Multiplicative theory of (additive) partitions

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October 27, 2020
Additive number theory
Additive number theory

Patterns and interconnections
Patterns and interconnections

- theory of *partitions*
Patterns and interconnections

- theory of partitions
- beautiful generating functions
- surprising bijections
- Ramanujan congruences
- combinatorics, algebra, analytic num. theory, mod. forms, stat. phys., QT, string theory, chemistry, ...
Birth of partition theory
Birth of partition theory

Ishango bone (Africa, ca. 20,000 B.C.E.)
Birth of partition theory

Ahmes papyrus (Egypt, ca. 2,000 B.C.E.)
Incan *quipu* (Peru, 2,000 B.C.E. - 1600s)
Birth of partition theory
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G. W. Leibniz (1600s)
Leibniz wondered about size of $p(n) := \# \text{ of partitions of } n$
Leibniz

- wondered about size of $p(n) := \# \text{ of partitions of } n$
- $p(n)$ is called the *partition function*
Birth of partition theory
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Leonhard Euler (1700s)
Birth of partition theory

Partition generating function (Euler)

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \left(1 - q^n\right) - 1$$

$$q \in \mathbb{C}, |q| < 1$$

Template for partition theory
product-sum generating functions
combinatorics encoded in exponents, coefficients
connected analysis to partitions
Birth of partition theory

Partition generating function (Euler)

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Birth of partition theory

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Proof
Birth of partition theory

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Proof

RHS
Birth of partition theory

Partition generating function (Euler)

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Proof

$$\text{RHS} = \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + q^{3n} + \ldots)$$
## Birth of partition theory

**Partition generating function (Euler)**

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\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1} \quad (q \in \mathbb{C}, |q| < 1)
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**Proof**

\[
\text{RHS} = \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + q^{3n} + \ldots) \quad \text{(geom. series)}
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Birth of partition theory

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= \prod_{n=1}^{\infty} (1 + q^n + q^{n+n} + q^{n+n+n} + \ldots)
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Birth of partition theory

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= \[\prod_{n=1}^{\infty} (1 + q^n + q^{n+n} + q^{n+n+n} + ...)\] (like, second grade)
Birth of partition theory

Partition generating function (Euler)

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= \( \prod_{n=1}^{\infty} (1 + q^n + q^{n+n} + q^{n+n+n} + ...) \) (like, second grade)
= 1 + q^1 + q^{1+1} + q^2 + q^{1+1+1} + q^{1+2} + q^3 + ...
Birth of partition theory

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\]

\[
= 1 + q^1 + q^{1+1} + q^2 + q^{1+1+1} + q^{1+2} + q^3 + \ldots = LHS
\]
I.e., most of classical number theory focuses on primes, divisors, Euler phi function $\phi(n)$, Möbius function $\mu(n)$, arithmetic functions, Dirichlet convolution, zeta functions, Dirichlet series, and L-functions.
Multiplicative number theory

I.e., most of classical number theory
**Multiplicative number theory**

**I.e., most of classical number theory**
- primes
- divisors
- Euler phi function $\varphi(n)$, Möbius function $\mu(n)$
- arithmetic functions, Dirichlet convolution
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Birth of multiplicative number theory

Ishango bone (Africa, ca. 20,000 B.C.E.)
Birth of multiplicative number theory

Ahmes papyrus (Egypt, ca. 2,000 B.C.E.)
Birth of multiplicative number theory

Eratosthenes, Euclid (Alexandria, ca. 300 B.C.E.)
Birth of multiplicative number theory

Euler

\[ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\text{Re}(s) > 1) \]

explicit zeta values → compute even powers of \( \pi \):

- \( \zeta(2) = \pi^2 / 6 \)
- \( \zeta(4) = \pi^4 / 90 \)
- \( \zeta(6) = \pi^6 / 945 \)

...
Birth of multiplicative number theory

Euler

- first to understand (Riemann) zeta function

\[ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\text{Re}(s) > 1) \]

