Indefinite Zeta Functions

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Outline

Part One: Hilbert's 12th Problem
My personal motivation for this work

Part Two and Three: Indefinite Theta and Indefinite Zeta
A general construction with potential applications beyond the motivating problem

Part Four: Kronecker Limit Formulas
An application of the general construction to one aspect of the motivating problem

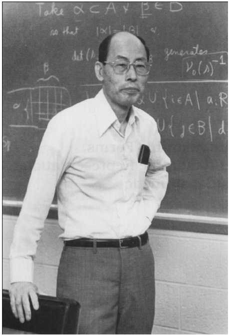
Part One: Hilbert’s 12th Problem

Hilbert's 12th problem

- List of 23 open problems published in 1900
- 12th problem asks for an "Extension of Kronecker's Theorem on Abelian Fields to any Algebraic Realm of Rationality."
- Kronecker's Theorem (Kronecker-Weber theorem) says that the abelian extensions of $\mathbb{Q}$ are generated by the values of $e(z) = e^{2\pi i z}$ at rational values of z.
- Given any base field ("realm of rationality"), Hilbert wanted "analytic functions" that play the role of $e(z)$.

Class field theory

- The imaginary quadratic case was mostly known to Hilbert and uses the theory of elliptic curves with complex multiplication (CM), due to Weber and others.
- Abstract class field theory, developed during the 1910s and 1920s by Takagi and others, constructs class fields in an indirect manner.
- Goro Shimura generalized CM theory to "CM base fields" by replacing elliptic curves with abelian varieties.



Stark conjectures

- Introduced 1971–1980 by Harold Stark
- Artin L-function $L(s, \rho)$ for irrep $\rho : \mathrm{Gal}(L/K) \to \mathbf{GL}_n(\mathbb{C})$
- Taylor series at $s = 0$: $L(s, \rho) = c_r s^r + \cdots$
- Leading coefficient c_r conjectured to be a product of an algebraic number and a "Stark regulator", a determinant of an $r \times r$ matrix of linear forms of logarithms of algebraic units.
- If L/K is an abelian, $L(s, \rho) = L(s, \chi)$ is a Hecke L-function—specified by data internal to K.
- Units are predicted to live in the corresponding class field.
- Partial answer to Hilbert's 12th problem in the "rank 1" case ($r = 1$), when we can recover the Stark units by exponentiation.
- The rank 1 abelian Stark conjectures remain open for any real quadratic field, e.g., $\mathbb{Q}(\sqrt{3})$.

This formula can be proved using calculus. Try it! Hint: Replace $\frac{1}{n}$ with $\frac{x^n}{n}$ and take a derivative.

Example

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} + - - + \cdots = \frac{1}{\sqrt{2}} \log\left(1 + \sqrt{2}\right)$$

The left-hand side is the value $L(1, \chi)$, where $\chi(n) = \left(\frac{2}{n}\right)$ is the Dirichlet character associated to the field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$. The right-hand side involves $\varepsilon = 1 + \sqrt{2}$, the fundamental unit of $\mathbb{Q}(\sqrt{2})$.

This formula is proved using the theory of complex multiplication for elliptic curves. The notation $e(z) := e^{2\pi iz}$.

Example

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{e(m/5) - e(2m/5)}{m^2 + mn + n^2} = \frac{2\pi}{\sqrt{3}} \log\left(\varepsilon^{1/5}\right)$$

where $\varepsilon = 29 + 12\sqrt{5} + 2\sqrt{6(65 + 29\sqrt{5})}$.

The left-hand side is a linear combination of Hecke L-values at $s = 1$ for $\mathbb{Q}(\sqrt{-3})$. The right-hand side involves an algebraic unit ε in the ray class field modulo (5) for $\mathbb{Q}(\sqrt{-3})$.

This example is related to the 5-torsion points of the elliptic curve $y^2 = x^3 + 1$.

This formula is an open conjecture!

Example

$$\sum_{m=1}^{\infty} \sum_{\substack{n \in \mathbb{Z} \\ -\frac{5}{3}m \leq n < \frac{5}{3}m}} \frac{e(4m/5) - e(m/5)}{3m^2 - n^2} = \frac{\pi}{i\sqrt{3}} \log(\varepsilon),$$

where $\varepsilon \approx 3.890861714$ is a root of the polynomial equation

$$\begin{aligned} x^8 &- (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 \\ &+ (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 \\ &- (8+5\sqrt{3})x + 1 = 0. \end{aligned}$$

The number ε is an algebraic unit in the narrow ray class field of $\mathbb{Q}(\sqrt{3})$ modulo 5.

