NILPOTENCE, SEMILINEARITY AND THE HAMILTONIAN PROPERTY

Almudena Colacito
Joint work with Constantine Tsinakis

Universität Bern

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COMMUTATIVITY

In the study of residuated lattices and lattice-ordered groups (short: ℓ-groups), **commutativity** carries along very strong structural properties, that allow us to exploit the structure theory of these algebras.

EXAMPLE (CONGRUENCE EXTENSION PROPERTY)

Let $\mathcal{V}$ be a variety of commutative residuated lattices. Then, for every algebra A in $\mathcal{V}$, for any subalgebra B of A and for any congruence θ on B, there exists a congruence $\overline{\theta}$ on A such that $\overline{\theta} \cap B^2 = \theta$.

EXAMPLE (DEDUCTION THEOREM)

Let $\mathcal{V}$ be a variety of commutative residuated lattices. Then, for any $\Sigma \cup \{\alpha, \beta\} \subseteq \mathrm{Fm}(Y)$,

$$\Sigma \cup \beta \models_{\mathcal{V}} \alpha \quad \Longleftrightarrow \quad \Sigma \models_{\mathcal{V}} (\beta \wedge \mathrm{e})^n \to \alpha \text{ for some } n \in \mathbb{N}.$$

THE HAMILTONIAN PROPERTY

We weaken commutativity in two ways, coming from group theory.

Hamiltonian groups are those non-Abelian groups for which every subgroup is normal.

Finite Hamiltonian groups were studied in 1895 by Dedekind.
The most familiar (and smallest) example of a Hamiltonian group is the quaternion group of order 8, denoted by Q8.
Dedekind named such groups after William R. Hamilton, who discovered quaternions in 1843.

The Hamiltonian Property

We weaken commutativity in two ways, coming from group theory.

Hamiltonian ℓ-groups are those ℓ-groups for which every convex ℓ-subgroup is normal.

Hamiltonian ℓ-groups were implicitly studied by Martinez in 1971.
They were first studied explicitly by Reilly in 1980; he studies them in relation with the class of nilpotent ℓ-groups.

NILPOTENCE

We weaken commutativity in two ways, coming from group theory.

Nilpotent groups have central series of finite length $c \in \mathbb{N}^+$, and they are 'almost Abelian', as they can be obtained from Abelian groups using extensions.

Abelian groups are those groups that have central series of length 1.

NILPOTENCE

We weaken commutativity in two ways, coming from group theory.

Nilpotent ℓ-groups of class $c \in \mathbb{N}^+$ are those ℓ-groups whose underlying group is nilpotent of class c.

The theory of nilpotent partially ordered groups, and nilpotent ℓ-groups started with Malcev (1951), Rhemtulla (1972), Martinez (1979), Hollister (1978).

The existing theory of partially ordered nilpotent groups consists primarily of theorems that were first proved for abelian groups. (Hollister, 1978)

A RELATED PROPERTY: SEMILINEARITY

We consider a related property.

Representable ℓ-groups are those ℓ-groups that are subdirect products of totally ordered groups.

Clifford (1940) and Lorenzen (1939) showed that Abelian ℓ-groups are subdirect products of totally ordered groups.
Birkhoff (1942) showed that representable ℓ-groups form a variety.

'Representable residuated lattices' are usually called **semilinear**.

EXAMPLE

MV-algebras are semilinear.

Hamiltonian & Nilpotent Lattice-Ordered Groups

Theorem (Martinez, 1972)

Hamiltonian ℓ-groups are representable.

Theorem (Kopytov, 1975; Reilly, 1983)

Nilpotent ℓ-groups are Hamiltonian.

Corollary (Kopytov, 1975; Hollister, 1978)

Nilpotent ℓ-groups are representable.

CANCELLATIVE RESIDUATED LATTICES

A residuated lattice is **cancellative** if its monoid reduct is cancellative—or equivalently, if it satisfies the equations

$$xy/y \approx x \approx y\backslash yx.$$

We mention that cancellative residuated lattices are **e-cyclic**, that is,

$$x\backslash \mathrm{e} \approx \mathrm{e}/x$$

holds in every cancellative residuated lattice.

EXAMPLE

The variety of ℓ-groups is a variety of cancellative residuated lattices.

