Inverse-free subreducts of lattice-ordered groups

Nick Galatos
University of Denver
(joint work with George Metcalfe and Almudena Colacito)

March, 2020

Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order.

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$,

Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, {}^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, {}^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$, either lexicographically

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$, either lexicographically or coorinatewise

Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$, either lexicographically or coorinatewise
- The order-bijections $\mathbf{Aut}(C, \leq)$ on a chain $(C, \leq)$.

Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$, either lexicographically or coorinatewise
- The order-bijections $\mathbf{Aut}(C, \leq)$ on a chain $(C, \leq)$. For example $\mathbf{Aut}(\mathbf{n})$, $\mathbf{Aut}(\mathbb{N})$, $\mathbf{Aut}(\mathbb{Z})$, $\mathbf{Aut}(\mathbb{R})$.

Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$, either lexicographically or coorinatewise
- The order-bijections $\mathbf{Aut}(C, \leq)$ on a chain $(C, \leq)$. For example $\mathbf{Aut}(\mathbf{n})$, $\mathbf{Aut}(\mathbb{N})$, $\mathbf{Aut}(\mathbb{Z})$, $\mathbf{Aut}(\mathbb{R})$.

Note: special case of a residuated lattice.

Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$, either lexicographically or coorinatewise
- The order-bijections $\mathbf{Aut}(C, \leq)$ on a chain $(C, \leq)$. For example $\mathbf{Aut(n)}$, $\mathbf{Aut}(\mathbb{N})$, $\mathbf{Aut}(\mathbb{Z})$, $\mathbf{Aut}(\mathbb{R})$.

Note: special case of a residuated lattice.

Fact (The lattice reducts of) ℓ-groups are distributive.

Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$, either lexicographically or coorinatewise
- The order-bijections $\mathbf{Aut}(C, \leq)$ on a chain $(C, \leq)$. For example $\mathbf{Aut}(\mathbf{n})$, $\mathbf{Aut}(\mathbb{N})$, $\mathbf{Aut}(\mathbb{Z})$, $\mathbf{Aut}(\mathbb{R})$.

Note: special case of a residuated lattice.

Fact (The lattice reducts of) ℓ-groups are distributive. Also, the De Morgan laws hold.

Lattice-ordered groups

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$, either lexicographically or coorinatewise
- The order-bijections $\mathbf{Aut}(C, \leq)$ on a chain $(C, \leq)$. For example $\mathbf{Aut}(\mathbf{n})$, $\mathbf{Aut}(\mathbb{N})$, $\mathbf{Aut}(\mathbb{Z})$, $\mathbf{Aut}(\mathbb{R})$.

Note: special case of a residuated lattice.

Fact (The lattice reducts of) ℓ-groups are distributive. Also, the De Morgan laws hold.

Holland's embedding theorem Every ℓ-group can be embedded in $\mathbf{Aut}(\mathbf{C})$, for some chain $\mathbf{C}$.

Subvarieties and decidability

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

Subvarieties and decidability

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

The variety of *representable* ℓ-groups (subdirect products of totally ordered ones) is properly between abelian and the whole variety.

Subvarieties and decidability

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

The variety of *representable* ℓ-groups (subdirect products of totally ordered ones) is properly between abelian and the whole variety. It is axiomatizeed by $yx \leq xyx \vee y$ and the decidability of the equational theory remains unknown.

Subvarieties and decidability

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

The variety of *representable* ℓ-groups (subdirect products of totally ordered ones) is properly between abelian and the whole variety. It is axiomatizeed by $yx \leq xyx \vee y$ and the decidability of the equational theory remains unknown.

Holland's generation theorem The variety of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$ (also by $\mathbf{Aut}(\mathbb{Q})$).

Subvarieties and decidability

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

The variety of *representable* ℓ-groups (subdirect products of totally ordered ones) is properly between abelian and the whole variety. It is axiomatizeed by $yx \leq xyx \vee y$ and the decidability of the equational theory remains unknown.

Holland's generation theorem The variety of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$ (also by $\mathbf{Aut}(\mathbb{Q})$).

Theorem (Holland - McCleary) The equational theory of ℓ-groups is decidable.

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

The variety of *representable* ℓ-groups (subdirect products of totally ordered ones) is properly between abelian and the whole variety. It is axiomatizeed by $yx \leq xyx \vee y$ and the decidability of the equational theory remains unknown.

Holland's generation theorem The variety of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$ (also by $\mathbf{Aut}(\mathbb{Q})$).

Theorem (Holland - McCleary) The equational theory of ℓ-groups is decidable. (Implemented online by P. Jipsen.)

Subvarieties and decidability

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

The variety of *representable* ℓ-groups (subdirect products of totally ordered ones) is properly between abelian and the whole variety. It is axiomatizeed by $yx \leq xyx \vee y$ and the decidability of the equational theory remains unknown.

Holland's generation theorem The variety of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$ (also by $\mathbf{Aut}(\mathbb{Q})$).

Theorem (Holland - McCleary) The equational theory of ℓ-groups is decidable. (Implemented online by P. Jipsen.)

If the equation is false then a finite partial description (a *diagram*) of an infinite counterexample is provided by the algorithm.

Subvarieties and decidability

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

The variety of *representable* ℓ-groups (subdirect products of totally ordered ones) is properly between abelian and the whole variety. It is axiomatizeed by $yx \leq xyx \vee y$ and the decidability of the equational theory remains unknown.

Holland's generation theorem The variety of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$ (also by $\mathbf{Aut}(\mathbb{Q})$).

Theorem (Holland - McCleary) The equational theory of ℓ-groups is decidable. (Implemented online by P. Jipsen.)

If the equation is false then a finite partial description (a *diagram*) of an infinite counterexample is provided by the algorithm. If it is true, the termination of the diagram search certifies that it is false.

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

The variety of *representable* ℓ-groups (subdirect products of totally ordered ones) is properly between abelian and the whole variety. It is axiomatizeed by $yx \leq xyx \vee y$ and the decidability of the equational theory remains unknown.

Holland's generation theorem The variety of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$ (also by $\mathbf{Aut}(\mathbb{Q})$).

Theorem (Holland - McCleary) The equational theory of ℓ-groups is decidable. (Implemented online by P. Jipsen.)

If the equation is false then a finite partial description (a *diagram*) of an infinite counterexample is provided by the algorithm. If it is true, the termination of the diagram search certifies that it is false.

Fact It is enough to decide equations of the form $1 \leq g_1 \vee \cdots \vee g_n$, or $g_1 \wedge \cdots \wedge g_n \leq 1$, where $g_1, \ldots, g_n$ are group terms.

The following implications/quasiequations/inference rules hold in ℓ-groups

$$\frac{s \wedge g \leq 1 \quad s \wedge k \leq 1}{s \wedge gk \leq 1}\ (\textsc{mix}) \qquad \frac{s \wedge gh \leq 1}{s \wedge g \wedge h \leq 1}\ (\textsc{split})$$

$$\frac{s \wedge gk \leq 1}{s \wedge ghh^{-1}k \leq 1}\ (\textsc{simp}) \qquad \frac{s \wedge gk \leq 1 \quad s \wedge nh \leq 1}{s \wedge gh \wedge nk \leq 1}\ (\textsc{com})$$

The following implications/quasiequations/inference rules hold in ℓ-groups

$$\frac{s \wedge g \leq 1 \quad s \wedge k \leq 1}{s \wedge gk \leq 1}\ (\textsc{mix}) \qquad \frac{s \wedge gh \leq 1}{s \wedge g \wedge h \leq 1}\ (\textsc{split})$$

$$\frac{s \wedge gk \leq 1}{s \wedge ghh^{-1}k \leq 1}\ (\textsc{simp}) \qquad \frac{s \wedge gk \leq 1 \quad s \wedge nh \leq 1}{s \wedge gh \wedge nk \leq 1}\ (\textsc{com})$$

The system $\mathrm{G}\ell$ consists of the axioms and rules:

$$\frac{g \text{ gp. valid}}{s \wedge g \leq 1}\ (\textsc{gv}) \qquad \frac{}{s \wedge h \wedge h^{-1} \leq 1}\ (\textsc{em})$$

$$\frac{s \wedge gh \leq 1 \quad s \wedge h^{-1}k \leq 1}{s \wedge gk \leq 1}\ (\textsc{cut}) \qquad \frac{s \leq 1}{s \wedge t \leq 1}\ (\textsc{ew})$$

Note that $(\textsc{mix})$ is an instance of $(\textsc{cut})$.

The following implications/quasiequations/inference rules hold in ℓ-groups

$$\frac{s \wedge g \leq 1 \quad s \wedge k \leq 1}{s \wedge gk \leq 1}\ (\textsc{mix}) \qquad \frac{s \wedge gh \leq 1}{s \wedge g \wedge h \leq 1}\ (\textsc{split})$$

$$\frac{s \wedge gk \leq 1}{s \wedge ghh^{-1}k \leq 1}\ (\textsc{simp}) \qquad \frac{s \wedge gk \leq 1 \quad s \wedge nh \leq 1}{s \wedge gh \wedge nk \leq 1}\ (\textsc{com})$$

The system $\mathrm{G}\ell$ consists of the axioms and rules:

$$\frac{g \text{ gp. valid}}{s \wedge g \leq 1}\ (\textsc{gv}) \qquad \frac{}{s \wedge h \wedge h^{-1} \leq 1}\ (\textsc{em})$$

$$\frac{s \wedge gh \leq 1 \quad s \wedge h^{-1}k \leq 1}{s \wedge gk \leq 1}\ (\textsc{cut}) \qquad \frac{s \leq 1}{s \wedge t \leq 1}\ (\textsc{ew})$$

Note that $(\textsc{mix})$ is an instance of $(\textsc{cut})$. Also the other three rules follow from $\mathrm{G}\ell$.

