The structure of Boolean commutative idempotent residuated lattices

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Lattices with a $\leq$-preserving idempotent binary operation

Let $(A, \wedge, \vee, \cdot)$ be a lattice with an **order-preserving idempotent binary operation** $\cdot$, i.e.,

$$xx = x, \quad \text{and} \quad x \leq y \implies xz \leq yz \text{ and } zx \leq zy.$$

Then $x \wedge y = (x \wedge y)(x \wedge y) \leq xy \leq (x \vee y)(x \vee y) = x \vee y$.

So idempotence implies that xy is in the interval $[x \wedge y, x \vee y]$.

Atomic Boolean algebras with an idempotent operator

Let $\mathbf{B} = (B, \wedge, \vee, \neg, \bot, \top, \cdot)$ be a Boolean algebra (BA) with an **idempotent normal binary operator** $\cdot$, i.e.,

$$xx = x, \quad x\bot = \bot = \bot x, \quad x(y \vee z) = xy \vee xz \quad \text{and} \quad (x \vee y)z = xz \vee yz.$$

$a \in B$ is an **atom** if a is a cover of $\bot$.

The set of atoms of $\mathbf{B}$ is denoted $\mathrm{At}(\mathbf{B})$.

$\mathbf{B}$ is **atomic** if for every $x \in B$ there exists an atom $a \leq x$,
or equivalently, if $\mathrm{At}(\mathbf{B})$ is **join-dense**, i.e., every $x \in B$ is a join of atoms.

Lemma

In an atomic Boolean algebra with an idempotent binary operator

$$ab \in \{\bot, a, b, a \vee b\} \quad \textit{for all } a, b \in At(\mathbf{B}).$$

Atomic BAs with normal operator and ternary relations

Lemma

1. *Let* $\mathbf{B}$ *be an atomic Boolean algebra with a normal binary operator,* $A = At(\mathbf{B})$ *and define a ternary relation* $R_{\mathbf{B}} \subseteq A^3$ *by* $R_{\mathbf{B}}(a, b, c) \iff a \le bc$*. Then for all* $x, y \in B$*,*

$$xy = \bigvee\{a : \exists b \le x \exists c \le y\ R(a, b, c)\}.$$

2. *Suppose* $R \subseteq A^3$ *is a ternary relation on a set* A*, and define* $\mathbf{B}_R = (\mathcal{P}(A), \cup, \cap, \neg, \emptyset, A, \cdot)$ *where for* $Y, Z \in P(A)$

$$Y \cdot Z = \{x : \exists y \in Y \exists z \in Z\ R(x, y, z)\}.$$

   Then $\mathbf{B}_R$ *is a complete atomic Boolean algebra with a normal binary operator.*

3. $R_{\mathbf{B}_R} \cong R$ *and if* $\mathbf{B}$ *is complete then* $\mathbf{B}_{R_{\mathbf{B}}} \cong \mathbf{B}$*.*

Characterizing the relations of idempotent BAs

How to characterize the relations R that arise from an idempotent $\mathbf{B}$?

Lemma

$\mathbf{B}_R$ has an idempotent binary operator if and only if $R(a, a, a)$ and $R(a, b, c) \implies a = b$ or $a = c$.

Proof.

Assume $\mathbf{B}_R$ is idempotent, $a, b, c \in A = At(\mathbf{B}_R)$ be atoms and $a \leq bc$.
By idempotence $bc \leq b \vee c$, so $a \leq b \vee c$.
Since a, b, c are atoms, it follows that $a = b$ or $a = c$.

Now suppose $R(a, a, a)$ and $(R(a, b, c) \Rightarrow a = b$ or $a = c)$ holds for all atoms $a, b, c \in A$. Then for any $x \in B$ we have $x \leq xx$ since $R(a, a, a)$ holds for all atoms $a \leq x$.
Now let a be an atom such that $a \leq xx$. Then $a \leq bc$ for some atoms $b, c \leq x$, therefore $R(a, b, c)$ holds and by assumption $a = b$ or $a = c$.
Hence $a \leq x$ and it follows that $xx = x$. □

Idempotence reduces R to two binary relations

R is said to be **idempotent** if $R(a,a,a)$, and $R(a,b,c) \implies a=b$ or $a=c$