Kronecker limit formulas

In the imaginary quadratic case (where Stark is known), *L*-values relate to special values of modular forms by Kronecker limit formulas.

The first limit formula is for the real analytic Eisenstein series (which specialize to linear combinations of Hecke *L*-functions of conductor 1 when τ is imaginary quadratic),

$$E(\tau, s) := \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{\mathrm{Im}(\tau)^s}{|m\tau + n|^{2s}}.$$

Theorem (Kronecker first limit formula)

$$\lim_{s\to 1}\left(E(\tau, s) - \frac{\pi}{s-1}\right) = \gamma - 2\log\left|2\sqrt{\mathrm{Im}(\tau)}\eta(\tau)\right|.$$

Here, $\eta(\tau)$ is the Dedekind eta function (a modular form of weight $\frac{1}{2}$), and γ is the Euler-Mascheroni constant.

Kronecker limit formulas

The second limit formula is for the twisted real analytic Eisenstein series for $(p_1, p_2) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ (which specialize to to linear combinations of Hecke L-functions of conductor N when τ is imaginary quadratic and $p_1, p_2 \in \frac{1}{N}\mathbb{Z}$),

$$E^*_{p_1,p_2}(\tau, s) := \frac{1}{2}\pi^{-s}\Gamma(s) \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} e\left(p_1 m + p_2 n\right) \frac{\operatorname{Im}(\tau)^s}{|m\tau + n|^{2s}}.$$

Theorem (Kronecker second limit formula)

$$E^*_{p_1,p_2}(\tau, 1) = -2 \log \left| \frac{\vartheta_{\frac{1}{2}+p_1, \frac{1}{2}-p_2}(\tau)}{\eta(\tau)} \right|.$$

Here ϑ is the Jacobi theta function, η is the Dedekind eta function, and the expression inside the absolute value is a modular function for $\Gamma(N)$ whenever $p_1, p_2 \in \frac{1}{N}\mathbb{Z}$.

Kronecker limit formulas for real quadratic fields

- One approach to the Stark conjectures is to find Kronecker limit formula for real quadratic fields.
- Hecke (1917), Herglotz (1923), Zagier (1975) found analogues of first limit formula.

- Shintani (1976) found an analogue of the second limit formula...
- ...and proved (1978) a special case of the Stark conjectures.
- All (except Hecke) interpolate between zeta functions in a similar way.

Kronecker limit formulas for real quadratic fields

- We introduce a new way of interpolating that preserves the functional equation...
- ...and obtain a new Kronecker limit formula (analogous to second).
- Gives a new, fast-converging analytic formula for (presumptive) Stark units...
- ...but does not (yet) help with proving algebraicity.

Part Two: Indefinite Theta Functions

Zwegers's thesis

- Indefinite theta functions were introduced in the PhD thesis of Sander Zwegers in 2002.
- He used them to build harmonic weak Maass forms whose holomorphic parts are Ramanujan's mock theta functions.
- Zwegers's work, and subsequent work of Kathrin Bringmann and Ken Ono, led to a renaissance in mock modular forms.

Generalizing Zwegers's theta

Let M be a real symmetric matrix of signature $(g-1,1)$ and $c_1, c_2 \in \mathbb{R}^g$ satisfying $c_j^\top A c_j < 0$. Zwegers's indefinite theta function is $\vartheta_M^{c_1,c_2}(z,\tau)$ for $z \in \mathbb{C}^g$ and $\tau \in \mathcal{H}$. We generalize it by...

- Replacing τM with a symmetric matrix $\Omega = N + iM$ such that M has signature $(g-1,1)$. Fairly straightforward.
- Allowing c_1, c_2 to take complex values. Not straightforward.
- To get a good transformation theory, the latter is required to once we do the former.

Definition (Siegel intermediate half-space)

For $0 \leq k \leq g$, we define the *Siegel intermediate half-space* of genus g and index k to be

$$\mathcal{H}_g^{(k)} = \{\Omega \in \mathbf{M}_g(\mathbb{C}) : \Omega = \Omega^\top \text{ and } \operatorname{Im}(\Omega) \text{ has signature } (g-k, k)\}.$$

Symplectic group action

The symplectic group $\mathbf{Sp}_{2g}(\mathbb{R})$ acts on the set of $g \times g$ complex symmetric matrices $\Omega = N + iM$ with M invertible, by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}.$$

Proposition

The $\mathcal{H}_g^{(k)}$ are the open orbits of this $\mathbf{Sp}_{2g}(\mathbb{R})$-action.

