Rota's vision and the Lehmer conjecture

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(joint work with M. Neuhauser)

Vanderbilt University Number Theory Seminar

RWTH Aachen

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Outline

Infinite Products and Generating Series

Let $r \in \mathbb{Z}$.

$$\sum_{n=0}^{\infty} a_n(r) X^n := \prod_{n=1}^{\infty} \left(1 - X^n\right)^r . \tag{1}$$

Important topic: Properties and formulas of the $a_n(r)$.

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- $r = 1$ Euler (Pentagonal numbers)
- $r = 3$ Jacobi (Triangle numbers)
- $r = -1$ Euler (Partition numbers)
- $r = 24$ Ramanujan tau-function
  Lehmer conjecture (1947): $\tau(n) := a_{n-1}(24) \neq 0$ for all $n \in \mathbb{N}$.

Rota and Lehmer

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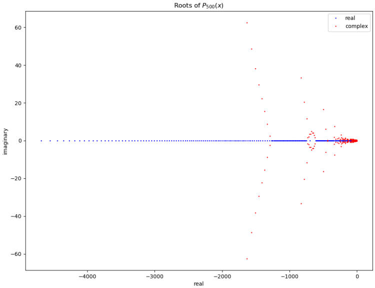
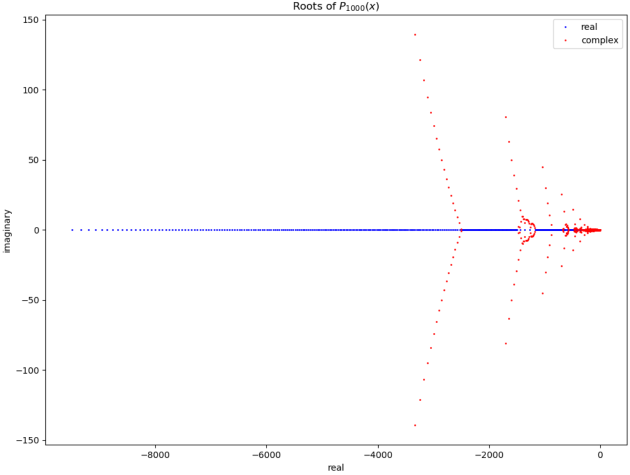


Let $P_0(x) = 1$ and $P_n(x) := \frac{x}{n} \sum_{k=1}^{n} \sigma(k)\, P_{n-k}(x)$.
Then $\tau(n) = P_{n-1}(-24)$.

Rota's way—combinatorics and roots



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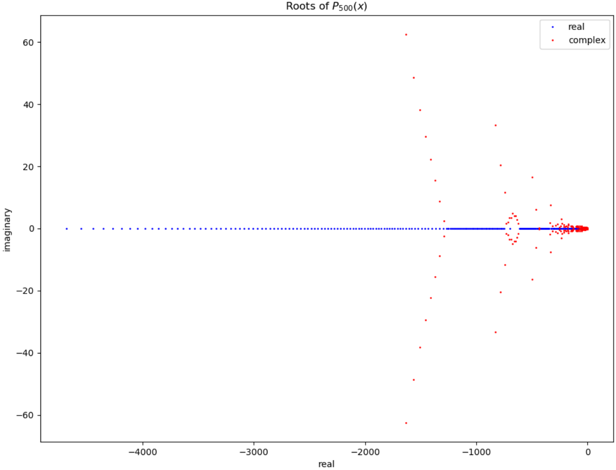

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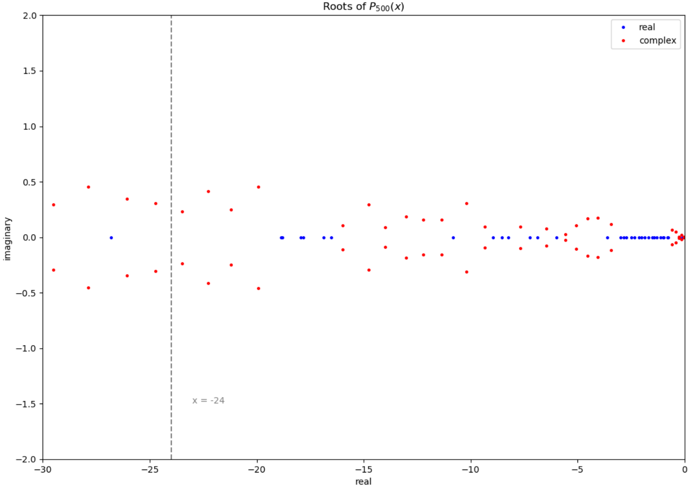

Vanishing results: $r \in \mathbb{N}$.