Derivable rules

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xxx^{-1}x^{-1} \wedge yy \leq 1}\ (\textsc{gv})}{xxx^{-1} \wedge yy \wedge x^{-1} \leq 1}\ (\textsc{split})}{xx \wedge yy \wedge x^{-1} \leq 1}\ (\textsc{split})}{xx \wedge yy \wedge x^{-1}y^{-1}y \leq 1}\ (\textsc{simp}) \quad \dfrac{\dfrac{\dfrac{}{xx \wedge yyy^{-1}y^{-1} \leq 1}\ (\textsc{gv})}{xx \wedge yyy^{-1} \wedge y^{-1} \leq 1}\ (\textsc{split})}{xx \wedge yy \wedge y^{-1} \leq 1}\ (\textsc{split})}{xx \wedge yy \wedge x^{-1}y^{-1} \leq 1}\ (\textsc{cut})$$

For $(\textsc{split})$, $(\textsc{simp})$ and $(\textsc{com})$ we have:

$$\dfrac{\dfrac{s \wedge gh}{s \wedge gh \wedge h}\ (\textsc{ew}) \quad \dfrac{}{s \wedge h^{-1} \wedge h}\ (\textsc{em})}{s \wedge g \wedge h}\ (\textsc{cut}) \qquad \dfrac{s \wedge gk \quad \dfrac{}{s \wedge k^{-1}hh^{-1}k}\ (\textsc{gv})}{s \wedge ghh^{-1}k}\ (\textsc{cut})$$

Derivable rules

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xxx^{-1}x^{-1} \wedge yy \le 1}\,(\mathrm{GV})}{xxx^{-1} \wedge yy \wedge x^{-1} \le 1}\,(\mathrm{SPLIT})}{xx \wedge yy \wedge x^{-1} \le 1}\,(\mathrm{SPLIT})}{xx \wedge yy \wedge x^{-1}y^{-1}y \le 1}\,(\mathrm{SIMP}) \qquad \dfrac{\dfrac{\dfrac{}{xx \wedge yyy^{-1}y^{-1} \le 1}\,(\mathrm{GV})}{xx \wedge yyy^{-1} \wedge y^{-1} \le 1}\,(\mathrm{SPLIT})}{xx \wedge yy \wedge y^{-1} \le 1}\,(\mathrm{SPLIT})}{xx \wedge yy \wedge x^{-1}y^{-1} \le 1}\,(\mathrm{CUT})$$

For (SPLIT), (SIMP) and (COM) we have:

$$\dfrac{\dfrac{s \wedge gh}{s \wedge gh \wedge h}\,(\mathrm{EW}) \qquad \dfrac{}{s \wedge h^{-1} \wedge h}\,(\mathrm{EM})}{s \wedge g \wedge h}\,(\mathrm{CUT}) \qquad\qquad \dfrac{s \wedge gk \qquad \dfrac{}{s \wedge k^{-1}hh^{-1}k}\,(\mathrm{GV})}{s \wedge ghh^{-1}k}\,(\mathrm{CUT})$$

$$\dfrac{\dfrac{\dfrac{s \wedge nh \le 1}{s \wedge gh \wedge nh \le 1}\,(\mathrm{EW})}{s \wedge gh \wedge nkk^{-1}h \le 1}\,(\mathrm{SIMP}) \qquad \dfrac{\dfrac{s \wedge gk \le 1}{s \wedge ghh^{-1}k \le 1}\,(\mathrm{SIMP})}{s \wedge gh \wedge h^{-1}k \le 1}\,(\mathrm{SPLIT})}{s \wedge gh \wedge nk \le 1}\,(\mathrm{CUT})$$

A decidable system

Theorem (G. - Metcalfe) The system $\mathrm{G}\ell$ provides an axiomatization for ℓ-groups.

Theorem (G. - Metcalfe) The system $\mathrm{G}\ell$ provides an axiomatization for ℓ-groups. Also, the following "resolution" rule is admissible.

$$\frac{s \wedge g \leq 1 \quad s \wedge g^{-1} \leq 1}{s \leq 1} \ (\text{RES})$$

where g is not group valid.

A decidable system

Theorem (G. - Metcalfe) The system $\mathrm{G}\ell$ provides an axiomatization for ℓ-groups. Also, the following "resolution" rule is admissible.

$$\frac{s \wedge g \leq 1 \quad s \wedge g^{-1} \leq 1}{s \leq 1}\ (\textsc{res})$$

where g is not group valid.

When exploring (upward) the possible proofs of a given inequality, the choices of the subterms in $(\textsc{cut})$ and in $(\textsc{res})$ can be restricted to a finite set given by the inequality (inspired by the diagrams in Holland's proof).

Theorem (G. - Metcalfe) The system $\mathrm{G}\ell$ provides an axiomatization for ℓ-groups. Also, the following "resolution" rule is admissible.

$$\frac{s \wedge g \leq 1 \quad s \wedge g^{-1} \leq 1}{s \leq 1} \ (\textsc{res})$$

where g is not group valid.

When exploring (upward) the possible proofs of a given inequality, the choices of the subterms in $(\textsc{cut})$ and in $(\textsc{res})$ can be restricted to a finite set given by the inequality (inspired by the diagrams in Holland's proof).

This yields decidability and actually the complexity of the resulting algorithm is co-NP complete.

A decidable system

Theorem (G. - Metcalfe) The system $\mathrm{G}\ell$ provides an axiomatization for ℓ-groups. Also, the following "resolution" rule is admissible.

$$\frac{s \wedge g \leq 1 \quad s \wedge g^{-1} \leq 1}{s \leq 1}\ (\textsc{res})$$

where g is not group valid.

When exploring (upward) the possible proofs of a given inequality, the choices of the subterms in $(\textsc{cut})$ and in $(\textsc{res})$ can be restricted to a finite set given by the inequality (inspired by the diagrams in Holland's proof).

This yields decidability and actually the complexity of the resulting algorithm is co-NP complete.

If the equation is true the derivation can be transformed into an equaltional-logic proof.

A decidable system

Theorem (G. - Metcalfe) The system $\mathrm{G}\ell$ provides an axiomatization for ℓ-groups. Also, the following "resolution" rule is admissible.

$$\frac{s \wedge g \leq 1 \quad s \wedge g^{-1} \leq 1}{s \leq 1}\ (\textsc{res})$$

where g is not group valid.

When exploring (upward) the possible proofs of a given inequality, the choices of the subterms in $(\textsc{cut})$ and in $(\textsc{res})$ can be restricted to a finite set given by the inequality (inspired by the diagrams in Holland's proof).

This yields decidability and actually the complexity of the resulting algorithm is co-NP complete.

If the equation is true the derivation can be transformed into an equaltional-logic proof.

As a by-product, this provides an alternative proof of Holland's generation theorem without using Holland's embedding theorem.

Theorem (G. - Metcalfe) The following is an alternative derivation system for ℓ-groups. Note that in the system no unexpected terms appear when reading the rules upwards.

$$\frac{}{1 \leq 1}\ (\text{EMP}) \qquad \frac{}{xx^{-1} \leq 1}\ (\text{ID}) \qquad \frac{hg \leq 1}{gh \leq 1}\ (\text{CYCLE}) \qquad \frac{s \leq 1}{s \wedge t \leq 1}\ (\text{EW})$$

$$\frac{s \wedge g \leq 1 \quad s \wedge h \leq 1}{s \wedge gh \leq 1}\ (\text{MIX}) \qquad \frac{s \wedge gk \leq 1 \quad s \wedge nh \leq 1}{s \wedge gh \wedge nk \leq 1}\ (\text{COM})$$

$$\frac{s \wedge gth \leq 1 \quad s \wedge gsh \leq 1}{s \wedge g(t \vee s)h \leq 1}\ (\vee) \qquad \frac{s \wedge gth \wedge gsh \leq 1}{s \wedge g(t \wedge s)h \leq 1}\ (\wedge)$$

A cut-free system

Theorem (G. - Metcalfe) The following is an alternative derivation system for ℓ-groups. Note that in the system no unexpected terms appear when reading the rules upwards.

$$\frac{}{1 \leq 1}\ (\textsc{emp}) \qquad \frac{}{xx^{-1} \leq 1}\ (\textsc{id}) \qquad \frac{hg \leq 1}{gh \leq 1}\ (\textsc{cycle}) \qquad \frac{s \leq 1}{s \wedge t \leq 1}\ (\textsc{ew})$$

$$\frac{s \wedge g \leq 1 \quad s \wedge h \leq 1}{s \wedge gh \leq 1}\ (\textsc{mix}) \qquad \frac{s \wedge gk \leq 1 \quad s \wedge nh \leq 1}{s \wedge gh \wedge nk \leq 1}\ (\textsc{com})$$

$$\frac{s \wedge gth \leq 1 \quad s \wedge gsh \leq 1}{s \wedge g(t \vee s)h \leq 1}\ (\vee) \qquad \frac{s \wedge gth \wedge gsh \leq 1}{s \wedge g(t \wedge s)h \leq 1}\ (\wedge)$$

Shortcoming Neither system allows for a good duality theory, as provided by residuated frames **(G. - Jipsen)**.