Moduli problem

- The space $\mathcal{H}_g^{(0)} / \mathbf{Sp}_2(\mathbb{Z})$ is the moduli space of principally polarized abelian varieties of dimension g.
- For $k > 0$, the points of $\mathcal{H}_g^{(k)} / \mathbf{Sp}_2(\mathbb{Z})$ correspond to certain non-algebraic complex tori of dimension g...
- ...except the action of $\mathbf{Sp}_2(\mathbb{Z})$ is not properly discontinuous.
- When $k = 1$, one may fix the action by adding an auxiliary parameter $c \in \mathbb{P}^{g-1}(\mathbb{C})$ such that $\overline{c}^\top \operatorname{Im}(\Omega) c < 0$ and letting $\mathbf{Sp}_2(\mathbb{Z})$ act on (Ω, c) by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot (\Omega, c) = \left((A\Omega + B)(C\Omega + D)^{-1}, (C\overline{\Omega} + D)c\right).$$

- Also works for (Ω, c_1, c_2).

Definition of definite (Riemann) theta function

This function was defined by Riemann and is known as the Riemann theta function.

Definition (Definite theta function)

Let $z \in \mathbb{C}^g$ and $\Omega = N + iM \in \mathcal{H}_g^{(0)}$. Define

$$\Theta(z;\Omega) = \sum_{n \in \mathbb{Z}^g} e\left(\frac{1}{2} n^\top \Omega n + n^\top z\right).$$

The sum will only converge if the bilinear form $Q_M(n) = \frac{1}{2} n^\top M n$ is positive definite—that is, if $\Omega \in \mathcal{H}_g^{(0)}$.

Definition (Incomplete Gaussian integral)

For any $\alpha \in \mathbb{C}$, set $\mathcal{E}(\alpha) = \displaystyle\int_0^\alpha e^{-\pi u^2}\, du.$

Definition (Indefinite theta function)

Let $z \in \mathbb{C}^g$ and $\Omega = N + iM \in \mathcal{H}_g^{(1)}$. Take $c_1, c_2 \in \mathbb{C}^g$ such that $\overline{c_j}^\top M c_j < 0$. Define

$$\Theta^{c_1,c_2}(z;\Omega) = \sum_{n \in \mathbb{Z}^g} \mathcal{E}\left(\frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-\frac{1}{2} c^\top M c}} \right) \Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2} n^\top \Omega n + n^\top z \right).$$

Note: $f(c)|_{c=c_1}^{c_2}$ means $f(c_2) - f(c_1)$.

Proposition (K)

The series defining the indefinite theta function converges absolutely.

We switch to a different notation because it will make our formulas nicer.

Definition (Definite theta null with real characteristics)

For "characteristics" $p, q \in \mathbb{R}^g$ and $\Omega \in \mathcal{H}_g^{(0)}$, set

$$\Theta_{p,q}(\Omega) = e\left(\frac{1}{2}q^\top \Omega q + p^\top q\right) \Theta\left(p + \Omega q, \Omega\right).$$

Definition (Indefinite theta null with real characteristics)

For "characteristics" $p, q \in \mathbb{R}^g$, $\Omega \in \mathcal{H}_g^{(1)}$, and $c_1, c_2 \in \mathbb{C}^g$ such that $\overline{c_j}^\top \operatorname{Im}(\Omega) c_j < 0$, set

$$\Theta_{p,q}^{c_1,c_2}(\Omega) = e\left(\frac{1}{2}q^\top \Omega q + p^\top q\right) \Theta^{c_1,c_2}\left(p + \Omega q; \Omega\right).$$

Let $p, q \in \mathbb{R}^g$ and $a, b \in \mathbb{Z}^g$.