- explicit zeta values imply computing even powers of \( \pi \):
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Birth of multiplicative number theory

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Product formula for zeta function (Euler)

\[\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\text{Re}(s) > 1)\]
Multiplicative number theory

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Template for study of L-functions
### Product formula for zeta function (Euler)

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### Template for study of L-functions

- “Euler product” generating functions
Product formula for zeta function (Euler)

\[ \sum_{n=0}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1} \quad (\text{Re}(s) > 1) \]

Template for study of L-functions

- “Euler product” generating functions
- connected analysis to prime numbers
Additive-multiplicative correspondence (Euler)

Partition generating function (encodes addition)

\[ \prod_{n=1}^{\infty} \left(1 - q^n\right) = \sum_{n=0}^{\infty} p(n) q^n, \quad |q| < 1 \]

Euler product formula (encodes primes / multiplic.)

\[ \prod_{p \in \mathbb{P}} \left(1 - p^{-s}\right) = \sum_{k=1}^{\infty} \frac{n^{-s}}{k^n}, \quad \text{Re}(s) > 1 \]

Proofs feel similar (multiply geometric series)
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- Proofs feel similar (multiply geometric series)
Modern vistas in partition theory

Alladi–Erdős (1970s)

Bijection between integer factorizations, prime partitions

Study properties of arithmetic functions

Question

Are other theorems in arithmetic images in prime partitions of combinatorial/set-theoretic meta-structures?
Modern vistas in partition theory

Alladi–Erdős (1970s)
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# Modern vistas in partition theory

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Modern vistas in partition theory

Andrews (1970s)

Theory of partition ideals inspired by lattice theory unifies classical results on generating functions, bijections suggestive of a universal algebra of partitions.

Question:
Is there an algebra of partitions generalizing arithmetic in integers (i.e., prime partitions)?
Andrews (1970s)

Theory of partition ideals

Modern vistas in partition theory
Modern vistas in partition theory

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Modern vistas in partition theory
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Modern vistas in partition theory

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Modern vistas in partition theory

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- Theory of partition ideals
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- Unifies classical results on generating functions, bijections
- Suggestive of a *universal algebra of partitions*

**Question**

Is there an *algebra of partitions* generalizing arithmetic in integers (i.e., prime partitions)?
Multiplicative theory of (additive) partitions

Philosophy of this talk

Exist multipl., division, arith. functions on partitions

Objects in classical multiplic. number theory

→ special cases of partition-theoretic structures

Expect arithmetic theorems

→ extend to partitions

Expect partition properties

→ properties of integers
Multiplicative theory of (additive) partitions

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Multiplicative theory of (additive) partitions

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Exist multipl., division, arith. functions on partitions

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Multiplicative theory of (additive) partitions

### Philosophy of this talk

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- Expect arithmetic theorems → extend to partitions
- Expect partition properties → properties of integers
Partition notations

Let $P$ denote the set of all integer partitions.

Let $\emptyset$ denote the empty partition.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, denote a nonempty partition, e.g. $\lambda = (3, 2, 2, 1)$.

Let $P_X$ denote partitions into elements $\lambda_i \in X \subseteq \mathbb{N}$, e.g. $P_P$ is the "prime partitions".
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Partition notations

\[ \ell(\lambda) := \text{length (number of parts)}. \]

\[ m_i(\lambda) := \text{multiplicity (or "frequency") of } i. \]

\[ |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r \text{ is size (sum of parts)}. \]

"\( \lambda \vdash n \)" means \( \lambda \) is a partition of \( n \).