Lemma

*An idempotent ternary relation $R \subseteq A^3$ is definitionally equivalent to a pair of **reflexive** binary relations $P,Q \subseteq A^2$ via the following definitions.*

Defining P,Q from R:

$$\text{(Pdef)} \quad P(x,y) \Leftrightarrow R(x,y,x) \qquad \text{(Qdef)} \quad Q(x,y) \Leftrightarrow R(x,x,y)$$

Defining R from P,Q:

$$\text{(Rdef)} \quad R(x,y,z) \Leftrightarrow (x=y \,\&\, Q(y,z)) \textit{ or } (x=z \,\&\, P(z,y)).$$

Commutativite idempotent R reduce to digraphs

R is said to be **commutative** if $\mathbf{B}_R$ satisfies $xy = yx$.

Equivalently, R is commutative if $R(a, b, c) \implies R(a, c, b)$.

Lemma

*A commutative idempotent ternary relation $R \subseteq A^3$ is definitionally equivalent to a **reflexive** binary relation $P \subseteq A^2$:*

$$P(x, y) \iff R(x, y, x) \quad (\iff R(x, x, y))$$

$$R(x, y, z) \iff (x = y \ \& \ P(y, z)) \text{ or } (x = z \ \& \ P(z, y)).$$

i.e., an idempotent R is commutative $\iff P = Q$.

Commutative idempotent residuated Boolean algebras

A **residuated Boolean algebra** or *r*-**algebra** $(\mathbf{B}_0, \cdot, \backslash, /)$ is a Boolean algebra $\mathbf{B}_0 = (B, \wedge, \vee, \neg, \bot, \top)$ with three binary operations such that

$$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$

Jónsson-Tarski 1952, Jónsson-Tsinakis 1993: *r*-algebras form a variety, and $\cdot$ is an operator.

Each of $\cdot, \backslash, /$ uniquely determines the other two.

Theorem

1. *Complete and atomic (ca-)r-algebras correspond to ternary relations.*
2. *Idempotent ca-r-algebras correspond to pairs* (P, Q) *of directed graphs.*
3. *Commutative idempotent ca-r-algebras correspond to directed graphs.*

Associativity and multiplicative identity

An ordered algebra is **subassociativity** if it satisfies $(xy)z \leq x(yz)$ and **supassociativite** if it satisfies $(xy)z \geq x(yz)$.

The operation $\cdot$ is **right unital** if for some $e \in B$, $xe = x$.
The operation $\cdot$ is **left unital** if for some $e \in B$, $ex = x$.

Theorem (Maddux 1982)

Let R be a ternary relation on a set A. Then $\mathbf{B}_R$ is

$$\begin{array}{lll} \textit{subassociative} & \Leftrightarrow & (R(u,x,y)\,\&\,R(w,u,z) \Rightarrow \exists v(R(v,y,z)\,\&\,R(w,x,v))) \\ \textit{right unital} & \Leftrightarrow & \exists I \subseteq A(x = y \Leftrightarrow \exists z \in I\, R(x,y,z)) \\ \textit{left unital} & \Leftrightarrow & \exists I \subseteq A(x = z \Leftrightarrow \exists y \in I\, R(x,y,z)) \end{array}$$

Associative idempotent ternary relations

R is called **subassociative** if $\mathbf{B}_R$ satisfies $(xy)z \le x(yz)$.

Theorem

An idempotent ternary relation $R \subseteq A^3$ is subassociative if and only if the corresponding reflexive relations P, Q satisfy

$$(\mathrm{P}_1) \quad P(x,y) \ \& \ P(y,z) \Rightarrow P(x,z) \qquad \textit{P-transitivity}$$
$$(\mathrm{P}_2) \quad Q(x,y) \ \& \ Q(x,z) \Rightarrow Q(y,z) \text{ or } P(z,y)$$
$$(\mathrm{P}_3) \quad P(x,y) \ \& \ Q(y,z) \ \& \ x \neq y \Rightarrow P(x,z)$$

To characterize supassociativity of R (i.e. $(xy)z \ge x(yz)$ for $\mathbf{B}_R$), it suffices to interchange P, Q in these conditions to obtain (P'_1), (P'_2), (P'_3).

Hence R is associative if and only if P, Q satisfy all six conditions.