Proposition (Elliptic trans., definite case; classical)

For $\Omega \in \mathcal{H}_g^{(0)}$,

$$\Theta_{p+a,q+b}(\Omega) = e\left(a^\top (q+b)\right) \Theta_{p,q}(\Omega).$$

Proposition (Elliptic trans., indefinite case; K)

For $\Omega \in \mathcal{H}_g^{(1)}$, $c_1, c_2 \in \mathbb{C}^g$, *and* $\overline{c_j}^\top \operatorname{Im}(\Omega) c_j < 0$,

$$\Theta_{p+a,q+b}^{c_1,c_2}(\Omega) = e\left(a^\top (q+b)\right) \Theta_{p,q}^{c_1,c_2}(\Omega).$$

If you ignore the c_j, these two equations are exactly the same.

Proposition (Modular trans., definite case; classical)

Let $\Omega \in \mathcal{H}_g^{(0)}$ and $p, q \in \mathbb{R}^g$. Let $A \in \mathbf{GL}_g(\mathbb{Z})$, $B \in \mathbf{M}_g(\mathbb{Z})$, and $B = B^\top$.

(1) $\Theta_{p,q}(A^\top \Omega A) = \Theta_{A^{-\top}p, Aq}(\Omega)$.

(2) $\Theta_{p,q}(\Omega + 2B) = e(-q^\top Bq)\Theta_{p+2Bq,q}(\Omega)$.

(3) $\Theta_{p,q}\left(-\Omega^{-1}\right) = \dfrac{e\big(p^\top q\big)}{\sqrt{\det(i\Omega^{-1})}}\Theta_{-q,p}(\Omega)$.

Theorem (Modular trans., indefinite case; K)

Let $\Omega = N + iM \in \mathcal{H}_g^{(1)}$, $c_1, c_2 \in \mathbb{C}^g$ *such that* $\overline{c_j}^\top M c_j < 0$, *and* $p, q \in \mathbb{R}^g$. *Let* $A \in \mathbf{GL}_g(\mathbb{Z})$, $B \in \mathbf{M}_g(\mathbb{Z})$, *and* $B = B^\top$.

(1) $\Theta_{p,q}^{c_1,c_2}(A^\top \Omega A) = \Theta_{A^{-\top}p,Aq}^{Ac_1,Ac_2}(\Omega)$.

(2) $\Theta_{p,q}^{c_1,c_2}(\Omega + 2B) = e(-q^\top Bq)\Theta_{p+2Bq,q}^{c_1,c_2}(\Omega)$.

(3) $\Theta_{p,q}^{c_1,c_2}(-\Omega^{-1}) = \frac{e(p^\top q)}{\sqrt{\det(i\Omega^{-1})}}\Theta_{-q,p}^{-\overline{\Omega}^{-1}c_1,-\overline{\Omega}^{-1}c_2}(\Omega)$.

The case when N is a constant multiple of M and $c_1, c_2 \in \mathbb{R}^g$ is due to Zwegers. If you ignore the c_j, these are exactly the same equations as on the previous slide.

Part Three: Indefinite Zeta Functions

Definition

Let $p, q \in \mathbb{R}^g$ and $\Omega \in \mathcal{H}_g^{(0)}$. For $\mathrm{Re}(s) > 1$, define

$$\hat{\zeta}_{p,q}(\Omega, s) = \int_0^\infty \Theta_{p,q}(t\Omega) t^s \frac{dt}{t}.$$

Definition

Let $p, q \in \mathbb{R}^g$ and $\Omega \in \mathcal{H}_g^{(1)}$. Consider $c_1, c_2 \in \mathbb{C}^g$ such that $\overline{c_j}^\top \operatorname{Im}(\Omega) c_j < 0$. For $\mathrm{Re}(s) > 1$, define

$$\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) = \int_0^\infty \Theta_{p,q}^{c_1,c_2}(t\Omega) t^s \frac{dt}{t}.$$

Theorem (Analytic continuation; K)

For any choice of $r > 0$, the following expression is an analytic continuation of $\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s)$ to the entire s-plane.

$$\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s) = \int_r^\infty \Theta^{c_1,c_2}_{p,q}(t\Omega)t^s \frac{dt}{t} + \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \int_{r^{-1}}^\infty \Theta^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}\left(t\left(-\Omega^{-1}\right)\right) t^{\frac{g}{2}-s} \frac{dt}{t}.$$

I have used this formula for computer calculations, as it may be used to compute the indefinite zeta function to arbitrary precision in polynomial time.

Symmetry about the line $s = \frac{g}{2}$.