Define \( \ell(\emptyset) = |\emptyset| = m_i(\emptyset) = 0, \emptyset \vdash 0. \)
Partition notations

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Multiplicative theory of (additive) partitions

Define $N(\lambda)$, the norm of $\lambda$, to be the product of the parts:

$$N(\lambda) := \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_r$$

Define $N(\emptyset) := 1$ (it is an empty product).
Multiplicative theory of (additive) partitions

New partition statistic

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See "The product of parts or 'norm' etc." (S-Sills)
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Multiplicative theory of (additive) partitions

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Multiplicative theory of (additive) partitions

Partition multiplication

For $\lambda, \gamma \in \mathcal{P}$ let $\lambda \gamma$ denote multiset union of the parts, e.g. $(3, 2)(2, 1) = (3, 2, 2, 1)$.

Identity is $\emptyset$.

Partitions into one part are like primes, FTA holds.
## Multiplicative theory of (additive) partitions

### Partition multiplication

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Multiplicative theory of (additive) partitions

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Multiplicative theory of (additive) partitions

Partition division (subpartitions)

For $\lambda, \delta \in \mathbb{P}$, let $\delta \mid \lambda$ mean all parts of $\delta$ are parts of $\lambda$, e.g. $(3, 2, 1) \mid (3, 2, 2, 1)$.

For $\delta \mid \lambda$, let $\lambda / \delta$ mean parts of $\delta$ deleted from $\lambda$, e.g. $(3, 2, 2, 1) / (3, 2, 1) = (2)$.

Replace $\mathbb{P}$ with $\mathbb{P}$ mult./div. in $\mathbb{Z}^+$
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Multiplicative theory of (additive) partitions

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Multiplicative theory of (additive) partitions

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Replace \mathcal{P} with \mathcal{P} \rightarrow \text{mult./div. in } \mathbb{Z}_+^{105}
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Multiplicative theory of (additive) partitions

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- Replace $\mathcal{P}$ with $\mathcal{P}_{\mathbb{P}}$. 


Multiplicative theory of (additive) partitions

Partition division (subpartitions)

- For $\lambda, \delta \in \mathcal{P}$, let $\delta | \lambda$ mean all parts of $\delta$ are parts of $\lambda$, e.g. $(3, 2, 1) | (3, 2, 2, 1)$.

- For $\delta | \lambda$, let $\lambda / \delta$ mean parts of $\delta$ deleted from $\lambda$, e.g. $(3, 2, 2, 1) / (3, 2, 1) = (2)$.

- Replace $\mathcal{P}$ with $\mathcal{P}_\mathbb{P} \rightarrow \text{mult./div. in } \mathbb{Z}^+$
Many arithmetic objects have partition counterparts.

Partition Möbius function

For $\lambda \in \mathbb{P}$, define

$$\mu(\lambda) = \begin{cases} 0 & \text{if } \lambda \text{ has any part repeated,} \\ (-1)^{\ell(\lambda)} & \text{otherwise.} \end{cases}$$

Replacing $\mathbb{P}$ with $\mathbb{P}^{\mathbb{P}}$ reduces to $\mu(N(\lambda))$, where $N(\lambda)$ is the norm (product of parts).
Many arithmetic objects have partition counterparts.
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\[ \mu_P(\lambda) := \begin{cases} 0 & \text{if } \lambda \text{ has any part repeated}, \\ (-1)^\ell(\lambda) & \text{otherwise} \end{cases} \]

Replacing \( P \) with \( \overline{P} \) reduces to \( \mu(N(\lambda)) \), where \( N(\lambda) \) is the norm (product of parts).
Many arithmetic objects have partition counterparts.

**Partition Möbius function**

For \( \lambda \in P \), define

\[
\mu_P(\lambda) =
\begin{cases} 
0 & \text{if } \lambda \text{ has any part repeated,} \\
(-1)^{\ell(\lambda)} & \text{otherwise}
\end{cases}
\]

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Replacing $\mathcal{P}$ with $\mathcal{P}_\mathbb{P}$ reduces to $\mu(N(\lambda))$, where $N(\lambda)$ is the norm (product of parts).
Parallel universe

Just as in classical cases, nice sums over “divisors”...

\[ \sum_{\delta} \mu(\lambda) P(\delta) = \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ 0 & \text{otherwise} \end{cases} \]
Just as in classical cases, nice sums over “divisors”...
Parallel universe

Just as in classical cases, nice sums over “divisors”...