Theorem (G. - Metcalfe) The following is an alternative derivation system for ℓ-groups. Note that in the system no unexpected terms appear when reading the rules upwards.

$$\frac{}{1 \leq 1}\ (\textsc{emp}) \qquad \frac{}{xx^{-1} \leq 1}\ (\textsc{id}) \qquad \frac{hg \leq 1}{gh \leq 1}\ (\textsc{cycle}) \qquad \frac{s \leq 1}{s \wedge t \leq 1}\ (\textsc{ew})$$

$$\frac{s \wedge g \leq 1 \quad s \wedge h \leq 1}{s \wedge gh \leq 1}\ (\textsc{mix}) \qquad \frac{s \wedge gk \leq 1 \quad s \wedge nh \leq 1}{s \wedge gh \wedge nk \leq 1}\ (\textsc{com})$$

$$\frac{s \wedge gth \leq 1 \quad s \wedge gsh \leq 1}{s \wedge g(t \vee s)h \leq 1}\ (\vee) \qquad \frac{s \wedge gth \wedge gsh \leq 1}{s \wedge g(t \wedge s)h \leq 1}\ (\wedge)$$

Shortcoming Neither system allows for a good duality theory, as provided by residuated frames **(G. - Jipsen)**. This is because the ℓ-group axioms are high in the substructural hierarchy **(G. - Ciabbatoni - Terui)**.

Inverse-free reducts

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Inverse-free reducts

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Theorem (Repnitskii) The inverse-free subreducts of abelian ℓ-groups are a proper subvariety of the *commutative* totally distributive ℓ-monoids. Actually, it is not finitely based.

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Theorem (Repnitskii) The inverse-free subreducts of abelian ℓ-groups are a proper subvariety of the *commutative* totally distributive ℓ-monoids. Actually, it is not finitely based.

Theorem (Colacito - G. - Metcalfe) The inverse-free subreducts of ℓ-groups are exactly the totally distributive ℓ-monoids.

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Theorem (Repnitskii) The inverse-free subreducts of abelian ℓ-groups are a proper subvariety of the *commutative* totally distributive ℓ-monoids. Actually, it is not finitely based.

Theorem (Colacito - G. - Metcalfe) The inverse-free subreducts of ℓ-groups are exactly the totally distributive ℓ-monoids.

Lemma (FEP) The totally distributive ℓ-monoids are generated by $\{\mathbf{End}(\mathbf{C}) : \mathbf{C}$ is a finite chain $\}$.

Inverse-free reducts

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Theorem (Repnitskii) The inverse-free subreducts of abelian ℓ-groups are a proper subvariety of the *commutative* totally distributive ℓ-monoids. Actually, it is not finitely based.

Theorem (Colacito - G. - Metcalfe) The inverse-free subreducts of ℓ-groups are exactly the totally distributive ℓ-monoids.

Lemma (FEP) The totally distributive ℓ-monoids are generated by $\{\mathbf{End}(\mathbf{C}) : \mathbf{C}$ is a finite chain $\}$.

Proof-idea By (Tsinakis/Paoli)/(Anderson-Edwards) if an inverse-free equation $s = t$ fails in tdℓ-monoids, it fails in the endomorphisms $\mathbf{End}(\mathbf{C})$ of a chain $\mathbf{C}$.

Inverse-free reducts

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Theorem (Repnitskii) The inverse-free subreducts of abelian ℓ-groups are a proper subvariety of the *commutative* totally distributive ℓ-monoids. Actually, it is not finitely based.

Theorem (Colacito - G. - Metcalfe) The inverse-free subreducts of ℓ-groups are exactly the totally distributive ℓ-monoids.

Lemma (FEP) The totally distributive ℓ-monoids are generated by $\{\mathbf{End}(\mathbf{C}) : \mathbf{C}$ is a finite chain $\}$.

Proof-idea By (Tsinakis/Paoli)/(Anderson-Edwards) if an inverse-free equation $s = t$ fails in tdℓ-monoids, it fails in the endomorphisms $\mathbf{End}(\mathbf{C})$ of a chain $\mathbf{C}$. So, there are $f_1, \dots, f_n \in End(\mathbf{C})$ such that $s(f_1, \dots, f_n) \neq t(f_1, \dots, f_n)$.

Inverse-free reducts

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Theorem (Repnitskii) The inverse-free subreducts of abelian ℓ-groups are a proper subvariety of the *commutative* totally distributive ℓ-monoids. Actually, it is not finitely based.

Theorem (Colacito - G. - Metcalfe) The inverse-free subreducts of ℓ-groups are exactly the totally distributive ℓ-monoids.

Lemma (FEP) The totally distributive ℓ-monoids are generated by $\{\mathbf{End}(\mathbf{C}) : \mathbf{C} \text{ is a finite chain }\}$.

Proof-idea By (Tsinakis/Paoli)/(Anderson-Edwards) if an inverse-free equation $s = t$ fails in tdℓ-monoids, it fails in the endomorphisms $\mathbf{End}(\mathbf{C})$ of a chain $\mathbf{C}$. So, there are $f_1, \ldots, f_n \in End(\mathbf{C})$ such that $s(f_1, \ldots, f_n) \neq t(f_1, \ldots, f_n)$. So, there is $p \in C$ such that $(p)s(f_1, \ldots, f_n) \neq (p)t(f_1, \ldots, f_n)$.

Inverse-free reducts

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Theorem (Repnitskii) The inverse-free subreducts of abelian ℓ-groups are a proper subvariety of the *commutative* totally distributive ℓ-monoids. Actually, it is not finitely based.

Theorem (Colacito - G. - Metcalfe) The inverse-free subreducts of ℓ-groups are exactly the totally distributive ℓ-monoids.

Lemma (FEP) The totally distributive ℓ-monoids are generated by $\{\mathbf{End}(\mathbf{C}) : \mathbf{C}$ is a finite chain $\}$.

Proof-idea By (Tsinakis/Paoli)/(Anderson-Edwards) if an inverse-free equation $s = t$ fails in tdℓ-monoids, it fails in the endomorphisms $\mathbf{End}(\mathbf{C})$ of a chain $\mathbf{C}$. So, there are $f_1, \ldots, f_n \in End(\mathbf{C})$ such that $s(f_1, \ldots, f_n) \neq t(f_1, \ldots, f_n)$. So, there is $p \in C$ such that $(p)s(f_1, \ldots, f_n) \neq (p)t(f_1, \ldots, f_n)$. Let $\mathbf{C}'$ be the finite subchain of $\mathbf{C}$ containing all the points involved in the evaluation of the failure and f_i''s the partial maps that are the restrictions of the f_i's.

Inverse-free reducts

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Theorem (Repnitskii) The inverse-free subreducts of abelian ℓ-groups are a proper subvariety of the *commutative* totally distributive ℓ-monoids. Actually, it is not finitely based.

Theorem (Colacito - G. - Metcalfe) The inverse-free subreducts of ℓ-groups are exactly the totally distributive ℓ-monoids.

Lemma (FEP) The totally distributive ℓ-monoids are generated by $\{\mathbf{End}(\mathbf{C}) : \mathbf{C}$ is a finite chain $\}$.

Proof-idea By (Tsinakis/Paoli)/(Anderson-Edwards) if an inverse-free equation $s = t$ fails in tdℓ-monoids, it fails in the endomorphisms $\mathbf{End}(\mathbf{C})$ of a chain $\mathbf{C}$. So, there are $f_1, \ldots, f_n \in End(\mathbf{C})$ such that $s(f_1, \ldots, f_n) \neq t(f_1, \ldots, f_n)$. So, there is $p \in C$ such that $(p)s(f_1, \ldots, f_n) \neq (p)t(f_1, \ldots, f_n)$. Let $\mathbf{C}'$ be the finite subchain of $\mathbf{C}$ containing all the points involved in the evaluation of the failure and f_i''s the partial maps that are the restrictions of the f_i's. Then we can extend the f_i''s to $\mathbf{End}(\mathbf{C}')$.

Inverse-free reducts

Lemma If an inverse-free equation fails in an $\mathbf{End}(C)$, where C is a finite chain, then it fails in $\mathbf{Aut_m}(\mathbb{Q})$, the inverse-free reduct of $\mathbf{Aut}(\mathbb{Q})$.

Lemma If an inverse-free equation fails in an $\mathbf{End}(C)$, where C is a finite chain, then it fails in $\mathbf{Aut_m}(\mathbb{Q})$, the inverse-free reduct of $\mathbf{Aut}(\mathbb{Q})$.

Proof-idea If an inverse-free equation $s = t$ fails in $\mathbf{End}(\mathbf{C})$, where C is a finite chain, then there are $f_1, \ldots, f_n \in End(C)$ and $p \in End(C)$ such that $(p)s(f_1, \ldots, f_n) \neq (p)t(f_1, \ldots, f_n)$.

Lemma If an inverse-free equation fails in an $\mathbf{End}(C)$, where C is a finite chain, then it fails in $\mathbf{Aut_m}(\mathbb{Q})$, the inverse-free reduct of $\mathbf{Aut}(\mathbb{Q})$.

Proof-idea If an inverse-free equation $s = t$ fails in $\mathbf{End}(\mathbf{C})$, where C is a finite chain, then there are $f_1, \ldots, f_n \in End(C)$ and $p \in End(C)$ such that $(p)s(f_1, \ldots, f_n) \neq (p)t(f_1, \ldots, f_n)$. We first create a finite chain $\overline{C}$ and partial injections $\overline{f}_1, \ldots, \overline{f}_n$ which still show the failure at some $p \in \overline{C}$. Then using the density of $\mathbb{Q}$ we embed $\overline{C}$ in $\mathbb{Q}$ and extend $\overline{f}_1, \ldots, \overline{f}_n$ to total bijections of $\mathbb{Q}$.