Corollary (Functional equation; K)

$$\hat{\zeta}_{p,q}^{c_1,c_2}\left(\Omega, \frac{g}{2} - s\right) = \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \hat{\zeta}_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}\left(-\Omega^{-1}, s\right).$$

Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let $\mathfrak{c}$ be a ideal in $\mathcal{O}_K$, and let S be a subset of the real embeddings of K.

Definition (Ray class group modulo $\{\mathfrak{c}, S\}$)

$$\mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K) = \frac{\{\text{fractional ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{c}\}}{\{a\mathcal{O}_K \text{ s.t. } a \equiv 1 \pmod{\mathfrak{c}} \text{ and } \rho(a) > 0 \text{ for } \rho \in S\}}$$

Zeta functions associated to ray classes

Definition

For $A \in \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$, the associated zeta function is

$$\zeta(s, A) = \sum_{\substack{\mathfrak{a} \leq \mathcal{O}_K \\ \mathfrak{a} \in A}} N(\mathfrak{a})^{-s}.$$

Let $R \in \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$ be the ideal class

$$R = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{c}} \text{ and } \rho(a) > 0 \text{ for } \rho \in S\}.$$

Definition

For $A \in \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$, the associated differenced zeta function is

$$Z_A(s) = \zeta(s, A) - \zeta(s, RA).$$

Real quadratic $Z_A(s)$ as an indefinite zeta function

Let K be a real quadratic field and $\mathfrak{c}$ a nonzero ideal in $\mathcal{O}_K$.

Theorem (Specialization of indefinite zeta; K)

For each $A \in \mathrm{Cl}_{\mathfrak{c},\{\infty_1,\infty_2\}}$ and integral ideal $\mathfrak{b} \in A^{-1}$, there exist $c_1, c_2 \in \mathbb{Q}^2$, M a rational symmetric matrix of signature $(1,1)$, and $q \in \mathbb{Q}^2$ such that,

$$(2\pi N(\mathfrak{b}))^{-s}\Gamma(s)Z_A(s) = \hat{\zeta}_{0,q}^{c_1,c_2}(iM, s).$$

The rank 1 Stark conjecture predicts that, for $B \in \mathrm{Cl}_{\mathfrak{c},\{\infty_2\}}$, $Z_B'(0) = \log(\varepsilon_B)$ for some algebraic unit ε_B generating a particular abelian extension of K.

Example

- Let $K = \mathbb{Q}(\sqrt{3})$, so $\mathcal{O}_K = \mathbb{Z}[\sqrt{3}]$, and let $\mathfrak{c} = 5\mathcal{O}_K$.
- The ray class group $\mathrm{Cl}_{\mathfrak{c},\{\infty_2\}} \cong \mathbb{Z}/8\mathbb{Z}$. Let I be the identity.
- The ray class group $\mathrm{Cl}_{\mathfrak{c},\{\infty_1,\infty_2\}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. Write $I = I_+ \sqcup I_-$, where I_+ is the identity element of $\mathrm{Cl}_{\mathfrak{c},\{\infty_1,\infty_2\}}$.
- We have $Z_I(s) = Z_{I_+}(s) + Z_{I_-}(s)$. But it turns out that $Z_{I_-}(s)$ is identically zero in this case, so $Z_I(s) = Z_{I_+}(s)$.
- For $q = \frac{1}{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $P = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$,

$$\begin{aligned} Z_I'(0) &= Z_{I_+}'(0) \\ &= \hat{\zeta}_{0,q}^{c_1,P^3c_1}(iM,0) \\ &= \hat{\zeta}_{0,q}^{c_1,Pc_1}(iM,0) + \hat{\zeta}_{0,q}^{Pc_1,P^2c_1}(iM,0) + \hat{\zeta}_{0,q}^{P^2c_1,P^3c_1}(iM,0) \\ &= \hat{\zeta}_{0,q_0}^{c_1,Pc_1}(iM,0) + \hat{\zeta}_{0,q_1}^{c_1,Pc_1}(iM,0) + \hat{\zeta}_{0,q_2}^{c_1,Pc_1}(iM,0), \end{aligned}$$

where $q_0 = \frac{1}{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q_1 = \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $q_2 = \frac{1}{5}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Example

- Using this, we can calculate $Z_l'(0) \approx 1.3586306534$ and $\exp(Z_l'(0)) \approx 3.8908617139$—apparently the root of the degree 8 polynomial we saw earlier.

$$
\begin{aligned}
x^8 &- (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 \\
&+ (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 \\
&- (8+5\sqrt{3})x + 1 = 0.
\end{aligned}
$$

- Indeed, the polynomial was found (as a factor of a degree 16 integer polynomial found) by the Mathematica `RootApproximant[]` function after calculating $\exp(Z_l'(0))$ to 100 decimal places. We then checked in Magma that it generates the expected class field.