Partition Möbius function
Just as in classical cases, nice sums over “divisors”...

\[
\sum_{\delta \mid \lambda} \mu_P(\delta) = \begin{cases} 
1 & \text{if } \lambda = \emptyset, \\
0 & \text{otherwise}
\end{cases}
\]
If we have \( f(\lambda) = \sum \delta |\lambda| g(\delta) \), we also have \( g(\lambda) = \sum \delta |\lambda| \mu P(\lambda/\delta) f(\delta) \).
If we have

\[ f(\lambda) = \sum_{\delta | \lambda} g(\delta) \]

we also have

\[ g(\lambda) = \sum_{\delta | \lambda} \mu_P(\lambda / \delta) f(\delta). \]
Classically, $\mu(n)$ has a close companion in $\varphi(n)$. 
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**Partition phi function**

For $\lambda \in \mathcal{P}$, define

$$\varphi(\lambda) = N(\lambda) \prod_{\lambda_i \in \lambda \text{ no repeats}} (1 - \lambda - 1)^i.$$
Classically, \( \mu(n) \) has a close companion in \( \varphi(n) \).

**Partition phi function**

For \( \lambda \in \mathcal{P} \), define

\[
\varphi_{\mathcal{P}}(\lambda) := N(\lambda) \prod_{\lambda_i \in \lambda \atop \text{no repeats}} (1 - \lambda_i^{-1}).
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Classically, $\mu(n)$ has a close companion in $\varphi(n)$.

**Partition phi function**

For $\lambda \in \mathcal{P}$, define

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- Replacing $\mathcal{P}$ with $\mathcal{P}_\mathcal{P}$ reduces to $\varphi(N(\lambda))$. 
Parallel universe

Phi function identities
Parallel universe

Phi function identities

\[ \sum_{\delta|\lambda} \varphi_{\mathcal{P}}(\delta) = N(\lambda), \quad \varphi_{\mathcal{P}}(\lambda) = N(\lambda) \sum_{\delta|\lambda} \frac{\mu_{\mathcal{P}}(\delta)}{N(\delta)} \]
Parallel universe

Phi function identities

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- Replacing \( \mathcal{P} \) with \( \mathcal{P}_\mathbb{P} \) reduces to classical cases.
Parallel universe

**Phi function identities**

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- Replacing \( \mathcal{P} \) with \( \mathcal{P}_\mathcal{P} \) reduces to classical cases.
- Many analogs of objects in multiplicative # theory...
Parallel universe

Partition Cauchy product

\[
\left( \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{\lambda} \right) \left( \sum_{\lambda \in \mathcal{P}} g(\lambda) q^{\lambda} \right) = \sum_{\lambda \in \mathcal{P}} q^{\lambda} \sum_{\delta | \lambda} f(\delta) g(\lambda/\delta)
\]
Parallel universe

Partition Cauchy product

\[
\left( \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{\lambda_1} \right) \left( \sum_{\lambda \in \mathcal{P}} g(\lambda) q^{\lambda_1} \right) = \sum_{\lambda \in \mathcal{P}} q^{\lambda_1} \sum_{\delta | \lambda} f(\delta) g(\lambda / \delta)
\]

Swapping order of summation

\[
\sum_{\lambda \in \mathcal{P}} f(\lambda) \sum_{\delta | \lambda} g(\delta) = \sum_{\lambda \in \mathcal{P}} g(\lambda) \sum_{\gamma \in \mathcal{P}} f(\lambda \gamma)
\]
Parallel universe

Partition Cauchy product

\[
\left( \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{\lambda} \right) \cdot \left( \sum_{\lambda \in \mathcal{P}} g(\lambda) q^{\lambda} \right) = \sum_{\lambda \in \mathcal{P}} q^{\lambda} \sum_{\delta | \lambda} f(\delta) g(\lambda / \delta)
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Other multiplicative objects generalize to partition theory...
Partition zeta functions

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^{-s}} \]

**Definition**

For \( P' \subsetneq P \), \( s \in \mathbb{C} \), define a partition zeta function

\[ \zeta_{P'}(s) := \sum_{\lambda \in P'} N(\lambda)^{-s} \]

where \( N(\lambda) \) is the norm (product of parts) of \( \lambda \), \( N(\emptyset) := 1 \).