Inverse-free reducts

Lemma If an inverse-free equation fails in an $\mathbf{End}(C)$, where C is a finite chain, then it fails in $\mathbf{Aut_m}(\mathbb{Q})$, the inverse-free reduct of $\mathbf{Aut}(\mathbb{Q})$.

Proof-idea If an inverse-free equation $s = t$ fails in $\mathbf{End}(\mathbf{C})$, where C is a finite chain, then there are $f_1, \ldots, f_n \in End(C)$ and $p \in End(C)$ such that $(p)s(f_1, \ldots, f_n) \neq (p)t(f_1, \ldots, f_n)$. We first create a finite chain $\overline{C}$ and partial injections $\overline{f}_1, \ldots, \overline{f}_n$ which still show the failure at some $p \in \overline{C}$. Then using the density of $\mathbb{Q}$ we embed $\overline{C}$ in $\mathbb{Q}$ and extend $\overline{f}_1, \ldots, \overline{f}_n$ to total bijections of $\mathbb{Q}$.

To create $\overline{C}$ we need to duplicate points $q \in C$ such that $f(p_1) = f(p_2) = q$ for some f, and then extend the functions to the new chain in an iterative way, ensuring that the failure remains. This is complex if done by hand.

Lemma If an inverse-free equation fails in an $\mathbf{End}(C)$, where C is a finite chain, then it fails in $\mathbf{Aut_m}(\mathbb{Q})$, the inverse-free reduct of $\mathbf{Aut}(\mathbb{Q})$.

Proof-idea If an inverse-free equation $s = t$ fails in $\mathbf{End}(\mathbf{C})$, where C is a finite chain, then there are $f_1, \ldots, f_n \in End(C)$ and $p \in End(C)$ such that $(p)s(f_1, \ldots, f_n) \neq (p)t(f_1, \ldots, f_n)$. We first create a finite chain $\overline{C}$ and partial injections $\overline{f}_1, \ldots, \overline{f}_n$ which still show the failure at some $p \in \overline{C}$. Then using the density of $\mathbb{Q}$ we embed $\overline{C}$ in $\mathbb{Q}$ and extend $\overline{f}_1, \ldots, \overline{f}_n$ to total bijections of $\mathbb{Q}$.

To create $\overline{C}$ we need to duplicate points $q \in C$ such that $f(p_1) = f(p_2) = q$ for some f, and then extend the functions to the new chain in an iterative way, ensuring that the failure remains. This is complex if done by hand. We define $\overline{C}$ to be all (non-empty) finite sequences $(p_0, p_1, \ldots, p_n)$ of elements from C, such that $p_0 = p$ and $(p_{i-1})g_i = p_i$, for all $1 \leq i \leq n$, where the g_i's are taken from the f_i's and $g_1 \cdots g_n$ is an initial monoid subterm in $s = t$ (in conjunctive/disjunctive form). The order on $\overline{C}$ is done lexicographically from right to left.

Inverse-free reducts of representable

Proposition (Colacito - G. - Metcalfe) The inverse-free subreducts of representatble ℓ-groups are not the whole variety of semilinear (subdirect product of chains) totally distributive ℓ-monoids.

Proposition (Colacito - G. - Metcalfe) The inverse-free subreducts of representatble ℓ-groups are not the whole variety of semilinear (subdirect product of chains) totally distributive ℓ-monoids.

Proof idea We define the terms

$$F = x_1x_2x_3 \wedge x_5x_4x_6 \wedge x_9x_7x_8, \;\; G = x_1x_4x_7 \vee x_5x_2x_8 \vee x_9x_6x_3,$$
$$F' = x_1x_3x_2 \wedge x_5x_6x_4 \wedge x_9x_8x_7, \; G' = x_1x_7x_4 \vee x_5x_8x_2 \vee x_9x_3x_6.$$

Proposition (Colacito - G. - Metcalfe) The inverse-free subreducts of representatble ℓ-groups are not the whole variety of semilinear (subdirect product of chains) totally distributive ℓ-monoids.

Proof idea We define the terms

$F = x_1x_2x_3 \wedge x_5x_4x_6 \wedge x_9x_7x_8$, $G = x_1x_4x_7 \vee x_5x_2x_8 \vee x_9x_6x_3$,
$F' = x_1x_3x_2 \wedge x_5x_6x_4 \wedge x_9x_8x_7$, $G' = x_1x_7x_4 \vee x_5x_8x_2 \vee x_9x_3x_6$.

We show that $F \wedge F' \leq G \vee G'$ fails in a *commutative* totally ordered monoid. (Note that in the commutative case $F = F'$ and $G = G'$.)

Inverse-free reducts of representable

Proposition (Colacito - G. - Metcalfe) The inverse-free subreducts of representatble ℓ-groups are not the whole variety of semilinear (subdirect product of chains) totally distributive ℓ-monoids.

Proof idea We define the terms

$F = x_1x_2x_3 \wedge x_5x_4x_6 \wedge x_9x_7x_8$, $G = x_1x_4x_7 \vee x_5x_2x_8 \vee x_9x_6x_3$,
$F' = x_1x_3x_2 \wedge x_5x_6x_4 \wedge x_9x_8x_7$, $G' = x_1x_7x_4 \vee x_5x_8x_2 \vee x_9x_3x_6$.

We show that $F \wedge F' \leq G \vee G'$ fails in a *commutative* totally ordered monoid. (Note that in the commutative case $F = F'$ and $G = G'$.)

We also prove that $F \wedge F' \leq G \vee G'$ holds in all totally ordered groups.

Proposition (Colacito - G. - Metcalfe) The inverse-free subreducts of representatble ℓ-groups are not the whole variety of semilinear (subdirect product of chains) totally distributive ℓ-monoids.

Proof idea We define the terms

$F = x_1x_2x_3 \wedge x_5x_4x_6 \wedge x_9x_7x_8$, $G = x_1x_4x_7 \vee x_5x_2x_8 \vee x_9x_6x_3$,
$F' = x_1x_3x_2 \wedge x_5x_6x_4 \wedge x_9x_8x_7$, $G' = x_1x_7x_4 \vee x_5x_8x_2 \vee x_9x_3x_6$.

We show that $F \wedge F' \leq G \vee G'$ fails in a *commutative* totally ordered monoid. (Note that in the commutative case $F = F'$ and $G = G'$.)

We also prove that $F \wedge F' \leq G \vee G'$ holds in all totally ordered groups. This is done by presenting a derivation in the system of **(G. - Metcalfe)** expanded by the *cycle* quasiequation $(1 \leq s \vee gh \;\Rightarrow\; 1 \leq s \vee hg)$, which holds in the free representable ℓ-group.

Inverse-free reducts of representable

Proposition (Colacito - G. - Metcalfe) The inverse-free subreducts of representatble ℓ-groups are not the whole variety of semilinear (subdirect product of chains) totally distributive ℓ-monoids.

Proof idea We define the terms

$F = x_1x_2x_3 \wedge x_5x_4x_6 \wedge x_9x_7x_8$, $G = x_1x_4x_7 \vee x_5x_2x_8 \vee x_9x_6x_3$,
$F' = x_1x_3x_2 \wedge x_5x_6x_4 \wedge x_9x_8x_7$, $G' = x_1x_7x_4 \vee x_5x_8x_2 \vee x_9x_3x_6$.

We show that $F \wedge F' \leq G \vee G'$ fails in a *commutative* totally ordered monoid. (Note that in the commutative case $F = F'$ and $G = G'$.)

We also prove that $F \wedge F' \leq G \vee G'$ holds in all totally ordered groups. This is done by presenting a derivation in the system of **(G. - Metcalfe)** expanded by the *cycle* quasiequation $(1 \leq s \vee gh \;\Rightarrow\; 1 \leq s \vee hg)$, which holds in the free representable ℓ-group.

Conjecture The inverse-free subreducts of representatble ℓ-groups do not form a finitely axiomatizable variety (over the semilinear (totally distributive) ℓ-monoids).

Inverse-free reducts of representable

Proposition (Colacito - G. - Metcalfe) The inverse-free subreducts of representatble ℓ-groups are not the whole variety of semilinear (subdirect product of chains) totally distributive ℓ-monoids.

Proof idea We define the terms

$$F = x_1x_2x_3 \wedge x_5x_4x_6 \wedge x_9x_7x_8, \;\; G = x_1x_4x_7 \vee x_5x_2x_8 \vee x_9x_6x_3,$$
$$F' = x_1x_3x_2 \wedge x_5x_6x_4 \wedge x_9x_8x_7, \; G' = x_1x_7x_4 \vee x_5x_8x_2 \vee x_9x_3x_6.$$

We show that $F \wedge F' \leq G \vee G'$ fails in a *commutative* totally ordered monoid. (Note that in the commutative case $F = F'$ and $G = G'$.)

We also prove that $F \wedge F' \leq G \vee G'$ holds in all totally ordered groups. This is done by presenting a derivation in the system of **(G. - Metcalfe)** expanded by the *cycle* quasiequation $(1 \leq s \vee gh \;\Rightarrow\; 1 \leq s \vee hg)$, which holds in the free representable ℓ-group.

Conjecture The inverse-free subreducts of representatble ℓ-groups do not form a finitely axiomatizable variety (over the semilinear (totally distributive) ℓ-monoids).

We should first axiomatize the variety of semilinear TDL-monoids.

