Semisimplicity, the excluded middle, and Glivenko theorems

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We investigate the relationship between three fundamental notions of logic and algebra, namely semisimplicity, the law of the excluded middle (LEM), and Glivenko theorems, which provide double negation translations between logics. These are all shown to be part of a single coherent circle of ideas: the semisimplicity of a logic is equivalent to some form of the LEM, and a Glivenko connection obtains between a logic and its semisimple companion.

Two notions of semisimplicity are at play here: syntactic semisimplicity, which states that each theory is an intersection of maximal non-trivial theories, and semantic semisimplicity, which states that each model is a subdirect product of simple models. The equivalence between syntactic semisimplicity and the LEM holds by a transparent syntactic argument, as does the Glivenko connection between a logic and its syntactic semisimple companion. Under suitable assumptions, syntactic semisimplicity then extends to semantic semisimplicity.

As an application, we show how this general theory yields a simple proof of the result of Kowalski that a variety of FL_{ew}-algebras is semisimple if and only if it satisfies the equation $x \vee \neg(x^n) \approx 1$ for some n, and a similar result for varieties of Boolean algebras with an operator. These can in fact be extended to FL_e-algebras and Heyting algebras. Moreover, several known Glivenko theorems are obtained as immediate corollaries. The inconsistency lemmas of Raftery and the dual of the notion of structural completeness are also implicated in the circle of ideas sketched above.