For \( 1 \not\in P' = P \setminus \{ \text{parts in } X \subset \mathbb{N} \} \rightarrow \) Euler product:

\[ \zeta_{P,X}(s) = \prod_{n \in X} \left( 1 - n^{-s} \right) \]
In analogy with $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$:

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- For $1 \notin \mathcal{P}' = \mathcal{P}_X$ (parts in $X \subset \mathbb{N}$) → Euler product:

$$\zeta_{\mathcal{P}_X}(s) = \prod_{n \in X} (1 - n^{-s})^{-1}$$
Fix $s = 2$, vary subset $\mathcal{P}'$.
Fix $s = 2$, vary subset $\mathcal{P}'$

Summing over partitions into prime parts:

$$\zeta_{\mathcal{P}_p}(2) = \zeta(2) = \frac{\pi^2}{6}$$

Summing over partitions into even parts:

$$\zeta_{\mathcal{P}_{\text{even}}}(2) = \frac{\pi}{2}$$

Summing over partitions into distinct parts:

$$\zeta_{\mathcal{P}_{\text{distinct}}}(2) = \frac{\sinh \pi}{\pi}$$
Partition zeta functions: nice identities

Partition analogs of classical identities

\[
\sum_{\lambda \in \mathcal{P}_X} \mu_{\mathcal{P}}(\lambda) N(\lambda)^{-s} = \frac{1}{\zeta_{\mathcal{P}_X}(s)}
\]

\[
\sum_{\lambda \in \mathcal{P}_X} \varphi_{\mathcal{P}}(\lambda) N(\lambda)^{-s} = \frac{\zeta_{\mathcal{P}_X}(s-1)}{\zeta_{\mathcal{P}_X}(s)}
\]

Takeaway from these examples

Different subsets of \( \mathcal{P} \) induce very different zeta values.

Classical zeta theorems \( \rightarrow \) partition zeta theorems.
Partition zeta functions: nice identities

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Takeaway from these examples

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- Classical zeta theorems \( \rightarrow \) partition zeta theorems
Pretty cool, but it would be a whole lot cooler if we had families of identities like Euler’s zeta values
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Question
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Do there exist (non-Riemann) partition zeta functions such that, for the “right” choice of \( P' \subsetneq P \)
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**Question**

Do there exist (non-Riemann) partition zeta functions such that, for the “right” choice of \( P' \not\subset P \),

\[ \zeta_{P'}(N) = \pi^M \times \text{rational number} \]
Partition zeta functions: nice family

Partitions of fixed length $k$

Define sum over partitions of fixed length $\ell(\lambda) = k$:

$$\zeta_P(\{s\}_k) := \sum_{\lambda \in P} \lambda - s(\text{Re}(s) > 1)$$
Partition zeta functions: nice family

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$$\zeta_P(\{s\}^k) \ := \sum_{\lambda \in \mathcal{P} \atop \ell(\lambda) = k} N(\lambda)^{-s} \quad (\text{Re}(s) > 1)$$
Theorem (S, 2016)

Summing over partitions of fixed length $k > 0$:

$$\zeta_P(\{2\}_k) = \pi^2 k \times \text{rational number}$$

Note: $k = 0$ suggests $\zeta(0) = -\frac{1}{2}$ (correct value)

Theorem (Ono-Rolen-S, 2017)

For $N \geq 1$:

$$\zeta_P(\{2N\}_k) = \pi^2 Nk \times \text{rational number}$$
Theorem (S, 2016)

Summing over partitions of fixed length $k > 0$:

$$\zeta_P(\{2\}^k) = \frac{2^{2k-1} - 1}{2^{2k-2}} \zeta(2k) = \pi^{2k} \times \text{rational number}$$
Partition zeta functions: nice family

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Some *other* partition zeta fcts. contin. to right half-plane.
Partition zeta functions: nice family

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Some *other* partition zeta functions continue to the right half-plane.