Theorem (Colacito - G. - Metcalfe) Among totally distributive ℓ-monoids the subvariety of all semilinear ones is axiomatized by the equation (esl)

$$z_1 x z_2 \wedge w_1 y w_2 \leq z_1 y z_2 \vee w_1 x w_2.$$

Theorem (Colacito - G. - Metcalfe) Among totally distributive ℓ-monoids the subvariety of all semilinear ones is axiomatized by the equation (esl)

$$z_1 x z_2 \wedge w_1 y w_2 \leq z_1 y z_2 \vee w_1 x w_2.$$

Theorem (G. - Horčík) A join-semilattice monoid can be embedded into the order endomorphisms $\mathbf{End}(\mathbf{C})$ of a chain $\mathbf{C}$ iff it satisfies

$$u \leq h \vee zx \ \ \& \ \ u \leq h \vee wy \Longrightarrow u \leq h \vee zy \vee wx.$$

In the lattice-ordered case this is equivalent to

$$(h \vee zx) \wedge (h \vee wy) \leq h \vee zy \vee wx.$$

In the distributive lattice-ordered case this is equivalent to

$$zx \wedge wy \leq zy \vee wx.$$

(The theorem also has versions for residuated lattices and for ℓ-groups: Holland's embeddding theorem.)

(Melier) For an monoid $\mathbf{M}$, $m \in M$ and subset I, we define

$$\frac{I}{m} = \{(x, y) \in M \times M : xmy \in I\}.$$

Also, we define a binary relation by

$a \sim_I b$ iff $\frac{I}{a} = \frac{I}{b}$ iff for all $z, w \in M$, $zaw \in I$ iff $zbw \in I$.

(Melier) For an monoid $\mathbf{M}$, $m \in M$ and subset I, we define

$$\frac{I}{m} = \{(x, y) \in M \times M : xmy \in I\}.$$

Also, we define a binary relation by

$$a \sim_I b \text{ iff } \tfrac{I}{a} = \tfrac{I}{b} \text{ iff for all } z, w \in M,\ zaw \in I \text{ iff } zbw \in I.$$

A *semilattice-monoid* (aka *idempotent semiring*) is a structure $\mathbf{M} = (M, \vee, \cdot, 1)$ such that $(M, \vee)$ is a join-semilattice, $(M, \cdot, 1)$ is a monoid and multiplication distributes over join on both sides.

(Melier) For an monoid $\mathbf{M}$, $m \in M$ and subset I, we define

$$\frac{I}{m} = \{(x, y) \in M \times M : xmy \in I\}.$$

Also, we define a binary relation by

$$a \sim_I b \text{ iff } \tfrac{I}{a} = \tfrac{I}{b} \text{ iff for all } z, w \in M,\ zaw \in I \text{ iff } zbw \in I.$$

A *semilattice-monoid* (aka *idempotent semiring*) is a structure $\mathbf{M} = (M, \vee, \cdot, 1)$ such that $(M, \vee)$ is a join-semilattice, $(M, \cdot, 1)$ is a monoid and multiplication distributes over join on both sides.

Lemma (Melier) If I is an ideal of a semilattice-monoid, then $\sim_I$ is a congruence. If $\mathbf{M}$ is a lattice and I is $\wedge$-prime, then $\sim_I$ is compatible with meet.

(Melier) For an monoid $\mathbf{M}$, $m \in M$ and subset I, we define

$$\frac{I}{m} = \{(x, y) \in M \times M : xmy \in I\}.$$

Also, we define a binary relation by

$$a \sim_I b \text{ iff } \frac{I}{a} = \frac{I}{b} \text{ iff for all } z, w \in M,\ zaw \in I \text{ iff } zbw \in I.$$

A *semilattice-monoid* (aka *idempotent semiring*) is a structure $\mathbf{M} = (M, \vee, \cdot, 1)$ such that $(M, \vee)$ is a join-semilattice, $(M, \cdot, 1)$ is a monoid and multiplication distributes over join on both sides.

Lemma (Melier) If I is an ideal of a semilattice-monoid, then $\sim_I$ is a congruence. If $\mathbf{M}$ is a lattice and I is $\wedge$-prime, then $\sim_I$ is compatible with meet.

In this case the quotient M/I is also a (lattice-ordered) semilattice-monoid.

Lemma (cf. G. - Horčík) The quotient M/I is a chain iff

$z_1 x z_2 \in I$ and $w_1 y w_2 \in I$ implies $z_1 y z_2 \in I$ or $w_1 x w_2 \in I$.

Lemma (cf. G. - Horčík) The quotient M/I is a chain iff

$$z_1xz_2 \in I \text{ and } w_1yw_2 \in I \text{ implies } z_1yz_2 \in I \text{ or } w_1xw_2 \in I.$$

Lemma A semilattice monoid is semilinear iff it satisfies the implication (sl)

$$u \leq h \vee z_1xz_2 \;\;\&\;\; u \leq h \vee w_1yw_2 \Longrightarrow u \leq h \vee z_1yz_2 \vee w_2xw_2$$

Lemma (cf. G. - Horčík) The quotient M/I is a chain iff

$z_1xz_2 \in I$ and $w_1yw_2 \in I$ implies $z_1yz_2 \in I$ or $w_1xw_2 \in I$.

Lemma A semilattice monoid is semilinear iff it satisfies the implication (sl)

$$u \leq h \vee z_1xz_2 \ \ \& \ \ u \leq h \vee w_1yw_2 \Longrightarrow u \leq h \vee z_1yz_2 \vee w_2xw_2$$

Proof idea
1. relatively maximal ideals produce linear quotients (and are $\wedge$-prime in the lattice case) and that
2. we have enough relatively maximal to separate points.

Lemma (cf. G. - Horčík) The quotient M/I is a chain iff

$$z_1 x z_2 \in I \text{ and } w_1 y w_2 \in I \text{ implies } z_1 y z_2 \in I \text{ or } w_1 x w_2 \in I.$$

Lemma A semilattice monoid is semilinear iff it satisfies the implication (sl)

$$u \leq h \vee z_1 x z_2 \ \& \ u \leq h \vee w_1 y w_2 \Longrightarrow u \leq h \vee z_1 y z_2 \vee w_2 x w_2$$

Proof idea
1. relatively maximal ideals produce linear quotients (and are $\wedge$-prime in the lattice case) and that
2. we have enough relatively maximal to separate points.

Lemma If a lattice-ordered semlilattice-monoid is distributive, then (sl) is equivalent to the equation (esl):
$z_1 x z_2 \wedge w_1 y w_2 \leq z_1 y z_2 \vee w_1 x w_2$.

Lemma (cf. G. - Horčík) The quotient M/I is a chain iff

$$z_1xz_2 \in I \text{ and } w_1yw_2 \in I \text{ implies } z_1yz_2 \in I \text{ or } w_1xw_2 \in I.$$

Lemma A semilattice monoid is semilinear iff it satisfies the implication (sl)

$$u \leq h \vee z_1xz_2 \;\;\&\;\; u \leq h \vee w_1yw_2 \Longrightarrow u \leq h \vee z_1yz_2 \vee w_2xw_2$$

Proof idea
1. relatively maximal ideals produce linear quotients (and are $\wedge$-prime in the lattice case) and that
2. we have enough relatively maximal to separate points.

Lemma If a lattice-ordered semlilattice-monoid is distributive, then (sl) is equivalent to the equation (esl):
$z_1xz_2 \wedge w_1yw_2 \leq z_1yz_2 \vee w_1xw_2$.

Note that (esl) implies $ee(yx) \wedge yxe \leq ex(yx) \vee yee$, namely $yx \leq xyx \vee y$, the equation that axiomatizes representable ℓ-groups.

$$\frac{x \Rightarrow a \quad u[a] \Rightarrow c}{u[x] \Rightarrow c}\ (\mathsf{cut}) \qquad \frac{}{a \Rightarrow a}\ (\mathsf{Id})$$

$$\frac{u[a] \Rightarrow c}{u[a \wedge b] \Rightarrow c}\ (\wedge \mathsf{L}\ell) \qquad \frac{u[b] \Rightarrow c}{u[a \wedge b] \Rightarrow c}\ (\wedge \mathsf{L}r) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \mathsf{R})$$

$$\frac{u[a] \Rightarrow c \quad u[b] \Rightarrow c}{u[a \vee b] \Rightarrow c}\ (\vee \mathsf{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}r)$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[x \circ (a \backslash b)] \Rightarrow c}\ (\backslash \mathsf{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \mathsf{R})$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[(b/a) \circ x] \Rightarrow c}\ (/\mathsf{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathsf{R})$$

$$\frac{u[a \circ b] \Rightarrow c}{u[a \cdot b] \Rightarrow c}\ (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot \mathsf{R})$$

$$\frac{u[\ ] \Rightarrow a}{u[1] \Rightarrow a}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathsf{R})$$

Basic substructural logics

A *sequent* is an expression $x \Rightarrow a$, where $a \in Fm$ and x is an element of $(Fm^*, \circ, \varepsilon)$, the free monoid/groupoid over Fm.

A *sequent* is an expression $x \Rightarrow a$, where $a \in Fm$ and x is an element of $(Fm^*, \circ, \varepsilon)$, the free monoid/groupoid over Fm.

Also, u is a monoid word with an empty space and in $u[x]$ the empty space is filled by x.

A *sequent* is an expression $x \Rightarrow a$, where $a \in Fm$ and x is an element of $(Fm^*, \circ, \varepsilon)$, the free monoid/groupoid over Fm.

Also, u is a monoid word with an empty space and in $u[x]$ the empty space is filled by x.

$$\frac{u[x \circ y] \Rightarrow c}{u[y \circ x] \Rightarrow c}\,(e) \qquad \text{(exchange)} \qquad xy \leq yx$$

A *sequent* is an expression $x \Rightarrow a$, where $a \in Fm$ and x is an element of $(Fm^*, \circ, \varepsilon)$, the free monoid/groupoid over Fm.