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Case r even

Serre: η^r lacunary iff $r \in S_{even} := \{2, 4, 6, 8, 10, 14, 26\}$.

Serre's Table

Atkin, Cohen	$r = 5$	$n = 1560, 1802, 1838, 2318, 2690, \ldots$
Atkin	$r = 7$	$n = 28017$
Newman	$r = 15$	$n = 53$

Extended Serre's Table

Serre's table extended

r	Sources n_0	$\mathcal{N}_r(n_0)$	Checked up to
5	1560, 1802, ...	$\{n_0 l^2 + 5 \cdot \frac{l^2-1}{24}, (l, 2 \cdot 3) = 1, l \in \mathbb{N}\}$	10^{10}
7	28017	$\{28017\, l^2 + 7\frac{l^2-1}{24}, (l, 2 \cdot 3) = 1, l \in \mathbb{N}\}$	10^{10}
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Let r be even and $r \notin S_{even}\{2, 4, 6, 8, 10, 14, 26\}$ all numerical checks give $a_n(r) \neq 0$.

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B. Heim, M. Neuhauser, A. Weisse: *Records on the vanishing of Fourier coefficients of powers of the Dedekind eta function.* Res. Number Theory (2018).

Even case

Numerical evidence, Maeda's conjecture, ...

Complex powers

Let $z \in \mathbb{C}$.

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Let $z \in \mathbb{C}$.

$$\prod_{n=1}^{\infty} \left(1 - X^n\right)^{-z} =: \sum_{n=0}^{\infty} P_n(z)\, X^n. \tag{2}$$

The $P_n(z)$ are polynomials of degree n.

Families of polynomials: Most of interest are of D'Arcais Type

Main Idea

$$\sum_{n=0}^{\infty} P_n(z) X^n = \prod_{n=1}^{\infty} (1 - X^n)^{-z}, \qquad (X \in \mathbb{C},\ |X| < 1). \tag{3}$$

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- $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x/2(x+3)$, $P_3(x) = x/3!\,(x+1)(x+8)$.

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- Note that $P_3(-8) = 0$ encodes the information, that the third coefficient of $\prod_n (1 - X^n)^8$ is vanishing.

Basic properties of $P_n(x)$

- Positive integer coefficients

$$P_n(x) = \frac{x}{n!} \sum_{k=0}^{n-1} a_k \, x^k, \qquad \text{where } a_k \in \mathbb{N} \text{ with } a_{n-1} = 1. \tag{4}$$

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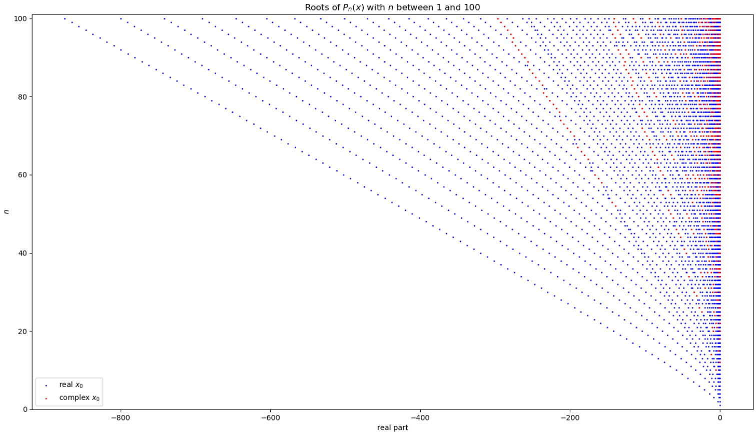
- Observation $(n \leq N = 1000)$

