The structure of Boolean commutative idempotent residuated lattices

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Let \((A, \wedge, \vee, \cdot)\) be a lattice with an order-preserving idempotent binary operation \(\cdot\), i.e.,

\[
xx = x, \quad \text{and} \quad x \leq y \implies xz \leq yz \quad \text{and} \quad zx \leq zy.
\]

Then \(x \wedge y = (x \wedge y)(x \wedge y) \leq xy \leq (x \vee y)(x \vee y) = x \vee y\).

So idempotence implies that \(xy\) is in the interval \([x \wedge y, x \vee y]\).
Atomic Boolean algebras with an idempotent operator

Let $B = (B, \land, \lor, \neg, \perp, \top, \cdot)$ be a Boolean algebra (BA) with an idempotent normal binary operator $\cdot$, i.e.,

\[
x x = x, \quad x \perp = \perp = \perp x, \quad x(y \lor z) = xy \lor xz \quad \text{and} \quad (x \lor y)z = xz \lor yz.
\]

$a \in B$ is an atom if $a$ is a cover of $\perp$.

The set of atoms of $B$ is denoted $\text{At}(B)$.

$B$ is atomic if for every $x \in B$ there exists an atom $a \leq x$, or equivalently, if $\text{At}(B)$ is join-dense, i.e., every $x \in B$ is a join of atoms.

**Lemma**

In an atomic Boolean algebra with an idempotent binary operator

\[
ab \in \{ \perp, a, b, a \lor b \} \quad \text{for all} \quad a, b \in \text{At}(B).
\]
Lemma

1. Let $\mathcal{B}$ be an atomic Boolean algebra with a normal binary operator, $A = \text{At}(\mathcal{B})$ and define a ternary relation $R_B \subseteq A^3$ by $R_B(a, b, c) \iff a \leq bc$. Then for all $x, y \in B$,

$$xy = \bigvee \{a : \exists b \leq x \exists c \leq y R(a, b, c)\}.$$

2. Suppose $R \subseteq A^3$ is a ternary relation on a set $A$, and define $\mathcal{B}_R = (\mathcal{P}(A), \cup, \cap, \neg, \emptyset, A, \cdot)$ where for $Y, Z \in \mathcal{P}(A)$

$$Y \cdot Z = \{x : \exists y \in Y \exists z \in Z R(x, y, z)\}.$$

Then $\mathcal{B}_R$ is a complete atomic Boolean algebra with a normal binary operator.

3. $R_{B_R} \cong R$ and if $\mathcal{B}$ is complete then $\mathcal{B}_{R_B} \cong \mathcal{B}$. 
Characterizing the relations of idempotent BAs

How to characterize the relations $R$ that arise from an idempotent $B$?

**Lemma**

$B_R$ has an idempotent binary operator if and only if $R(a, a, a)$ and $R(a, b, c) \implies a = b$ or $a = c$.

**Proof.**

Assume $B_R$ is idempotent, $a, b, c \in A = \text{At}(B_R)$ be atoms and $a \leq bc$. By idempotence $bc \leq b \lor c$, so $a \leq b \lor c$.

Since $a, b, c$ are atoms, it follows that $a = b$ or $a = c$.

Now suppose $R(a, a, a)$ and $(R(a, b, c) \implies a = b$ or $a = c$) holds for all atoms $a, b, c \in A$. Then for any $x \in B$ we have $x \leq xx$ since $R(a, a, a)$ holds for all atoms $a \leq x$.

Now let $a$ be an atom such that $a \leq xx$. Then $a \leq bc$ for some atoms $b, c \leq x$, therefore $R(a, b, c)$ holds and by assumption $a = b$ or $a = c$.

Hence $a \leq x$ and it follows that $xx = x$. 

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Boolean commutative idempotent RLs

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Idempotence reduces $R$ to two binary relations

$R$ is said to be **idempotent** if $R(a, a, a)$, and $R(a, b, c) \implies a = b$ or $a = c$

**Lemma**

An idempotent ternary relation $R \subseteq A^3$ is definitionally equivalent to a pair of reflexive binary relations $P, Q \subseteq A^2$ via the following definitions.

**Defining $P, Q$ from $R$:**

(Pdef) $P(x, y) \iff R(x, y, x)$  
(Qdef) $Q(x, y) \iff R(x, x, y)$

**Defining $R$ from $P, Q$:**

(Rdef) $R(x, y, z) \iff (x = y \& Q(y, z)) \text{ or } (x = z \& P(z, y))$. 
Commutativity idempotent $R$ reduce to digraphs

$R$ is said to be **commutative** if $B_R$ satisfies $xy = yx$.

Equivalently, $R$ is commutative if $R(a, b, c) \implies R(a, c, b)$.

