...,x_n) N w \cdots r_k(x_1,...,x_n) N w}{x_1 \circ \cdots \circ x_n N w} (\varepsilon)$$

Admissibility in $\mathbf{W}_{\mathtt{M}}$

 $\mathbf{W}_{\mathtt{M}} \models (\varepsilon) \text{ is equivalently stated as}$ $\frac{ur_1(x_1, ..., x_n)v \in \operatorname{Acc}(\mathtt{M}) \quad \cdots \quad ur_k(x_1, ..., x_n)v \in \operatorname{Acc}(\mathtt{M})}{u \cdot x_1 \cdots x_n \cdot v \in \operatorname{Acc}(\mathtt{M})} (\varepsilon)$

Theorem (Horčík 2015)

Let ε be a simple equation that "always contains a square as a subword" on its RHS. Then RL + ε has an undecidable word problem witnessed in its ordered monoid fragment.

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Theorem (Galatos and S.)

Let ε be a *spineless* simple equation. Then $RL + \varepsilon$ has an undecidable word problem witnessed in its ordered monoid fragment and $CRL + \varepsilon$ has an undecidable word problem witnessed in its idempotent semiring fragment.

Thank you!