Part Four: Kronecker Limit Formulas

Kronecker limit formula for definite zeta functions

Let $p_1, p_2 \in \mathbb{R}^2$ with $0 \le p_1, p_2 < 1$. For $\tau \in \mathcal{H}$, set

$$f_{p_1,p_2}(\tau) = e\left(-\frac{p_2}{2}\right) u_\tau^{\frac{p_1^2}{2}+\frac{1}{12}} \left(v_\tau^{\frac{1}{2}} - v_\tau^{-\frac{1}{2}}\right) \prod_{d=1}^{\infty} \left(1 - u_\tau^d v_\tau\right)\left(1 - u_\tau^d v_\tau^{-1}\right)$$
$$= \frac{e\left(\left(p_1 - \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right) \vartheta_{\frac{1}{2}+p_2, \frac{1}{2}-p_1}(\tau)}{\eta(\tau)},$$

where $u_\tau = e(\tau)$, $v_\tau = e(p_2 - p_1\tau)$, ϑ is the Jacobi theta function, and η is the Dedekind eta function. Let $\operatorname{Log} f_{p_1,p_2}$ is the branch satisfying

$$(\operatorname{Log} f_{p_1,p_2})(\tau) \sim \pi i \left(p_1^2 - p_1 + \frac{1}{6}\right)\tau \text{ as } \tau \to i\infty.$$

Theorem (Generalized second KLF at $s = 1$; K)

Let $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2$ *with* $0 \leq p_1, p_2 < 1$, *and let* $\Omega = N + iM \in \mathcal{H}_2^{(0)}$. *Let* $z = \tau^+$ *and* $z = \tau^-$ *be the solutions of* $Q_\Omega \begin{pmatrix} z \\ 1 \end{pmatrix} = 0$ *in the upper and lower half-planes, respectively. Then,*

$$\hat{\zeta}_{p,0}(\Omega, 1) = \frac{-1}{\sqrt{\det(-i\Omega)}} \left(\left(\operatorname{Log} f_{p_1,p_2}\right)(\tau^+) + \left(\operatorname{Log} f_{p_1,p_2}\right)(-\tau^-) \right).$$

Suppose $\Omega = N + iM \in \mathcal{H}_2^{(1)}$ and $c \in \mathbb{C}^2$ such that $\overline{c}^\top Mc < 0$. Let $\Lambda_c = \Omega - \frac{i}{Q_M(c)} Mcc^\top M \in \mathcal{H}_2^{(0)}$. For $v \in \mathbb{C}^2$, set

$$\kappa_\Omega^c(v) = \frac{c^\top Mv}{4\pi i \sqrt{-Q_M(c)}\, Q_\Omega(v) \sqrt{-2iQ_{\Lambda_c}(v)}}.$$

The function $\varphi_{p_1,p_2} : \mathcal{H} \to \mathbb{C}$ is defined by a product expansion,

$$\varphi_{p_1,p_2}(\xi) := (1 - e(p_1\xi_t + p_2)) \prod_{d=1}^{\infty} \frac{1 - e\left((d + p_1)\xi + p_2\right)}{1 - e\left((d - p_1)\xi - p_2\right)},$$

and its logarithm $\left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi)$ is the unique continuous branch with the property

$$\lim_{\xi \to i\infty} \left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi) = \begin{cases} \log(1 - e(p_2)) & \text{if } p_1 = 0, \\ 0 & \text{if } p_1 \neq 0. \end{cases}$$

Here $\log(1 - e(p_2))$ is the standard principal branch.