**Lemma**

A commutative idempotent ternary relation $R \subseteq A^3$ is definitionally equivalent to a **reflexive** binary relation $P \subseteq A^2$:

$$P(x, y) \iff R(x, y, x) \quad (\iff R(x, x, y))$$

$$R(x, y, z) \iff (x = y \& P(y, z)) \text{ or } (x = z \& P(z, y)).$$

i.e., an idempotent $R$ is commutative $\iff P = Q$. 
A residuated Boolean algebra or \( r \)-algebra \((B_0, \cdot, \backslash, /)\) is a Boolean algebra \( B_0 = (B, \land, \lor, \neg, \bot, \top)\) with three binary operations such that

\[
xy \leq z \iff x \leq z/y \iff y \leq x\backslash z.
\]

Jónsson-Tarski 1952, Jónsson-Tsinakis 1993: \( r \)-algebras form a variety, and \( \cdot \) is an operator.

Each of \( \cdot, \backslash, / \) uniquely determines the other two.

**Theorem**

1. Complete and atomic (ca-)\( r \)-algebras correspond to ternary relations.
2. Idempotent ca-\( r \)-algebras correspond to pairs \((P, Q)\) of directed graphs.
3. Commutative idempotent ca-\( r \)-algebras correspond to directed graphs.
An ordered algebra is **subassociativity** if it satisfies \((xy)z \leq x(yz)\) and **supassociativity** if it satisfies \((xy)z \geq x(yz)\).

The operation \(\cdot\) is **right unital** if for some \(e \in B\), \(xe = x\).

The operation \(\cdot\) is **left unital** if for some \(e \in B\), \(ex = x\).

**Theorem (Maddux 1982)**

Let \(R\) be a ternary relation on a set \(A\). Then \(\mathcal{B}_R\) is

- **subassociative** \(\iff\) \((R(u, x, y) \& R(w, u, z) \Rightarrow \exists v (R(v, y, z) \& R(w, x, v)))\)
- **right unital** \(\iff\) \(\exists I \subseteq A(x = y \iff \exists z \in I \ (R(x, y, z)))\)
- **left unital** \(\iff\) \(\exists I \subseteq A(x = z \iff \exists y \in I \ (R(x, y, z)))\)
**Theorem**

An idempotent ternary relation $R \subseteq A^3$ is subassociative if and only if the corresponding reflexive relations $P, Q$ satisfy

1. $(P_1)$ \quad $P(x, y) \land P(y, z) \Rightarrow P(x, z)$ \quad $P$-transitivity
2. $(P_2)$ \quad $Q(x, y) \land Q(x, z) \Rightarrow Q(y, z)$ or $P(z, y)$
3. $(P_3)$ \quad $P(x, y) \land Q(y, z) \land x \neq y \Rightarrow P(x, z)$

To characterize supassociativity of $R$ (i.e. $(xy)z \geq x(yz)$ for $B_R$), it suffices to interchange $P, Q$ in these conditions to obtain $(P_1'), (P_2'), (P_3')$.

Hence $R$ is associative if and only if $P, Q$ satisfy all six conditions.
Corollary

An idempotent commutative ternary relation $R \subseteq A^3$ is associative if and only if the corresponding relation $P$ satisfy

- (Refl) $P(x, x)$
- (Trans) $P(x, y) \& P(y, z) \Rightarrow P(x, z)$
- (Forest) $P(x, y) \& P(x, z) \Rightarrow P(y, z)$ or $P(z, y)$

i.e., each branch \{y : P(x, y)\} is a linear preorder.

Relations with these three properties are called **preorder forests**.

A connected preorder forest is called a **preorder tree** or **labelled tree**.
A semilattice operator on a lattice is a binary operation that is associative, commutative, idempotent and distributes over joins in each argument.

Theorem

Complete atomic Boolean algebras with a semilattice operator are in 1-1 correspondence with preorder forests.

For example a discrete poset or antichain \((A, =)\) corresponds to the Boolean algebra of all subsets of \(A\) with semilattice operator \(xy = x \land y = x \cap y\).

A full preorder \((A, A^2)\) corresponds to the Boolean algebra of all subsets of \(A\) with semilattice operator \(xy = x \lor y = x \cup y\) for nonempty \(x, y\) and \(\emptyset\) otherwise.
Rooted preorder forests

- A preorder forest is a disjoint union of one or more preorder trees.
- A **rooted** preorder tree has an element $r$ such that $P(x, r)$ for all $x$.
- A **rooted** preorder forest is a disjoint union of rooted preorder trees.
- Finite preorder forests are always rooted

How many preorder forests are there with $n$ elements (up to isomorphism)?

Let $pt(n)$ [$pf(n)$] = number of preorder trees [forests] with $n$ elements.

<table>
<thead>
<tr>
<th>$n$ =</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>OEIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$pt(n)$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>13</td>
<td>37</td>
<td>108</td>
<td>332</td>
<td>1042</td>
<td>A036249</td>
</tr>
<tr>
<td>$pf(n)$</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>24</td>
<td>71</td>
<td>224</td>
<td>710</td>
<td>2318</td>
<td>A052855</td>
</tr>
</tbody>
</table>

$pt(n + 1) = pt(n) + pf(n)$ and $pf(n) =$ Euler transform of $pt(n)$
Why is $pt(n+1) = pt(n) + pf(n)$?

Given a preorder forest, we add a single new root (maximum element) to get a preorder tree with $n + 1$ elements.

Given a preorder tree, we add another element to the root preorder class to get a preorder tree with $n + 1$ elements.