Also, u is a monoid word with an empty space and in $u[x]$ the empty space is filled by x.

$$\frac{u[x \circ y] \Rightarrow c}{u[y \circ x] \Rightarrow c}\ (e) \qquad \text{(exchange)} \qquad xy \leq yx$$

$$\frac{u[x \circ x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \text{(contraction)} \qquad x \leq x^2$$

A *sequent* is an expression $x \Rightarrow a$, where $a \in Fm$ and x is an element of $(Fm^*, \circ, \varepsilon)$, the free monoid/groupoid over Fm.

Also, u is a monoid word with an empty space and in $u[x]$ the empty space is filled by x.

$$\frac{u[x \circ y] \Rightarrow c}{u[y \circ x] \Rightarrow c}\ (e) \qquad \text{(exchange)} \qquad xy \leq yx$$

$$\frac{u[x \circ x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \text{(contraction)} \qquad x \leq x^2$$

$$\frac{u[\,] \Rightarrow c}{u[x] \Rightarrow c}\ (lw) \qquad \text{(left weakening)} \qquad x \leq 1$$

Basic substructural logics

A *sequent* is an expression $x \Rightarrow a$, where $a \in Fm$ and x is an element of $(Fm^*, \circ, \varepsilon)$, the free monoid/groupoid over Fm.

Also, u is a monoid word with an empty space and in $u[x]$ the empty space is filled by x.

$$\frac{u[x \circ y] \Rightarrow c}{u[y \circ x] \Rightarrow c}\ (e) \qquad \text{(exchange)} \qquad xy \leq yx$$

$$\frac{u[x \circ x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \text{(contraction)} \qquad x \leq x^2$$

$$\frac{u[\,] \Rightarrow c}{u[x] \Rightarrow c}\ (lw) \qquad \text{(left weakening)} \qquad x \leq 1$$

If the sequent s is provable in $\mathbf{FL}$ from the set of *sequents* S, we write $S \vdash_{\mathbf{FL}} s$.

Basic substructural logics

A *sequent* is an expression $x \Rightarrow a$, where $a \in Fm$ and x is an element of $(Fm^*, \circ, \varepsilon)$, the free monoid/groupoid over Fm.

Also, u is a monoid word with an empty space and in $u[x]$ the empty space is filled by x.

$$\frac{u[x \circ y] \Rightarrow c}{u[y \circ x] \Rightarrow c}\ (e) \qquad \text{(exchange)} \qquad xy \leq yx$$

$$\frac{u[x \circ x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \text{(contraction)} \qquad x \leq x^2$$

$$\frac{u[\,] \Rightarrow c}{u[x] \Rightarrow c}\ (lw) \qquad \text{(left weakening)} \qquad x \leq 1$$

If the sequent s is provable in $\mathbf{FL}$ from the set of *sequents* S, we write $S \vdash_{\mathbf{FL}} s$.

We want to add:

$$a \wedge (b \vee c) \Rightarrow (a \wedge b) \vee c \qquad \text{(distributivity)} \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

We change the notion of a sequent.

We change the notion of a sequent.
The left-hand side x of a sequent is now an element of $(Fm^\gamma, \circ, \varepsilon, \textcircled{\wedge})$, the free bi-groupoid over Fm.

We change the notion of a sequent.
The left-hand side x of a sequent is now an element of $(Fm^\gamma, \circ, \varepsilon, \bigcircle{\wedge})$, the free bi-groupoid over Fm.

Also, u is a polynomial over this signature.

We change the notion of a sequent.
The left-hand side x of a sequent is now an element of $(Fm^\gamma, \circ, \varepsilon, \owedge)$, the free bi-groupoid over Fm.

Also, u is a polynomial over this signature.

We add the rules:

$$\frac{u[x \owedge (y \owedge z)] \Rightarrow c}{u[(x \owedge y) \owedge z] \Rightarrow c}\ (\owedge a) \qquad \frac{u[x \owedge y] \Rightarrow c}{u[y \owedge x] \Rightarrow c}\ (\owedge e)$$

$$\frac{u[x] \Rightarrow c}{u[x \owedge y] \Rightarrow c}\ (\owedge i) \qquad \frac{u[x \owedge x] \Rightarrow c}{u[x] \Rightarrow c}\ (\owedge c)$$

And replace ($\wedge$L) by:

$$\frac{u[a \owedge b] \Rightarrow c}{u[a \wedge b] \Rightarrow c}\ (\wedge \mathsf{L})$$

Residuated frames

A *distributive residuated frame* is a structure
$\mathbf{W} = (W, W', N, \circ, 1, \owedge)$ where W and W' are sets $N \subseteq W \times W'$,

A *distributive residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, 1, \owedge)$ where W and W' are sets $N \subseteq W \times W'$, $((W, \circ, 1)$ is a monoid) and for all $x, y \in W$, $w \in W'$ and $\bullet \in \{\circ, \owedge\}$ there exist subsets $x \mathbin{\backslash\!\backslash} w, w \mathbin{/\!/} y \subseteq W'$ such that

$$(x \bullet y) \; N \; w \;\Leftrightarrow\; y \; N \; (x \mathbin{\backslash\!\backslash} w) \;\Leftrightarrow\; x \; N \; (w \mathbin{/\!/} y)$$

$$\frac{x \owedge (y \owedge w) N z}{\overline{(x \owedge y) \owedge w N z}}\ (\owedge a) \qquad \frac{x \owedge y N z}{y \owedge x N z}\ (\owedge e)$$

$$\frac{xNz}{\overline{x \owedge y N z}}\ (\owedge i) \qquad \frac{x \owedge x N z}{x N z}\ (\owedge c)$$

A *distributive residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, 1, \bigcirc\!\!\!\!\wedge)$ where W and W' are sets $N \subseteq W \times W'$, $((W, \circ, 1)$ is a monoid) and for all $x, y \in W$, $w \in W'$ and $\bullet \in \{\circ, \bigcirc\!\!\!\!\wedge\}$ there exist subsets $x \bbslash w, w \sslash y \subseteq W'$ such that

$$(x \bullet y)\ N\ w \ \Leftrightarrow\ y\ N\ (x \bbslash w)\ \Leftrightarrow\ x\ N\ (w \sslash y)$$

$$\frac{x \bigcirc\!\!\!\!\wedge (y \bigcirc\!\!\!\!\wedge w) N z}{(x \bigcirc\!\!\!\!\wedge y) \bigcirc\!\!\!\!\wedge w N z}\ (\bigcirc\!\!\!\!\wedge a) \qquad \frac{x \bigcirc\!\!\!\!\wedge y N z}{y \bigcirc\!\!\!\!\wedge x N z}\ (\bigcirc\!\!\!\!\wedge e)$$

$$\frac{x N z}{x \bigcirc\!\!\!\!\wedge y N z}\ (\bigcirc\!\!\!\!\wedge i) \qquad \frac{x \bigcirc\!\!\!\!\wedge x N z}{x N z}\ (\bigcirc\!\!\!\!\wedge c)$$

Corollary. (G-Jipsen) If $\mathbf{W}$ is a distributive residuated frame then the *Galois algebra* $\mathbf{W}^+$ of closed elements of $\mathcal{P}(W, \circ, 1)$ is a distributive residuated lattice.

A *distributive residuated frame* is a structure $\mathbf{W} = (W, W', N, \circ, 1, \owedge)$ where W and W' are sets $N \subseteq W \times W'$, ($(W, \circ, 1)$ is a monoid) and for all $x, y \in W$, $w \in W'$ and $\bullet \in \{\circ, \owedge\}$ there exist subsets $x \bbslash w, w \sslash y \subseteq W'$ such that

$$(x \bullet y)\ N\ w \ \Leftrightarrow\ y\ N\ (x \bbslash w) \ \Leftrightarrow\ x\ N\ (w \sslash y)$$

$$\frac{x \owedge (y \owedge w) N z}{\overline{(x \owedge y) \owedge w N z}}\ (\owedge a) \qquad \frac{x \owedge y N z}{y \owedge x N z}\ (\owedge e)$$

$$\frac{xNz}{\overline{x \owedge y N z}}\ (\owedge i) \qquad \frac{x \owedge x N z}{x N z}\ (\owedge c)$$

Corollary. **(G-Jipsen)** If $\mathbf{W}$ is a distributive residuated frame then the *Galois algebra* $\mathbf{W}^+$ of closed elements of $\mathcal{P}(W, \circ, 1)$ is a distributive residuated lattice. Also, $\owedge_N$ is intersection.

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

$(x_1^2 \wedge y) \vee (x_1 x_2 \wedge y) \vee (x_2 x_1 \wedge y) \vee (x_2^2 \wedge y) \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee y x_1 \vee y x_2$

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

$(x_1^2 \wedge y) \vee (x_1 x_2 \wedge y) \vee (x_2 x_1 \wedge y) \vee (x_2^2 \wedge y) \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee y x_1 \vee y x_2$

$x_1 x_2 \wedge y \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee y x_1 \vee y x_2$

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

$(x_1^2 \wedge y) \vee (x_1x_2 \wedge y) \vee (x_2x_1 \wedge y) \vee (x_2^2 \wedge y) \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee yx_1 \vee yx_2$

$x_1x_2 \wedge y \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee yx_1 \vee yx_2$

$$\frac{x_1 \wedge y \leq v \quad x_2 \wedge y \leq v \quad yx_1 \leq v \quad yx_2 \leq v}{x_1x_2 \wedge y \leq v}$$

Equations

Idea: Express equations over $\{\wedge, \vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\wedge, \vee, \cdot, 1\}$ we distribute products and meets over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: $\{\wedge, \cdot, 1\}$-terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 \wedge y \leq (x \wedge y) \vee yx$

$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2)$

$(x_1^2 \wedge y) \vee (x_1 x_2 \wedge y) \vee (x_2 x_1 \wedge y) \vee (x_2^2 \wedge y) \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee y x_1 \vee y x_2$

$x_1 x_2 \wedge y \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee y x_1 \vee y x_2$

$$\frac{x_1 \wedge y \leq v \quad x_2 \wedge y \leq v \quad y x_1 \leq v \quad y x_2 \leq v}{x_1 x_2 \wedge y \leq v}$$

$$\frac{x_1 \owedge y \; N \; z \quad x_2 \owedge y \; N \; z \quad y \circ x_1 \; N \; z \quad y \circ x_2 \; N \; z}{x_1 \circ x_2 \owedge y \; N \; z} \; R(\varepsilon)$$

Structural rules

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\wedge, \cdot, 1\}$-terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} \; (R(\varepsilon))$$

where the t_i's are evaluated in $(W, \circ, \varepsilon)$. Such a rule is called *simple* if all variables in t_0 are distinct.