Prelinearity

We call a residuated lattice **prelinear** if it satisfies the prelinearity laws

$$(x\backslash y \wedge \mathrm{e}) \vee (y\backslash x \wedge \mathrm{e}) \approx \mathrm{e} \approx (x/y \wedge \mathrm{e}) \vee (y/x \wedge \mathrm{e}).$$

Example

MV-algebras and GMV-algebras are prelinear.

Example

The variety of ℓ-groups is prelinear.

Every prelinear residuated lattice satisfies:

$$x\backslash(y \vee z) \approx (x\backslash y) \vee (x\backslash z) \quad \text{and} \quad (y \vee z)/x \approx (y/x) \vee (z/x).$$

STRUCTURE THEORY

A subalgebra H of a residuated lattice L is said to be *convex* if it is order-convex, that is, $c \in H$ whenever $a \leq c \leq b$ for $a, b \in H$.

A convex subalgebra H of a residuated lattice L is said to be **normal** if

$$(b\backslash ab) \wedge \mathrm{e} \in H \quad \text{and} \quad (ba/b) \wedge \mathrm{e} \in H, \quad \text{for all } a \in H, b \in L.$$

The lattice of convex normal subalgebras of a residuated lattice L is (isomorphic to) the lattice of congruences of L.

HAMILTONIAN RESIDUATED LATTICES

A residuated lattice L is **Hamiltonian** if every convex subalgebra of L is normal.

EXAMPLE

Commutative residuated lattices are Hamiltonian.

EXAMPLE

Hamiltonian ℓ-groups are Hamiltonian residuated lattices.

Hamiltonian Residuated Lattices

In the context of commutative residuated lattices, the prelinearity laws provide an equational basis for semilinearity.

HAMILTONIAN RESIDUATED LATTICES

In the context of commutative residuated lattices, the prelinearity laws provide an equational basis for semilinearity.

As it turns out, the Hamiltonian property and e-cyclicity suffice.

LEMMA

Hamiltonian prelinear cancellative residuated lattices are semilinear.

PROOF OUTLINE. Subdirectly irreducible algebras L have a least non-trivial congruence. Two incomparable elements $a, b \in L$ would lead to the existence of two non-trivial congruences with trivial intersection. □

HAMILTONIAN & NILPOTENT LATTICE-ORDERED GROUPS

The class of Hamiltonian ℓ-groups forms a *torsion class*.

THEOREM (REILLY, 1983)

The variety of **weakly Abelian ℓ-groups***, defined by*

$$(x \wedge \mathrm{e})^2 y \leq yx,$$

is the largest variety of Hamiltonian ℓ-groups.

Hamiltonian Residuated Lattices

Theorem (Botur, Kühr, Liu and Tsinakis, 2016)

A variety $\mathcal{V}$ of e*-cyclic residuated lattices is Hamiltonian if and only if there exists $n, m \in \mathbb{N}$ such that $\mathcal{V}$ satisfies the identities*

$$(x \wedge \mathrm{e})^n y \leq yx \quad \textit{and} \quad y(x \wedge \mathrm{e})^m \leq xy.$$

Theorem (Botur, Kühr, Liu and Tsinakis, 2016)

There exists no largest variety of Hamiltonian e*-cyclic residuated lattices.*

HAMILTONIAN RESIDUATED LATTICES

THEOREM

There exists a largest variety of Hamiltonian prelinear cancellative residuated lattices, defined by the equations:

$$(x \wedge \mathrm{e})^2 y \leq yx \quad \textit{and} \quad y(x \wedge \mathrm{e})^2 \leq xy.$$

PROOF OUTLINE. Otherwise, we could build a non-Hamiltonian direct product of chains. $\square$

Amalgamation Property

Theorem (Glass, Saracino and Wood, 1984)

The variety of weakly Abelian ℓ-groups fails the amalgamation property.

Corollary

The variety of weakly Abelian prelinear cancellative residuated lattices fails the amalgamation property.

proof outline. The set of invertible elements of a semilinear residuated lattice L is a subalgebra of L, and an ℓ-group. □

NILPOTENT MONOIDS

An ℓ-group is **nilpotent of class** $c \in \mathbb{N} - \{0\} = \mathbb{N}^+$ if its group reduct is nilpotent of class c, that is, it has a central series of length c.