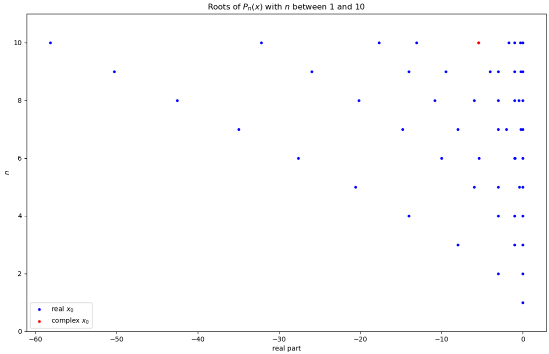
$$P_n(x) = \frac{x}{n!} \prod_{k=1}^{d_n} (x + r_k) \cdot \text{ irred. polynomial}/\mathbb{Q}, \qquad \text{where } r_k \in \mathbb{N}. \tag{5}$$

Basic properties of $P_n(x)$

- Root distribution

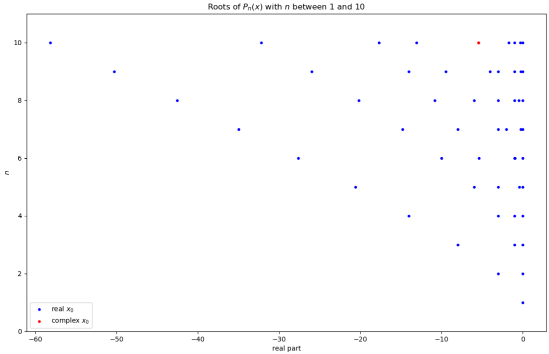


Properties of $P_n(x)$



- Open: Simplicity of roots

Properties of $P_n(x)$



- Open: Simplicity of roots
- Open: Hurwitz polynomials

Algebraic approach

We have already observed that roots are algebraic numbers and negative if real.
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Theorem ('18, H., Luca, Neuhauser)

Suppose that ξ_m is a mth root of unity and there exist an $n \in \mathbb{N}$, such that $P_n(\xi_m) = 0$. Then $\xi_m = -1$.

Fundamental recursion formula

Definition

Let $g : \mathbb{N} \longrightarrow \mathbb{C}$ normalized arithmetic function. Then

$$P_n^g(x) := \frac{x}{n} \sum_{k=1}^{n} g(k) P_{n-k}(x), \qquad (P_0(x) := 1). \tag{6}$$

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Observation

Let $g(n) = \sigma(n) = \sum_{d|n} d$. Then $P_n(x) = P_n^{\sigma}(x)$.

Laguerre polynomials

Observation

Let $g(n) = id(n)$. Then $P_n^{id}(x) = \frac{x}{n} L_{n-1}^{(1)}(-x)$, where $L_n^{(\alpha)}(x)$ is the α-associated Laguerre polynomial of degree n.

Growth Condition and Non-vanishing Zone

Theorem: Heim, Neuhauser 2020

Let $z \in \mathbb{C}$ and let $\mid z \mid > \kappa(n-1)$, $\kappa := 10.82$, then

$$\mid P_n(z) \mid > \frac{\mid z \mid}{2n} \mid P_{n-1}(z) \mid . \qquad (7)$$

This implies $P_n(z) \neq 0$ for $|z| > \kappa\,(n-1)$.

See HN 2020 : On the growth and zeros of polynomials attached to arithmetic functions. arXiv:2101.04654.