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\wedge, \cdot, 1\}$-terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} \ (R(\varepsilon))$$

where the t_i's are evaluated in $(W, \circ, \varepsilon)$. Such a rule is called *simple* if all variables in t_0 are distinct.

Theorem. If $(\mathbf{W}, \mathbf{B})$ is a Gentzen frame and ε an equation over $\{\wedge, \vee, \cdot, 1\}$, then $(\mathbf{W}, \mathbf{B})$ satisfies R(ε) iff $\mathbf{W}^+$ satisfies ε.

(The linearity of the denominator of $R(\varepsilon)$ plays an important role in the proof.)

Structural rules

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\wedge, \cdot, 1\}$-terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} \ (R(\varepsilon))$$

where the t_i's are evaluated in $(W, \circ, \varepsilon)$. Such a rule is called *simple* if all variables in t_0 are distinct.

Theorem. If $(\mathbf{W}, \mathbf{B})$ is a Gentzen frame and ε an equation over $\{\wedge, \vee, \cdot, 1\}$, then $(\mathbf{W}, \mathbf{B})$ satisfies R(ε) iff $\mathbf{W}^+$ satisfies ε.

(The linearity of the denominator of $R(\varepsilon)$ plays an important role in the proof.)

Theorem. (G. - Jipsen) Every system obtained from $\mathbf{DFL}$ by adding simple rules has the cut elimination property.

Derivation systems

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

Derivation systems

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

The system $\mathbf{DRL}$ supports the addition of equations such as l-distributivity of multiplication over meet: $xz \wedge xw \leq x(z \wedge w)$.

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

The system $\mathbf{DRL}$ supports the addition of equations such as l-distributivity of multiplication over meet: $xz \wedge xw \leq x(z \wedge w)$.

This can then be replaced by its linearized version
$xz \wedge yw \leq x(z \wedge w) \vee y(z \wedge w)$

Derivation systems

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

The system $\mathbf{DRL}$ supports the addition of equations such as l-distributivity of multiplication over meet: $xz \wedge xw \leq x(z \wedge w)$.

This can then be replaced by its linearized version $xz \wedge yw \leq x(z \wedge w) \vee y(z \wedge w)$ and then by a quasiequation $x(x \wedge w) \leq c \,\&\, y(z \wedge w) \leq c \Longrightarrow xz \wedge yw \leq c$. With this modification we still have completeness of the system without needing (cut).

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

The system $\mathbf{DRL}$ supports the addition of equations such as l-distributivity of multiplication over meet: $xz \wedge xw \leq x(z \wedge w)$.

This can then be replaced by its linearized version $xz \wedge yw \leq x(z \wedge w) \vee y(z \wedge w)$ and then by a quasiequation $x(x \wedge w) \leq c \,\&\, y(z \wedge w) \leq c \Longrightarrow xz \wedge yw \leq c$. With this modification we still have completeness of the system without needing (cut).

Note that this implies a version of the communication rule: $xz \wedge yw \leq xw \vee yz$

Derivation systems

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

The system $\mathbf{DRL}$ supports the addition of equations such as l-distributivity of multiplication over meet: $xz \wedge xw \leq x(z \wedge w)$.

This can then be replaced by its linearized version $xz \wedge yw \leq x(z \wedge w) \vee y(z \wedge w)$ and then by a quasiequation $x(x \wedge w) \leq c \,\&\, y(z \wedge w) \leq c \Longrightarrow xz \wedge yw \leq c$. With this modification we still have completeness of the system without needing (cut).

Note that this implies a version of the communication rule: $xz \wedge yw \leq xw \vee yz$ and $xw \leq c \,\&\, yz \leq c \Longrightarrow xz \wedge yw \leq c$.

Derivation systems

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

The system $\mathbf{DRL}$ supports the addition of equations such as l-distributivity of multiplication over meet: $xz \wedge xw \leq x(z \wedge w)$.

This can then be replaced by its linearized version $xz \wedge yw \leq x(z \wedge w) \vee y(z \wedge w)$ and then by a quasiequation $x(x \wedge w) \leq c \;\&\; y(z \wedge w) \leq c \Longrightarrow xz \wedge yw \leq c$. With this modification we still have completeness of the system without needing (cut).

Note that this implies a version of the communication rule: $xz \wedge yw \leq xw \vee yz$ and $xw \leq c \;\&\; yz \leq c \Longrightarrow xz \wedge yw \leq c$.

We can do the same for the semilinear case by transforming the equation (esl) $z_1xz_2 \wedge w_1yw_2 \leq z_1yz_2 \vee w_1xw_2$

Derivation systems

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

The system $\mathbf{DRL}$ supports the addition of equations such as l-distributivity of multiplication over meet: $xz \wedge xw \leq x(z \wedge w)$.

This can then be replaced by its linearized version $xz \wedge yw \leq x(z \wedge w) \vee y(z \wedge w)$ and then by a quasiequation $x(x \wedge w) \leq c \;\&\; y(z \wedge w) \leq c \Longrightarrow xz \wedge yw \leq c$. With this modification we still have completeness of the system without needing (cut).

Note that this implies a version of the communication rule: $xz \wedge yw \leq xw \vee yz$ and $xw \leq c \;\&\; yz \leq c \Longrightarrow xz \wedge yw \leq c$.

We can do the same for the semilinear case by transforming the equation (esl) $z_1xz_2 \wedge w_1yw_2 \leq z_1yz_2 \vee w_1xw_2$ into the quasiequation $z_1yz_2 \leq c \;\&\; w_1xw_2 \leq c \Longrightarrow z_1xz_2 \wedge w_1yw_2 \leq c$.

Derivation systems

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

The system $\mathbf{DRL}$ supports the addition of equations such as l-distributivity of multiplication over meet: $xz \wedge xw \leq x(z \wedge w)$.

This can then be replaced by its linearized version $xz \wedge yw \leq x(z \wedge w) \vee y(z \wedge w)$ and then by a quasiequation $x(x \wedge w) \leq c \;\&\; y(z \wedge w) \leq c \Longrightarrow xz \wedge yw \leq c$. With this modification we still have completeness of the system without needing (cut).

Note that this implies a version of the communication rule: $xz \wedge yw \leq xw \vee yz$ and $xw \leq c \;\&\; yz \leq c \Longrightarrow xz \wedge yw \leq c$.

We can do the same for the semilinear case by transforming the equation (esl) $z_1xz_2 \wedge w_1yw_2 \leq z_1yz_2 \vee w_1xw_2$ into the quasiequation $z_1yz_2 \leq c \;\&\; w_1xw_2 \leq c \Longrightarrow z_1xz_2 \wedge w_1yw_2 \leq c$.

Also, we can also transform the commutativity equation $xy = yx$.

Removing inverses

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one.

Removing inverses

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide inverse-free equations.

Removing inverses

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide inverse-free equations.

Question Is it enough to decide inverse-free equations in ℓ-groups?

Removing inverses

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide inverse-free equations.

Question Is it enough to decide inverse-free equations in ℓ-groups?

Theorem (Colacito - G. - Metcalfe) For ℓ-group terms c, d, g, h and fresh variabe x,

$$\mathsf{LG} \models h \wedge cg^{-1}d \leq u \;\Leftrightarrow\; \mathsf{LG} \models gxh \wedge gxcx \wedge d \leq gxu.$$

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide inverse-free equations.

Question Is it enough to decide inverse-free equations in ℓ-groups?

Theorem (Colacito - G. - Metcalfe) For ℓ-group terms c, d, g, h and fresh variabe x,

$$\mathsf{LG} \models h \wedge cg^{-1}d \leq u \;\Leftrightarrow\; \mathsf{LG} \models gxh \wedge gxcx \wedge d \leq gxu.$$

There is a lose analogy with the *density rule* in proof-theory.

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide inverse-free equations.

Question Is it enough to decide inverse-free equations in ℓ-groups?

Theorem (Colacito - G. - Metcalfe) For ℓ-group terms c, d, g, h and fresh variabe x,

$$\mathsf{LG} \models h \wedge cg^{-1}d \leq u \;\Leftrightarrow\; \mathsf{LG} \models gxh \wedge gxcx \wedge d \leq gxu.$$

There is a lose analogy with the *density rule* in proof-theory.

Corollary. Every equation in ℓ-groups is equivalent to one of the form $r_1 \wedge \cdots \wedge r_n \leq r_0$, where the r_i's monoid terms.

Removing inverses

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide inverse-free equations.

Question Is it enough to decide inverse-free equations in ℓ-groups?