The variety of nilpotent groups of class c is commonly defined with respect to the variety of groups by the group equations

$$[[[x_0, x_1], x_2], \ldots, x_c] \approx \mathrm{e},$$

where $[x, y] = x^{-1}y^{-1}xy$ is the *commutator*.

NILPOTENT MONOIDS

We consider the following *semigroup equations*

$$L_c\colon\ q_c(x,y,\bar{z}) \approx q_c(y,x,\bar{z}),$$

where $\bar{z}$ is $z_1, z_2, \ldots$, and $q_c(x,y,\bar{z})$ is inductively defined as follows:

$$\begin{array}{rcl} q_1(x,y,\bar{z}) & := & xy \\ q_{c+1}(x,y,\bar{z}) & := & q_c(x,y,\bar{z})z_c q_c(y,x,\bar{z}). \end{array}$$

THEOREM (NEUMANN AND TAYLOR, 1963)

For any group G *and any* $c \in \mathbb{N}^+$, *the equation* L_c *holds in* G *if and only if* G *is a nilpotent group of class* c.

NILPOTENT MONOIDS

EXAMPLE

- For $c = 1$, the equation L_1 is $xy \approx yx$, equivalent to $x^{-1}y^{-1}xy \approx \mathrm{e}$.
- For $c = 2$, the equation L_2 is $xyzyx \approx yxzxy$, which is less obviously equivalent to $y^{-1}x^{-1}yxz^{-1}x^{-1}y^{-1}xyz \approx \mathrm{e}$.
- More generally, the equation L_c is equivalent to

$$[[[x_0, x_1], x_2], \dots, x_c] \approx \mathrm{e}$$

over the variety of ℓ-groups.

SUBREDUCTS OF NILPOTENT GROUPS

A group G is a **group of left quotients** of a monoid M if :

- M is a submonoid of G.
- Every element of G is of the form $a^{-1}b$, for some $a, b \in M$.

We call a monoid with a group of left quotients an *Ore* monoid.

LEMMA (NEUMANN AND TAYLOR, 1963)

For any monoid M*, and any* $c \in \mathbb{N}^+$*, the following are equivalent:*

(1) M *can be embedded into a nilpotent group of class c.*

(2) M *is nilpotent of class c and cancellative.*

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QUESTION

When can a monoid be embedded into a nilpotent ℓ-group of class $c \in \mathbb{N}^+$?

Nilpotent Residuated Lattices

If we call *nilpotent of class c* those monoids satisfying L_c, we say that a residuated lattice L is **nilpotent of class** c if its monoid reduct $\langle L, \cdot, \mathrm{e} \rangle$ is nilpotent of class c, and **nilpotent** if it is nilpotent of some class c.

Example

Nilpotent ℓ-groups are nilpotent residuated lattices.

Example

The negative cone of a nilpotent ℓ-group of class c is a nilpotent residuated lattice of class c.

A CATEGORICAL EQUIVALENCE

A **conucleus** σ on a residuated lattice is a *interior operator*

$$\sigma\colon \langle L, \leq\rangle \to \langle L, \leq\rangle,$$

satisfying $\sigma(\mathrm{e}) = \mathrm{e}$ and $\sigma(a)\sigma(b) \leq \sigma(ab)$, for all $a, b \in L$.

If L is a residuated lattice and σ a co-nucleus on L, then the structure

$$\sigma[\mathrm{L}] = \langle \sigma[L], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, \mathrm{e}\rangle,$$

where $a \wedge_\sigma b := \sigma(a \wedge b)$, $a\backslash_\sigma b := \sigma(a\backslash b)$, and $a/_\sigma b := \sigma(a/b)$ for all $a, b \in \sigma[L]$, is a *residuated lattice*.

A Categorical Equivalence

For any $c \in \mathbb{N}^+$ and any class-c nilpotent cancellative residuated lattice L with lattice order $\leq$, the binary relation $\preceq \subseteq G(L) \times G(L)$ defined by

$$a^{-1}b \preceq c^{-1}d \quad \Longleftrightarrow \quad \text{there exist } m, n \in L \text{ s.t. } mb \leq nd \text{ and } ma = nc$$

where $a, b, c, d \in L$, is the unique partial order on $G(L)$ that extends $\leq$ and makes $\mathrm{G}(\mathrm{L})$ into a nilpotent ℓ-group of class c.