**Natural questions**
Partition zeta functions: nice family

**Theorem (Ono-Rolen-S., 2017)**

Some *other* partition zeta functions continue to the right half-plane.

**Natural questions**

General $\zeta_P(\{s\}^k)$? Analytic continuation? Poles? Roots?
Theorem (S.-Sills, 2020)

For $\Re(s) > 1$:

$$\zeta_P(\{s\}_k) = \sum_{\lambda \vdash k} \zeta(\lambda \mu_1 \zeta(2s) \mu_2 \zeta(3s) \mu_3 \cdots \zeta(ks) \mu_k \mathcal{N}(\lambda) \mu_1! \mu_2! \mu_3! \cdots \mu_k!$$

sum over partitions $\lambda$ of size $k$ on RHS inherits continuation from $\zeta(\lambda \mu_1 \cdots \zeta(ks) \mu_k)$.

poles at $s = 1, 1/2, 1/3, 1/4, \ldots, 1/k$.

trivial roots at $s \in -2\mathcal{N}$.

Note: $\zeta_P(\{s\}_k)$ not zero at roots of $\zeta(\lambda \mu_1 \cdots \zeta(ks) \mu_k)$ for $k > 1$. 


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For $\Re(s) > 1$:

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- sum over partitions \( \lambda \) of size \( k \) on RHS
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- sum over partitions $\lambda$ of size $k$ on RHS
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\]

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*Note:* $\zeta_P(\{s\}^k)$ not zero at roots of $\zeta(s)$ for $k > 1$
MacMahon-partition zeta correspondence

Proof mimics Sills’ combinatorial proof (2019) of MacMahon’s partial fraction decomposition...
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MacMahon’s partial fraction decomposition

For $|q| < 1$:

$$\prod_{n=1}^{k} (1 - q^n)^{-1} = \sum_{\lambda|k} \frac{(1 - q^2)^{-m_2} \cdots q^{km_k} (1 - q^k)^{-m_k}}{N(\lambda) \ m_1! \ m_2! \cdots \ m_k!}$$

Although here we really want to use...
MacMahon's partial fraction decomposition times $q^k$

\[ q^k \prod_{n=1}^{k} (1 - q^n)^{-1} = \sum_{\ell(\lambda) = k} q^{\lambda} \]

\[ = \sum_{\lambda \vdash k} \frac{q^{m_1}(1 - q)^{-m_1} \cdot q^{2m_2}(1 - q^2)^{-m_2} \cdots q^{km_k}(1 - q^k)^{-m_k}}{N(\lambda) \ m_1! \ m_2! \ \cdots \ \ m_k!} \]
MacMahon's partial fraction decomposition times $q^k$

$$q^k \prod_{n=1}^{k} (1 - q^n)^{-1} = \sum_{\ell(\lambda) = k} q^{\ell(\lambda)}$$

$$= \sum_{\lambda \vdash k} \frac{q^m (1 - q)^{-m_1} \cdot q^{2m_2} (1 - q^2)^{-m_2} \cdots q^{km_k} (1 - q^k)^{-m_k}}{N(\lambda) \cdot m_1! \cdot m_2! \cdots m_k!}$$

- LHS generates all partitions with largest part $k$
MacMahon’s partial fraction decomposition times $q^k$

\[
q^k \prod_{n=1}^{k} (1 - q^n)^{-1} = \sum_{\ell(\lambda) = k} q^{\lambda} = \sum_{\lambda \vdash k} q^{\lambda} \prod_{n=1}^{k} (1 - q^n)^{-1} \prod_{m_1}^{m_k} (1 - q^{m_i})^{-m_i} \cdot (1 - q^{k})^{-m_k} \cdot \frac{N(\lambda)}{m_1! \cdot m_2! \cdots m_k!}
\]