Associative **commutative** idempotent ternary relations

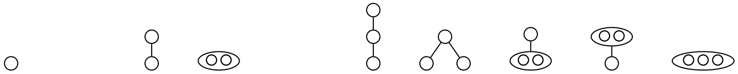
Corollary

An idempotent commutative ternary relation $R \subseteq A^3$ *is associative if and only if the corresponding relation* P *satisfy*

$$\begin{aligned} &(\text{Refl}) && P(x,x) \\ &(\text{Trans}) && P(x,y)\ \&\ P(y,z) \Rightarrow P(x,z) \\ &(\text{Forest}) && P(x,y)\ \&\ P(x,z) \Rightarrow P(y,z) \text{ or } P(z,y) \\ & && \textit{i.e., each branch } \{y : P(x,y)\} \textit{ is a linear preorder.} \end{aligned}$$

Relations with these three properties are called **preorder forests**.

A connected preorder forest is called a **preorder tree** or **labelled tree**.



Boolean algebras with a semilattice operator

A **semilattice operator** on a lattice is a binary operation that is **associative, commutative, idempotent** and distributes over joins in each argument.

Theorem

Complete atomic Boolean algebras with a semilattice operator are in 1-1 correspondence with preorder forests.

For example a discrete poset or antichain $(A, =)$ corresponds to the Boolean algebra of all subsets of A with semilattice operator $xy = x \wedge y = x \cap y$.

A **full** preorder (A, A^2) corresponds to the Boolean algebra of all subsets of A with semilattice operator $xy = x \vee y = x \cup y$ for nonempty x, y and $\emptyset$ otherwise.

Rooted preorder forests

- A preorder forest is a disjoint union of one or more preorder trees.
- A **rooted** preorder tree has an element r such that $P(x, r)$ for all x.
- A **rooted** preorder forest is a disjoint union of rooted preorder trees.
- Finite preorder forests are always rooted

How many preorder forests are there with n elements (up to isomorphism)?

Let $\text{pt}(n)$ $[\text{pf}(n)]$ = number of preorder trees [forests] with n elements.

$n =$	1	2	3	4	5	6	7	8	OEIS
$\text{pt}(n)$	1	2	5	13	37	108	332	1042	A036249
$\text{pf}(n)$	1	3	8	24	71	224	710	2318	A052855

$\text{pt}(n+1) = \text{pt}(n) + \text{pf}(n)$ and $\text{pf}(n) = \text{Euler transform of pt}(n)$

Why is $\mathrm{pt}(n+1) = \mathrm{pt}(n) + \mathrm{pf}(n)$?

Given a preorder forest, we add a single new root (maximum element) to get a preorder tree with $n+1$ elements.

Given a preorder tree, we add another element to the root preorder class to get a preorder tree with $n+1$ elements.

All preorder trees obtained in this way are nonisomorphic.

Given any preorder tree with $n+1$ elements, if it has a single maximal element, remove it to get a preorder forest with n elements,

and if it has several elements in the maximal preorder class, remove one of them to get a preorder tree with n elements.

Euler transform

Given $a_1, a_2, \ldots, a_n$ the Euler transform b_n is calculated by:

$$c_n = \sum_{d|n} d a_d \quad \text{and} \quad b_n = \frac{1}{n}(c_n + \sum_{k=1}^{n-1} c_k b_{n-k})$$

For example, given $a_1 = 1, a_2 = 2, a_3 = 5$ and $b_1 = 1, b_2 = 3$ we calculate

$c_1 = 1 \cdot a_1 = 1$
$c_2 = 1 \cdot a_1 + 2 \cdot a_2 = 1 + 4 = 5$
$c_3 = 1 \cdot a_1 + 3 \cdot a_3 = 1 + 15 = 16$

$b_3 = \frac{1}{3}(c_3 + c_1 b_2 + c_2 b_1) = \frac{1}{3}(16 + 3 + 5) = \frac{24}{3} = 8 = \mathrm{pf}(3)$

Preorder forests with singletons roots

A preorder forest P has **singleton roots** if it is rooted and for all roots r and all x, $P(r, x) \implies r = x$, i.e., each root is a singleton preorder class.

Lemma

Let P be a preorder forest on a set A, with associated ternary relation R, and let I be the union of all root preorder classes.

Then $\mathbf{B}_R$ has I as identity element if and only if P has singleton roots.

Proof.