Theorem (Colacito - G. - Metcalfe) For ℓ-group terms c, d, g, h and fresh variabe x,

$$\mathsf{LG} \models h \wedge cg^{-1}d \leq u \;\Leftrightarrow\; \mathsf{LG} \models gxh \wedge gxcx \wedge d \leq gxu.$$

There is a lose analogy with the *density rule* in proof-theory.

Corollary. Every equation in ℓ-groups is equivalent to one of the form $r_1 \wedge \cdots \wedge r_n \leq r_0$, where the r_i's monoid terms.

Therefore to decide (inverse-including) equations in ℓ-groups, we only need to be able to decide (inverse-free) equations in TDL-monoids.

Removing inverses

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide inverse-free equations.

Question Is it enough to decide inverse-free equations in ℓ-groups?

Theorem (Colacito - G. - Metcalfe) For ℓ-group terms c, d, g, h and fresh variabe x,

$$\mathsf{LG} \models h \wedge cg^{-1}d \leq u \;\Leftrightarrow\; \mathsf{LG} \models gxh \wedge gxcx \wedge d \leq gxu.$$

There is a lose analogy with the *density rule* in proof-theory.

Corollary. Every equation in ℓ-groups is equivalent to one of the form $r_1 \wedge \cdots \wedge r_n \leq r_0$, where the r_i's monoid terms.

Therefore to decide (inverse-including) equations in ℓ-groups, we only need to be able to decide (inverse-free) equations in TDL-monoids.

Hybrid system: Given an ℓ-group equation we apply (upward) instances of the density rule until we obtain an inverse-free equation. Then we continue in the system $\mathbf{TDLM}$.

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide inverse-free equations.

Question Is it enough to decide inverse-free equations in ℓ-groups?

Theorem (Colacito - G. - Metcalfe) For ℓ-group terms c, d, g, h and fresh variabe x,

$$\mathsf{LG} \models h \wedge cg^{-1}d \leq u \;\Leftrightarrow\; \mathsf{LG} \models gxh \wedge gxcx \wedge d \leq gxu.$$

There is a lose analogy with the *density rule* in proof-theory.

Corollary. Every equation in ℓ-groups is equivalent to one of the form $r_1 \wedge \cdots \wedge r_n \leq r_0$, where the r_i's monoid terms.

Therefore to decide (inverse-including) equations in ℓ-groups, we only need to be able to decide (inverse-free) equations in TDL-monoids.

Hybrid system: Given an ℓ-group equation we apply (upward) instances of the density rule until we obtain an inverse-free equation. Then we continue in the system $\mathbf{TDLM}$.

We can use *residuated frames* for totally distributive ℓ-monoids.

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide inverse-free equations.

Question Is it enough to decide inverse-free equations in ℓ-groups?

Theorem (Colacito - G. - Metcalfe) For ℓ-group terms c, d, g, h and fresh variabe x,

$$\mathsf{LG} \models h \wedge cg^{-1}d \leq u \;\Leftrightarrow\; \mathsf{LG} \models gxh \wedge gxcx \wedge d \leq gxu.$$

There is a lose analogy with the *density rule* in proof-theory.

Corollary. Every equation in ℓ-groups is equivalent to one of the form $r_1 \wedge \cdots \wedge r_n \leq r_0$, where the r_i's monoid terms.

Therefore to decide (inverse-including) equations in ℓ-groups, we only need to be able to decide (inverse-free) equations in TDL-monoids.

Hybrid system: Given an ℓ-group equation we apply (upward) instances of the density rule until we obtain an inverse-free equation. Then we continue in the system $\mathbf{TDLM}$.

We can use *residuated frames* for totally distributive ℓ-monoids.

This links well to the cut-free system of **(G.-Metcalfe)**.

(Pre)orders on the free group

Fact The lattice order of any ℓ-group is the intersection of all of its total-order extensions that are *right orders* (orders compatible with right multiplication).

(Pre)orders on the free group

Fact The lattice order of any ℓ-group is the intersection of all of its total-order extensions that are *right orders* (orders compatible with right multiplication).

Fact Every total right order on a group is determined by its positive (and/or negative) cone.

(Pre)orders on the free group

Fact The lattice order of any ℓ-group is the intersection of all of its total-order extensions that are *right orders* (orders compatible with right multiplication).

Fact Every total right order on a group is determined by its positive (and/or negative) cone.

Fact Total orders on the *free abelian group* on two generators are in bijective correspondence with lines through the origin with irrational slope together with (counted twice) lines through the origin with rational slope.

Theorem (Colacito - Metcalfe) The following are equivalent

1. $\{t_1, \ldots, , t_n\}$ does not extend to the positive cone of a right order on the free group over X.
2. $\models_{LG} 1 \leq t_1 \vee \cdots \vee t_n$

Theorem (Colacito - Metcalfe) The following are equivalent

1. $\{t_1, \ldots, , t_n\}$ does not extend to the positive cone of a right order on the free group over X.
2. $\models_{LG} 1 \leq t_1 \vee \cdots \vee t_n$

We extend the above theorem to:

Theorem (Colacito - G. - Metcalfe) Let $\Sigma \cup \{t_1, \ldots, , t_n\}$ be a set of group terms over the set X. The following are equivalent

1. There is no total right preorder of the *free group* over X that makes the normal closure of Σ positive and $\{t_1, \ldots, , t_n\}$ strictly negative.
2. $\Sigma \models_{\mathbf{Aut}(\mathbb{Q})} e \leq t_1 \vee \cdots \vee t_n$

Proof idea: we look at cyclic group actions on chains and at pointed valuations.

(Pre)orders on the free group

Theorem (Colacito - Metcalfe) The following are equivalent

1. $\{t_1, \ldots, , t_n\}$ does not extend to the positive cone of a right order on the free group over X.
2. $\models_{LG} 1 \leq t_1 \vee \cdots \vee t_n$

We extend the above theorem to:

Theorem (Colacito - G. - Metcalfe) Let $\Sigma \cup \{t_1, \ldots, , t_n\}$ be a set of group terms over the set X. The following are equivalent

1. There is no total right preorder of the *free group* over X that makes the normal closure of Σ positive and $\{t_1, \ldots, , t_n\}$ strictly negative.
2. $\Sigma \models_{\mathbf{Aut}(\mathbb{Q})} e \leq t_1 \vee \cdots \vee t_n$

Proof idea: we look at cyclic group actions on chains and at pointed valuations.

Q: Is there an analogue for the free monoid?

Theorem (Colacito - G. - Metcalfe) The following are equivalent

1. $\{s_1 < t_1, \ldots, s_n < t_n\}$ extends to a right order on the free monoid over X.
2. $\{s_1 < t_1, \ldots, s_n < t_n\}$ extends to a right order on the *free group* over X.
3. $\mathsf{LG} \not\models e \leq s_1^{-1}t_1 \vee \cdots \vee s_n^{-1}t_n$.
4. $\not\models_{LG} y_1 s_1 \wedge \cdots \wedge y_n s_n \leq y_1 t_1 \vee \cdots \vee y_n t_n$.
5. $\not\models_{TDLM} y_1 s_1 \wedge \cdots \wedge y_n s_n \leq y_1 t_1 \vee \cdots \vee y_n t_n$.

The variables $y_1, y_2, \ldots, y_n$ are not contained in the s_i's and t_i's.

Free group and free monoid

Theorem (Colacito - G. - Metcalfe) The following are equivalent

1. $\{s_1 < t_1, \ldots, s_n < t_n\}$ extends to a right order on the free monoid over X.
2. $\{s_1 < t_1, \ldots, s_n < t_n\}$ extends to a right order on the *free group* over X.
3. $\mathsf{LG} \not\models e \leq s_1^{-1}t_1 \vee \cdots \vee s_n^{-1}t_n$.
4. $\not\models_{LG} y_1 s_1 \wedge \cdots \wedge y_n s_n \leq y_1 t_1 \vee \cdots \vee y_n t_n$.
5. $\not\models_{TDLM} y_1 s_1 \wedge \cdots \wedge y_n s_n \leq y_1 t_1 \vee \cdots \vee y_n t_n$.

The variables $y_1, y_2, \ldots, y_n$ are not contained in the s_i's and t_i's.

(3) $\Leftrightarrow$ (4) For one direction substitute $y_i := s_i^{-1}$.

Theorem (Colacito - G. - Metcalfe) The following are equivalent

1. $\{s_1 < t_1, \ldots, s_n < t_n\}$ extends to a right order on the free monoid over X.
2. $\{s_1 < t_1, \ldots, s_n < t_n\}$ extends to a right order on the *free group* over X.
3. LG $\not\models e \leq s_1^{-1}t_1 \vee \cdots \vee s_n^{-1}t_n$.
4. $\not\models_{LG} y_1s_1 \wedge \cdots \wedge y_ns_n \leq y_1t_1 \vee \cdots \vee y_nt_n$.
5. $\not\models_{TDLM} y_1s_1 \wedge \cdots \wedge y_ns_n \leq y_1t_1 \vee \cdots \vee y_nt_n$.

The variables $y_1, y_2, \ldots, y_n$ are not contained in the s_i's and t_i's.

(3) $\Leftrightarrow$ (4) For one direction substitute $y_i := s_i^{-1}$.
For the other direction we have

$$1 \leq (y_1s_1 \wedge \cdots \wedge y_ns_n)^{-1}(y_1t_1 \vee \cdots \vee y_nt_n),$$

which is equivalent to

$$1 \leq s_1^{-1}t_1 \vee \cdots \vee s_n^{-1}t_n \vee \bigvee_{i \neq j} s_i^{-1}y_i^{-1}y_jt_j.$$