- LHS generates all partitions with largest part $k$
- Conjugation $\rightarrow$ also generates partitions with length $k$
MacMahon's partial fraction decomposition times $q^k$

$$q^k \prod_{n=1}^{k} (1 - q^n)^{-1} = \sum_{\ell(\lambda) = k} q^{||\lambda||}$$

$$= \sum_{\lambda \vdash k} \frac{q^{m_1}(1 - q)^{-m_1} \cdot q^{2m_2}(1 - q^2)^{-m_2} \cdots q^{km_k}(1 - q^k)^{-m_k}}{N(\lambda) \ m_1! \ m_2! \cdots \ m_k!}$$

- LHS generates all partitions with largest part $k$
- Conjugation $\rightarrow$ also gen. partitions w/ length $k$
- RHS = comb. geom. series over partitions of size $k$
MacMahon-partition zeta correspondence

MacMahon’s partial fraction decomposition times $q^k$

$$q^k \prod_{n=1}^{k} (1 - q^n)^{-1} = \sum_{\ell(\lambda)=k} q^{|\lambda|}$$

$$= \sum_{\lambda \vdash k} \frac{q^{m_1} (1 - q)^{-m_1} \cdot q^{2m_2} (1 - q^2)^{-m_2} \cdots q^{km_k} (1 - q^k)^{-m_k}}{N(\lambda) \cdot m_1! \cdot m_2! \cdots m_k!}$$

Compare and contrast

$$\zeta_P(\{s\}^k) = \sum_{\ell(\lambda)=k} N(\lambda)^{-s} = \sum_{\lambda \vdash k} \frac{\zeta(s)^{m_1} \zeta(2s)^{m_2} \cdots \zeta(ks)^{m_k}}{N(\lambda) \cdot m_1! \cdot m_2! \cdots m_k!}$$
### MacMahon-partition zeta correspondence

<table>
<thead>
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<th>Analogous zeta fctn component</th>
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</table>

Multiplication of terms of either shape $q^n$ or $n^{-s}$ generates partitions in exactly the same way:

$$q^{\lambda_1} q^{\lambda_2} q^{\lambda_3} \ldots q^{\lambda_r} = q^{|\lambda|} \longleftrightarrow \lambda_1^{-s} \lambda_2^{-s} \lambda_3^{-s} \ldots \lambda_r^{-s} = N(\lambda)^{-s}$$
<table>
<thead>
<tr>
<th>Gen fctn component</th>
<th>Analogous zeta fctn component</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^{\lambda}$</td>
<td>$N(\lambda)^{-s}$</td>
</tr>
<tr>
<td>$\frac{q^i}{1-q^i}$</td>
<td>$\zeta(js)$</td>
</tr>
<tr>
<td>$\frac{q^k}{\prod_{j=1}^{k}(1-q^j)}$</td>
<td>$\zeta_P({s}^k)$</td>
</tr>
</tbody>
</table>

The term $q^{jn}$ in geom series $\sum_{n=1}^{\infty} q^{jn}$ and resp. term $n^{-js}$ of $\zeta(js)$ both encode partition $(n)^j := (n, n, ..., n)$ ($j$ times):

$$\frac{q^i}{1-q^i} = \sum_{n=1}^{\infty} q^{\|(n)^j\|} \iff \zeta(js) = \sum_{n=1}^{\infty} N((n)^j)^{-s}.$$
Correspondence says $q^j$ and $\zeta(js)$ are interchangeable as gen fctns for partitions $(n, n, ..., n)$ ($j$ reps / same part).
MacMahon-partition zeta correspondence

Geometric series-zeta function duality

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Multiplicative theory of (additive) partitions

Applications

New combinatorial bijections
Computing coefficients of $q$-series, mock mod. forms
Statistical physics
Computational chemistry
Computing arithmetic densities
Computing $\pi$ (quite inefficiently, to boot!)
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- New combinatorial bijections
- Computing coefficients of $q$-series, mock mod. forms
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- Computational chemistry
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Definition

The arithmetic density of a subset \( S \subseteq \mathbb{Z}^+ \) is
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\lim_{N \to \infty} \frac{\#\{ n \in S \mid n \leq N \}}{N},
\]
if the limit exists.