Further, the map $\sigma_{\mathrm{L}} \colon \mathrm{G}(\mathrm{L}) \to \mathrm{G}(\mathrm{L})$ defined by

$$\sigma_{\mathrm{L}}(a^{-1}b) = a \backslash b, \quad \text{for all } a, b \in L,$$

is a conucleus on $\mathrm{G}(\mathrm{L})$, and $\sigma_{\mathrm{L}}[\mathrm{G}(\mathrm{L})]$ is isomorphic to L.

SUBREDUCTS OF NILPOTENT LATTICE-ORDERED GROUPS

THEOREM

For any monoid M*, and any* $c \in \mathbb{N}^+$*, the following are equivalent:*

(1) M *can be embedded into a nilpotent* ℓ*-group of class* c*.*

(2) M *is nilpotent of class* c*, cancellative, and has unique roots.*
(If $a^n = b^n$ *for* $n \in \mathbb{N}^+$*, then* $a = b$*.)*

PROOF OUTLINE. (1) $\Rightarrow$ (2) This is based on the fact that every torsion-free nilpotent group does have unique roots.

Subreducts of Nilpotent Lattice-Ordered Groups

Theorem

For any monoid M*, and any* $c \in \mathbb{N}^+$*, the following are equivalent:*

(1) M *can be embedded into a nilpotent* ℓ*-group of class c.*

(2) M *is nilpotent of class c, cancellative, and has unique roots.*
(If $a^n = b^n$ *for* $n \in \mathbb{N}^+$*, then* $a = b$*.)*

(3) *The group* $\mathrm{G}(\mathrm{M})$ *of quotients of* M *exists, is nilpotent class c, and torsion-free.*

PROOF OUTLINE. (2) $\Rightarrow$ (3) This relies on two facts: the set of torsion elements of a nilpotent group is a (normal) subgroup of G and, as such, it intersects the center non-trivially.

Subreducts of Nilpotent Lattice-Ordered Groups

Theorem

For any monoid M*, and any* $c \in \mathbb{N}^+$*, the following are equivalent:*

(1) M *can be embedded into a nilpotent* ℓ*-group of class* c*.*

(2) M *is nilpotent of class* c*, cancellative, and has unique roots.*
(If $a^n = b^n$ *for* $n \in \mathbb{N}^+$*, then* $a = b$*.)*

(3) *The group* $\mathrm{G}(\mathrm{M})$ *of quotients of* M *exists, is nilpotent class* c*, and torsion-free.*

(4) M *can be embedded into a nilpotent cancellative residuated lattice of class* c*.*

PROOF OUTLINE. (3) $\Rightarrow$ (4) Every torsion-free nilpotent group is orderable, and nilpotent ℓ-groups of class c are nilpotent cancellative residuated lattices of class c.

Subreducts of Nilpotent Lattice-Ordered Groups

Theorem

For any monoid M*, and any* $c \in \mathbb{N}^{+}$*, the following are equivalent:*

(1) M *can be embedded into a nilpotent* ℓ*-group of class* c*.*

(2) M *is nilpotent of class* c*, cancellative, and has unique roots.*
(If $a^n = b^n$ *for* $n \in \mathbb{N}^{+}$*, then* $a = b$*.)*

(3) *The group* $\mathrm{G}(\mathrm{M})$ *of quotients of* M *exists, is nilpotent class* c*, and torsion-free.*

(4) M *can be embedded into a nilpotent cancellative residuated lattice of class* c*.*

PROOF OUTLINE. (4) $\Rightarrow$ (1) This is based on the ‘categorical equivalence’.

NILPOTENT RESIDUATED LATTICES

THEOREM

Nilpotent cancellative residuated lattices are Hamiltonian.

PROOF OUTLINE. We make use of the categorical equivalence, and show that every nilpotent cancellative residuated lattices satisfies

$$(x \wedge \mathrm{e})^2 y \leq yx \quad \text{and} \quad y(x \wedge \mathrm{e})^2 \leq xy. \qquad \square$$

THEOREM

Nilpotent prelinear cancellative residuated lattices are semilinear.