Fundamental recursion formula

Definition

Let $g, h : \mathbb{N} \longrightarrow \mathbb{C}$ normalized arithmetic function and let h be non-vanishing. Then

$$P_n^{g,h}(x) := \frac{x}{h(n)} \sum_{k=1}^{n} g(k) P_{n-k}^{g,h}(x), \qquad (P_0^{g,h}(x) := 1). \tag{8}$$

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Theorem: H-N

Let $Q_n(x)$ be the family of polynomials attached to σ and $h(n) = 1$. Let $z \in \mathbb{C}$ and let $|z| > \kappa$, $\kappa := 10.82$, then

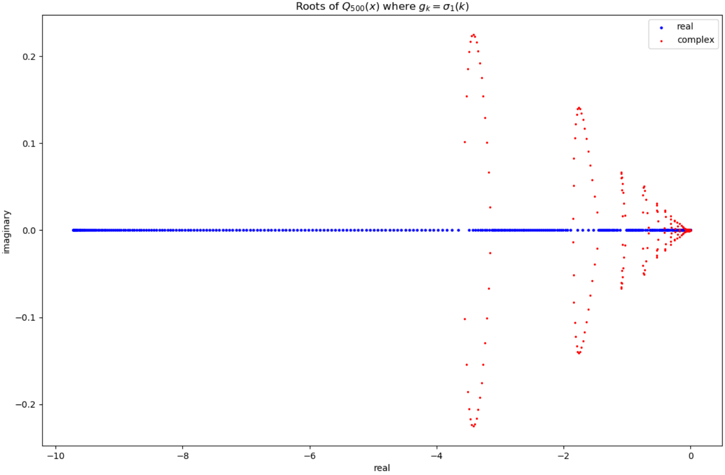
$$|Q_n(z)| > \frac{|z|}{2} |Q_{n-1}(z)|. \tag{9}$$

This implies $Q_n(z) \neq 0$ for $|z| > \kappa$.

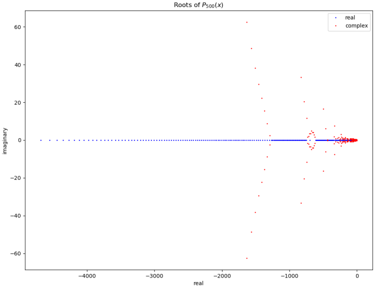
Chebychev polynomials of the second kind

Let $g(n) = id(n) = n$. Then $Q_n^{id}(x) = x\, U_{n-1}(x/2+1)$, Chebychev polynomial of the second kind.

Rota's way—combinatorics and roots



P and Q polynomials $n = 500$



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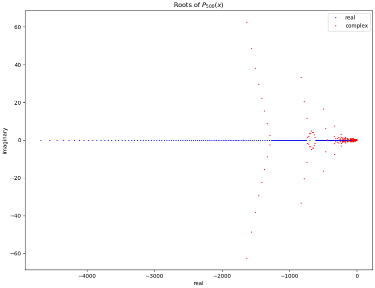
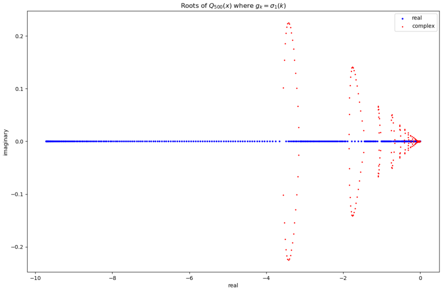
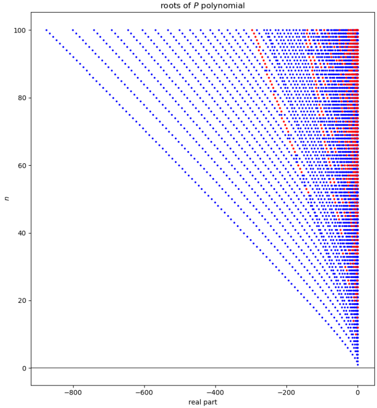
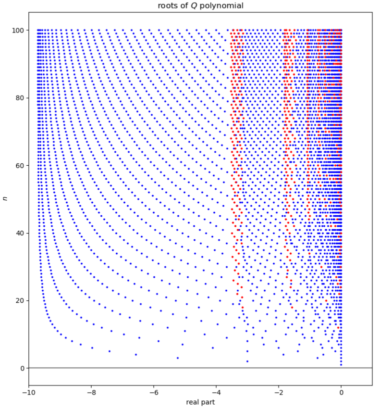