Theorem (KLF for indefinite zeta functions at $s = 1$; K)

Let $\Omega = N + iM = \in \mathcal{H}_2^{(1)}$, $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2$, *and* $c_1, c_2 \in \mathbb{C}^2$ *such that* $\overline{c_j}^\top \operatorname{Im}\Omega c_j < 0$. *For* $c = c_1, c_2$, *factor the quadratic form*

$$Q_{\Lambda_c}\begin{pmatrix} \xi \\ 1 \end{pmatrix} = \alpha(c)(\xi - \tau^+(c))(\xi - \tau^-(c)),$$

where $\tau^+(c)$ *is in the upper half-plane and* $\tau^-(c)$ *is in the lower half-plane. Then,*

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = I^+(c_2) - I^-(c_2) - I^+(c_1) + I^-(c_1), \ \textit{where}$$

$$I^\pm(c) = -\operatorname{Li}_2(e(\pm p_1))\kappa_\Omega^c\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2i\int_0^\infty (\operatorname{Log}\varphi_{p_1,\pm p_2})(\pm\tau^\pm(c) + it)\kappa_\Omega^c\begin{pmatrix} \pm(\tau^\pm(c) + it) \\ 1 \end{pmatrix} dt.$$

(In)definite KLF—proof sketch

- Compute Fourier series in ξ for $\Theta_{(T^\xi)^\top p, T^{-\xi} q}\left(t\left(T^\xi\right)^\top \Omega T^\xi\right)$ or $\Theta_{(T^\xi)^\top p, T^{-\xi} q}^{T^{-\xi} c_1, T^{-\xi} c_2}\left(t\left(T^\xi\right)^\top \Omega T^\xi\right)$. The kth Fourier coefficient is a sum over divisors of k.
- Take Mellin transform term-by-term to compute Fourier series for $\hat{\zeta}_{(T^\xi)^\top p, T^{-\xi} q}\left(\left(T^\xi\right)^\top \Omega T^\xi, s\right)$ or $\hat{\zeta}_{(T^\xi)^\top p, T^{-\xi} q}^{T^{-\xi} c_1, T^{-\xi} c_2}\left(\left(T^\xi\right)^\top \Omega T^\xi, s\right)$.
- Plug in $\xi = 0$ and $s = 1$. The only remaining big idea in the definite case is the use of the Jacobi triple product formula to rewrite $f_{p_1, p_2}(\tau) = \frac{e\left(\left(p_1 - \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right)\vartheta_{\frac{1}{2}+p_2, \frac{1}{2}-p_1}(\tau)}{\eta(\tau)}$.

Indefinite KLF—proof sketch

- In indefinite case, we obtain the following expression for the kth Fourier coefficient of the theta function when $k \neq 0$.

$$b_k(\xi) = \sum_{n|k} |n| \int_{-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} nt^{1/2}\right) \cdot e\left(Q_\Omega\begin{pmatrix} \xi \\ 1 \end{pmatrix} n^2 t + p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} n\right) e(-k\xi)\, d\xi.$$

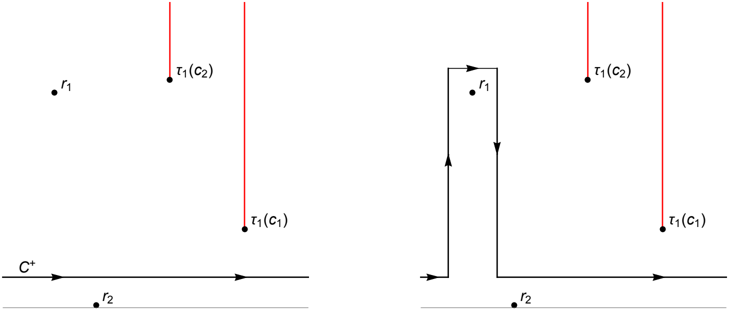
- Before taking the Mellin transform, we must shift some contours of integration up and others down so that we get a convergent expression afterwards.

$$b_k(\xi) = \sum_{n_2|k} |n| \int_{-\infty+i\lambda(\frac{k}{n},n)}^{\infty+i\lambda(\frac{k}{n},n)} \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} nt^{1/2}\right) \cdot e\left(Q_\Omega\begin{pmatrix} \xi \\ 1 \end{pmatrix} n^2 t + p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} n\right) e(-k\xi)\, d\xi.$$

Indefinite KLF—proof sketch

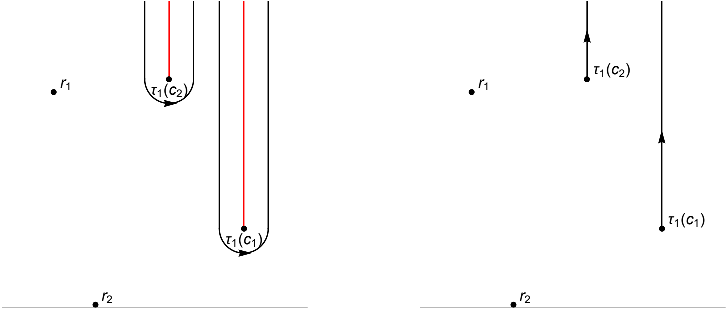
Combine into one integral over a horizontal line (plus Li_2 term).