All preorder trees obtained in this way are nonisomorphic.

Given any preorder tree with $n + 1$ elements, if it has a single maximal element, remove it to get a preorder forest with $n$ elements,

and if it has several elements in the maximal preorder class, remove one of them to get a preorder tree with $n$ elements.
Euler transform

Given \( a_1, a_2, \ldots, a_n \) the Euler transform \( b_n \) is calculated by:

\[
c_n = \sum_{d \mid n} d a_d \quad \text{and} \quad b_n = \frac{1}{n} \left( c_n + \sum_{k=1}^{n-1} c_k b_{n-k} \right)
\]

For example, given \( a_1 = 1, a_2 = 2, a_3 = 5 \) and \( b_1 = 1, b_2 = 3 \) we calculate

\[
c_1 = 1 \cdot a_1 = 1 \\
c_2 = 1 \cdot a_1 + 2 \cdot a_2 = 1 + 4 = 5 \\
c_3 = 1 \cdot a_1 + 3 \cdot a_3 = 1 + 15 = 16
\]

\[
b_3 = \frac{1}{3} (c_3 + c_1 b_2 + c_2 b_1) = \frac{1}{3} (16 + 3 + 5) = \frac{24}{3} = 8 = \text{pf}(3)
\]
A preorder forest $P$ has **singleton roots** if it is rooted and for all roots $r$ and all $x$, $P(r, x) \iff r = x$, i.e., each root is a singleton preorder class.

**Lemma**

Let $P$ be a preorder forest on a set $A$, with associated ternary relation $R$, and let $I$ be the union of all root preorder classes.

Then $B_R$ has $I$ as identity element if and only if $P$ has singleton roots.

**Proof.**

Suppose $a \neq b$ are in the same root preorder class. Then $ab = a \lor b$ hence \{a\}I \neq \{a\}.

Conversely, suppose \{a\}I \neq \{a\} for some $a \in I$. Since $a \leq aa$ it follows that \{a, b\} \subseteq \{a\}I$ for some $b \in A - \{a\}$. Then $b \leq ai$ for some $i \in I$, and by idempotence $b = i$. Hence $P(b, a)$ holds, and since $b \in I$ the preorder class of $a$ is not a singleton.
A residuated Boolean monoid or \( rm \)-algebra is an associative unital \( r \)-algebra. They are residuated lattices with a Boolean lattice reduct.

Commutative \( rm \)-algebras are also known as Boolean bunched implication algebras.

**Theorem**

*Complete and atomic idempotent commutative \( rm \)-algebras are definitionally equivalent to preorder forests with singleton roots.*

Hence all finite idempotent commutative \( rm \)-algebras can be constructed by enumerating preorder forests with singleton roots.
A preorder tree with singleton root and \( n \) elements is obtained by adding a new root to a preorder forest.

Let \( pt_1(n) = \) number of preorder trees with singleton root and \( n \) elements.

\[
\begin{array}{|c|cccccccc|}
\hline
n = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \text{OEIS} \\
\hline
pt(n) & 1 & 2 & 5 & 13 & 37 & 108 & 332 & 1042 & A036249 \\
pf(n) & 1 & 3 & 8 & 24 & 71 & 224 & 710 & 2318 & A052855 \\
pt_1(n) & 1 & 1 & 3 & 8 & 24 & 71 & 224 & 710 & \\
pf_1(n) & 1 & 2 & 5 & 14 & 41 & 127 & \\
\hline
\end{array}
\]

Here \( pf_1(n) \) is also the Euler transform of \( pt_1(n) \).
Enumerating finite preorder forests

Figure: Preorder forests with singleton roots: 1, 2, 5, 14, 41, 127, ...
Generalizing to distributive residuated lattices

Replace complete atomic Boolean algebras by **complete perfect distributive lattices**.

Replace the set of atoms by the poset \((J(D), \leq)\) of completely join-irreducible elements with partial order induced by \(D\).

A ternary relation \(R(a, b, c)\) is defined on \(J(D)\) as before: \(a \leq bc\).

This relation is captured by binary relations \(P, Q\) if \(D\) satisfies the formula

\[
xy = x \land y \text{ or } xy = x \text{ or } xy = y \text{ or } xy = x \lor y \text{ for all } x, y \in J(D)
\]

This is a class of idempotent distributive residuated lattices (but not all of them). It includes the class of **conservative residuated lattices**.
For a poset $A = (A, \leq)$, let $P \subseteq A^2$ be a preorder forest with singleton roots and let $R$ be the associated ternary relation.

Define $D_R = Dn(A) = \{ \downarrow X : X \subseteq A \}$ with operation

$$(\downarrow X)(\downarrow Y) = \downarrow \{ a : a \leq bc \text{ for some } b \in X, c \in Y \}.$$

Then $D_R$ is a commutative distributive idempotent residuated lattice if and only if $R$ satisfies the down-up-up-up property:

$$R(x, y, z), \ u \leq x, \ y \leq v \text{ and } z \leq w \implies R(u, v, w).$$

Find a simple characterization of this property in term of $P$ and $\leq$.

Thank you!