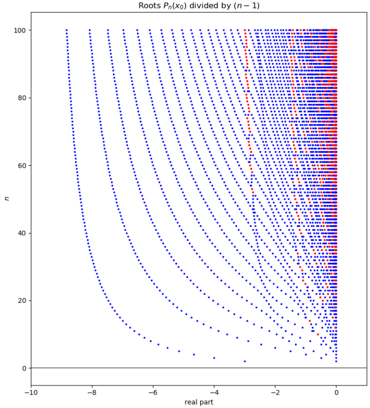


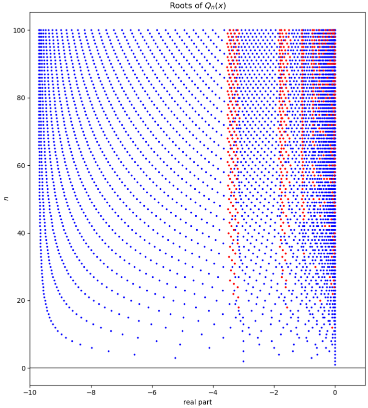
P and Q polynomials: roots (until $n = 100$)



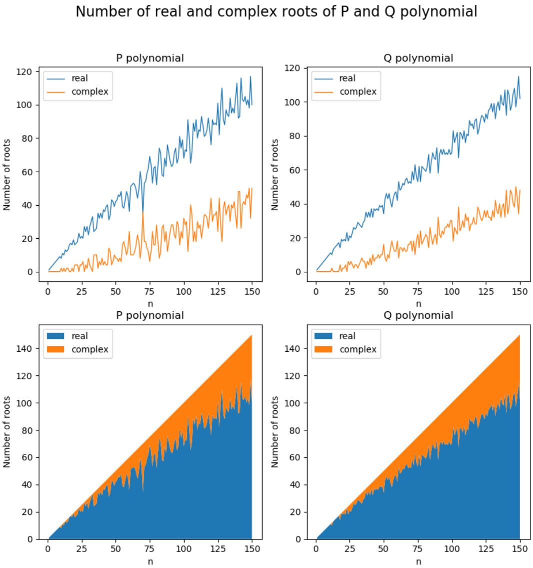


P and Q polynomials: roots (until $n = 100$) $x_0^{(P)}/(n-1)$





P and Q polynomials: type of roots (until $n = 150$)



Two families of polynomials P_n and Q_n.

Summary.

$$P_n^g(x) = \frac{x}{n} \sum_{k=1}^{n} g(k)\, P_{n-k}^g(x) \tag{10}$$

$$Q_n^g(x) = x \sum_{k=1}^{n} g(k)\, Q_{n-k}^g(x) \tag{11}$$

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Let $g(n) = \sum_{d|n} d$.

- $P_n(z) \neq 0$ for $|z| > 10.82\,(n-1)$ and $Q_n(z) \neq 0$ for $|z| > 10.82$.

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- Hurwitz polynomials. We observed numerically, and have high evidence that the roots of both polynomials (up to $z = 0$), have simple roots with negative real part.

$Q_n(x)$ Volterra type difference equation, $x_0 \in \mathbb{C}$.

Basic equation

Let $B(n) = x_0\, g(n+1)$ and $x_0 := Q_1(x_0)$. Then

$$x(n+1) = Ax(n) + \sum_{k=0}^{n} B(n+1-k)\, x(k), \qquad (A=0). \tag{12}$$

Note that this implies $Q_n(x_0) = x(n)$.

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The Z-transformation leads to

Identity

$$\frac{1}{1 - x_0 \sum_{n=1}^{\infty} g(n) z^{-n}} = Z(x(n)). \tag{13}$$

Applications

Theorem

Let $q := e^{2\pi i\tau}$, τ in the complex upper half-space, $z := q^{-1}$. Let j be Klein's absolute invariant and $g(n)$ essentially the coefficients of j. Then

$$\frac{1}{j(\tau)} = q \sum_{n=0}^{\infty} Q_n^g(-744) q^n, \tag{14}$$

where $(-1)^n Q_n^g(-744) > 0$.