$$\hat{\zeta}^{c_1,c_2}_{p,0}(\Omega,1) = -\left(\mathrm{Li}_2(e(p_1)) - \mathrm{Li}_2(e(-p_1))\right)\kappa^{c_1,c_2}_{\Omega}\begin{pmatrix}1\\0\end{pmatrix}$$
$$+\int_{C+}\left(-\left(\mathrm{Log}\,\varphi_{p_1,-p_2}\right)(\xi)\cdot\kappa^{c_1,c_2}_{\Omega}\begin{pmatrix}-\xi\\1\end{pmatrix}\right.$$
$$\left.+\left(\mathrm{Log}\,\varphi_{p_1,p_2}\right)(\xi)\cdot\kappa^{c_1,c_2}_{\Omega}\begin{pmatrix}\xi\\1\end{pmatrix}\right)d\xi.$$



Indefinite KLF—proof sketch

- After moving above the zeros of the $Q_{\Lambda_{c_j}}\begin{pmatrix} \pm\xi \\ 1 \end{pmatrix}$, the integral can be split up into pieces for c_1 and c_2.
- Finally, we collapse the contours onto the branch cuts of $\kappa_{\Omega}^{c_j}\begin{pmatrix} \xi \\ 1 \end{pmatrix}$.



Example

- Continue our running example with $K = \mathbb{Q}(\sqrt{3})$ and $\mathfrak{c} = 5\mathcal{O}_K$.
- We use the Kronecker limit formula for indefinite zeta functions to compute $Z_I'(0)$, where I is the principal ray class in $\mathrm{Cl}_{\mathfrak{c},\{\infty_2\}}$.
- By previous considerations, $Z_I'(0) = \hat{\zeta}_{0,q}^{c_1, P^3 c_1}(iM, 0)$.
- Use functional equation to write as $Z_I'(0) = \frac{-i}{2\sqrt{3}} \hat{\zeta}_{-q,0}^{c_1, \tilde{P}^3 c_1}(iM^{-1}, 1)$. Here, $\tilde{P} = MPM^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

Example

If we use the indefinite KLF directly, the branch cut of $\kappa_\Omega^{\tilde{P}^3 c_1}\begin{pmatrix}\xi \\ 1\end{pmatrix}$ starts very close to the real axis, at $\xi = \frac{-2340+i\sqrt{3}}{4053}$. Convergence is slow in practical terms. Instead, split into three pieces:

$$\hat{\zeta}_{-q,0}^{c_1,\tilde{P}^3 c_1}(-\Omega^{-1},1) = \hat{\zeta}_{-q_0,0}^{c,\tilde{P}c}(-\Omega^{-1},1) + \hat{\zeta}_{-q_1,0}^{c,\tilde{P}c}(-\Omega^{-1},1) + \hat{\zeta}_{-q_2,0}^{c,\tilde{P}c}(-\Omega^{-1},1),$$

where $c = \begin{pmatrix}1 \\ 3\end{pmatrix}$, $q_0 = \frac{1}{5}\begin{pmatrix}1 \\ 0\end{pmatrix}$, $q_1 = \frac{1}{5}\begin{pmatrix}2 \\ 1\end{pmatrix}$, and $q_2 = \frac{1}{5}\begin{pmatrix}2 \\ 4\end{pmatrix}$.
Now, branch cuts start at $\frac{\pm 3+i\sqrt{3}}{6}$, and convergence is rapid.

$$I_0(\tilde{P}c) - I_0(c) \approx -0.0592384392 + 3.6568783902i$$
$$I_1(\tilde{P}c) - I_1(c) \approx -1.3373302109 + 0.5247781254i$$
$$I_2(\tilde{P}c) - I_2(c) \approx 2.6405758737 + 0.5247781254i$$

Obtain $Z_l'(0) \approx 1.3586306534$, just as before.

Thanks & Questions

Thank you for attending my talk! Thank you to Larry Rolen and the other organizers.

Questions?