Suppose $a \neq b$ are in the same root preorder class. Then $ab = a \vee b$ hence $\{a\}I \neq \{a\}$.
Conversely, suppose $\{a\}I \neq \{a\}$ for some $a \in I$. Since $a \leq aa$ it follows that $\{a, b\} \subseteq \{a\}I$ for some $b \in A - \{a\}$. Then $b \leq ai$ for some $i \in I$, and by idempotence $b = i$. Hence $P(b, a)$ holds, and since $b \in I$ the preorder class of a is not a singleton. □

Boolean idempotent residuated lattices

A **residuated Boolean monoid** or *rm*-algebra is an associative unital *r*-algebra. They are **residuated lattices with a Boolean lattice reduct**.

Commutative *rm*-algebras are also known as **Boolean bunched implication algebras**.

Theorem

Complete and atomic idempotent commutative rm-algebras are definitionally equivalent to preorder forests with singleton roots.

Hence all finite idempotent commutative *rm*-algebras can be constructed by enumerating preorder forests with singleton roots.

Counting finite preorder forests with singleton roots

A preorder tree with singleton root and n elements is obtained by adding a new root to a preorder forest.

Let $\text{pt}_1(n)$ = number of preorder trees with singleton root and n elements.

$\text{pf}_1(n)$ = number of preorder forests with singleton root and n elements.

$n =$	1	2	3	4	5	6	7	8	OEIS
$\text{pt}(n)$	1	2	5	13	37	108	332	1042	A036249
$\text{pf}(n)$	1	3	8	24	71	224	710	2318	A052855
$\text{pt}_1(n)$	1	1	3	8	24	71	224	710	
$\text{pf}_1(n)$	1	2	5	14	41	127			

Here $\text{pf}_1(n)$ is also the Euler transform of $\text{pt}_1(n)$.

Enumerating finite preorder forests

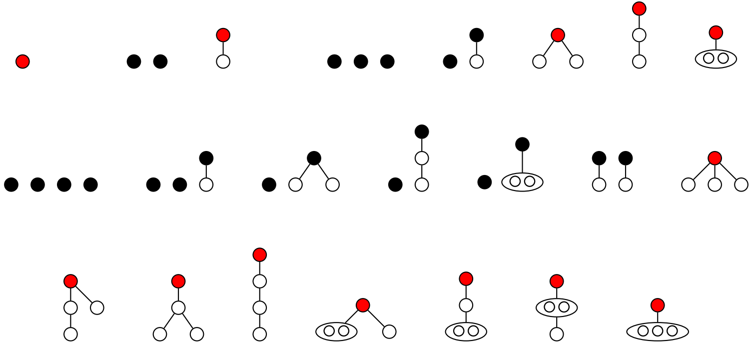


Figure: Preorder forests with singleton roots: 1, 2, 5, 14, 41, 127,. . .

Generalizing to distributive residuated lattices

Replace complete atomic Boolean algebras by **complete perfect distributive lattices**.

Replace the set of atoms by the poset $(J(\mathbf{D}), \leq)$ of completely join-irreducible elements with partial order induced by **D**.

A ternary relation $R(a, b, c)$ is defined on $J(\mathbf{D})$ as before: $a \leq bc$.

This relation is captured by binary relations P, Q if **D** satisfies the formula

$$xy = x \wedge y \text{ or } xy = x \text{ or } xy = y \text{ or } xy = x \vee y \text{ for all } x, y \in J(\mathbf{D})$$

This is a class of idempotent distributive residuated lattices (but not all of them). It includes the class of **conservative residuated lattices**.

Commutative distributive idempotent residuated lattices

For a poset $\mathbf{A} = (A, \leq)$, let $P \subseteq A^2$ be a preorder forest with singleton roots and let R be the associated ternary relation.

Define $\mathbf{D}_R = Dn(\mathbf{A}) = \{{\downarrow}X : X \subseteq A\}$ with operation

$$({\downarrow}X)({\downarrow}Y) = {\downarrow}\{a : a \leq bc \text{ for some } b \in X, c \in Y\}.$$

Then $\mathbf{D}_R$ is a commutative distributive idempotent residuated lattice if and only if R satisfies the down-up-up property:

$$R(x, y, z),\ u \leq x,\ y \leq v \text{ and } z \leq w \implies R(u, v, w).$$

Find a simple characterization of this property in term of P and $\leq$.

Thank you!

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