AMALGAMATION PROPERTY

THEOREM (POWELL AND TSINAKIS, 1989)

The variety of nilpotent ℓ-groups of class $c \in \mathbb{N}^+$ fails the amalgamation property.

COROLLARY

The variety of nilpotent prelinear cancellative residuated lattices of class $c \in \mathbb{N}^+$ fails the amalgamation property.

INTEGRAL SEMILINEAR RESIDUATED LATTICES

THEOREM

Every representable ℓ-group admits a total order.

Integral Semilinear Residuated Lattices

Theorem

Every representable ℓ-group admits a total order.

Theorem

Every integral semilinear cancellative residuated lattice admits a residuated total order.

Proof outline. We consider the lexicographic order and observe that, since e is the greatest element, residuals always exist. □

Integral Semilinear Residuated Lattices

We call a variety of integral semilinear cancellative residuated lattices a *semigroup variety* if it is defined relative to semilinear cancellative residuated lattices by *semigroup equations*.

For any variety $\mathcal{V}$ of residuated lattices, we write $\mathcal{M}_{\mathcal{V}}$ for the quasivariety of monoid subreducts of $\mathcal{V}$.

INTEGRAL SEMILINEAR RESIDUATED LATTICES

Every residuated chain in $\mathcal{V}$ whose monoid reduct is a finitely generated monoid is the homomorphic image of a residuated chain whose monoid reducts is a finitely generated free object in $\mathcal{M}_{\mathcal{V}}$.

INTEGRAL SEMILINEAR RESIDUATED LATTICES

Every semigroup variety of integral semilinear cancellative residuated lattices is generated by the class of residuated chains whose monoid reduct is a finitely generated monoid.

Every residuated chain in $\mathcal{V}$ whose monoid reduct is a finitely generated monoid is the homomorphic image of a residuated chain whose monoid reducts is a finitely generated free object in $\mathcal{M}_{\mathcal{V}}$.

Integral Semilinear Residuated Lattices

Every integral total order on a finitely generated monoid is residuated. (A. Přenosil)

For any semigroup variety $\mathcal{V}$ of integral semilinear cancellative residuated lattices, every finitely generated monoid from $\mathcal{M}_{\mathcal{V}}$ is the reduct of a totally ordered element of $\mathcal{V}$.

Every semigroup variety of integral semilinear cancellative residuated lattices is generated by the class of residuated chains whose monoid reduct is a finitely generated monoid.

Every residuated chain in $\mathcal{V}$ whose monoid reduct is a finitely generated monoid is the homomorphic image of a residuated chain whose monoid reducts is a finitely generated free object in $\mathcal{M}_{\mathcal{V}}$.

THE NILPOTENT CASE

For $\mathcal{V}$ the variety of integral nilpotent prelinear cancellative residuated lattices of class $c \in \mathbb{N}^+$, the n-generated free object in $\mathcal{M}_{\mathcal{V}}$ is the submonoid generated by the variables in the free nilpotent group $\mathrm{F}_c(n)$ of class c, for all $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, the submonoid generated by the variables in $\mathrm{F}_c(n)$ is the n-generated free object in the quasivariety of subreducts of nilpotent cancellative residuated lattices (or nilpotent ℓ-groups).

FURTHER QUESTIONS

- Can we represent the free nilpotent prelinear cancellative residuated lattice as a subdirect product of all residuated chains whose monoid reduct is the free nilpotent monoid?
- We characterized the monoid subreducts of nilpotent ℓ-groups of class $c \in \mathbb{N}^+$. Can we characterize those monoids that embed into the negative cone of a nilpotent ℓ-groups of class $c \in \mathbb{N}^+$?
  (This has to do with extending partial (right) orders to total orders on torsion-free nilpotent groups.)
- Can we show that nilpotent prelinear cancellative residuated lattices are semilinear and Hamiltonian, without relying on the analogue result for nilpotent ℓ-groups?
- What other semigroup properties are preserved by the (ℓ-)groups of left quotients?
- For cancellative residuated lattices, can we characterize the property of prelinearity in terms of properties of the conucleus σ?