Examples

Integers ≡ \( r \) (mod \( t \)) have density \( 1/t \).

Square-free integers have density \( 6/\pi^2 = 1/\zeta(2) \).
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Integers $\equiv r \pmod{t}$ have density $1/t$.

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- Integers $\equiv r \pmod{t}$ have density $1/t$
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Well-known relation between arithmetic density and zeta-type sums

If a subset $T \subseteq \mathbb{P}$ has arithmetic density in $\mathbb{P}$, its density is equal to the Dirichlet density of $T$.

$$\lim_{s \to 1} \sum_{p \in T} p^{-s} - \sum_{p \in \mathbb{P}} p^{-s}$$
Application: arithmetic densities

Classical computation

- Well-known relation between arithmetic density and zeta-type sums
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$$\lim_{s \to 1} \frac{\sum_{p \in T} p^{-s}}{\sum_{p \in \mathbb{P}} p^{-s}}$$
Theorem (Ono-S-Wagner, 2018)

The arith. density of integers \( r \mod t \) is equal to

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"sm" is the smallest part of \( \lambda \) extends work of Alladi (1977), Locus Dawsey (2017).
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Theorem (Ono-S-Wagner, 2018)

The arith. density of $k$th power-free integers is equal to

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**Proofs**

$q$-binomial thm + partition bijection + complex analysis
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another partition-zeta connection
Application: arithmetic densities

Theorem (Ono-S-Wagner, 2020)

For $d_S$ arith. density of a $q$-commensurate subset $S \subseteq \mathbb{N}$:

$$- \lim_{q \to 1} \sum_{\substack{\lambda \in \mathcal{P} \\ \text{sm}(\lambda) \in S}} \mu_{\mathcal{P}}(\lambda) q^{|\lambda|} = d_S.$$
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- here $\ast$ is partition convolution, $\varphi(n)$ classical phi
- second formula extends work of Wang (2020)
Application: arithmetic densities

Theorem (Ono-S-Wagner, 2020)

Proof requires a new theory of \( q \)-series density computations based on “\( q \)-density” statistic.:

\[
d_S(q) := \frac{\sum_{\sm(\lambda) \in S} q^{\lambda}}{\sum_{\lambda} q^{\lambda}} = (1 - q) \sum_{\sm(\lambda) \in S} q^{\lambda}.
\]

<table>
<thead>
<tr>
<th>Natural number ( n )</th>
<th>Partition ( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prime factors of ( n )</td>
<td>Parts of ( \lambda )</td>
</tr>
<tr>
<td>Square-free integers</td>
<td>Partitions into distinct parts</td>
</tr>
<tr>
<td>( \mu(n) )</td>
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<tr>
<td>( p_{\min}(n) )</td>
<td>( \sm(\lambda) )</td>
</tr>
<tr>
<td>( p_{\max}(n) )</td>
<td>( \lg(\lambda) )</td>
</tr>
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<td>( n^{-s} )</td>
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Partition-theoretic multiverse

Philosophy of this talk (again)

Exist multipl., division, arith. functions on partitions

→

Objects in classical multiplic. number theory
→ extend to partitions

Expect arithmetic theorems
→

Expect partition properties
→ properties of integers

Work in progress

With Akande, Beckwith, Dawsey, Hendon, Jameson, Just, Ono, Rolen, M. Schneider, Sellers, Sills, Wagner, ... you?
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Thank you for listening :)

Gratitude