Actually we prove that $\gamma_2(\tau)$, the cubic root of j has this property.
Remark: See also recent results on $1 \,/\, E_k$ on reciprocal Eisenstein series. IJNT 2021 and arXiv.

Summary and next steps

Fundamental equations

$$\sum_{n=0}^{\infty} P_n^g(x) X^n = \exp\left(x \sum_{n=1}^{\infty} g(n) \frac{X}{n} \right) \quad (15)$$

$$\sum_{n=0}^{\infty} Q_n^g(x) X^n = \frac{1}{1 - x_0 \sum_{n=1}^{\infty} g(n) X^n}. \quad (16)$$

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Identities

$$P_n^g(x) = \frac{x}{n} \sum_{k=1}^{n} g(k) P_{n-k}^g(x), \quad (P_0^g(x) = 1) \tag{17}$$

$$Q_n^g(x) = x \sum_{k=1}^{n} g(k) Q_{n-k}^g(x), \quad (Q_0^g(x) = 1) \tag{18}$$

In 1916, Ramanujan[1] published the following table:

n	$\tau(n)$	n	$\tau(n)$
1	+1	16	+987136
2	−24	17	−6905934
3	+252	18	+2727432
4	−1472	19	+10661420
5	+4830	20	−7109760
6	−6048	21	−4219488
7	−16744	22	−12830688
8	+84480	23	+18643272
9	−113643	24	+21288960
10	−115920	25	−25499225
11	+534612	26	+13865712
12	−370944	27	−73279080
13	−577738	28	+24647168
14	+401856	29	+128406630
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[1] On certain arithmetical functions, Transactions of the Cambridge Philosophical Society, XXII, No.9, 159-184

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Ramanujan observed

- Congruences
- Multiplicative properties: Hecke Theory (Mordell, Hecke)
- Growth conditions: Ramanujan-Petersson conjecture (Deligne)

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Non-vanishing & (non)-sign changes

In 1947, Lehmer conjectured that $\tau(n) \neq 0$ for all natural numbers. The conjecture is still open.

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9	−113643	24	+21288960
10	−115920	25	−25499225
11	+534612	26	+13865712
12	−370944	27	−73279080
13	−577738	28	+24647168
14	+401856	29	+128406630
15	+1217160	30	−29211840

Non-vanishing & (non)-sign changes

In 1947, Lehmer conjectured that $\tau(n) \neq 0$ for all natural numbers. The conjecture is still open.

n	$\tau(n)$	n	$\tau(n)$
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2	−24	17	−6905934
3	+252	18	+2727432
4	−1472	19	+10661420
5	+4830	20	−7109760
6	−6048	21	−4219488
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Hot topic since 20 years:
Sign changes.

- Infinite sign changes (Murty, Knopp, Kohnen, Pribtin, ...)
- First non-sign change. New results. See Conference Proceedings 2020, Ramakrishnan, H., Sahu (Article H-N).

Results of Kostant and Han

Kostant 2004

Let $\mathfrak{g}$ be a simple complex Lie-algebra (different from A_1, A_2, G_2). Then

$$\prod_{n=1}^{\infty} (1 - q^n)^{\dim \mathfrak{g}} = \sum_{n=0}^{\infty} a_n(\dim \mathfrak{g})\, q^n \tag{19}$$

has non-vanishing coefficients up to the dual Coxeter number $h^\vee$ and their sign is strictly alternating.

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Root System A_{m-1}

$\dim \mathfrak{g} = m^2 - 1$ and $(-1)^n a_n > 0$ for $n \leq \max\{4, m\}$.

Nekrasov-Okounkov Hook Length Formula

Random partitions and the Seiberg-Witten theory led Nekrasov-Okounkov (2003, appeared 2006) to

NO Formula

Let λ run through all partitions. Let $|\lambda|$ be the size, and $\mathcal{H}(\lambda)$ be the multiset of hook lengths. Then

$$\sum_{\lambda} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 + \frac{z}{h^2}\right) = \prod_{n=1}^{\infty} (1 - q^n)^{-(z+1)} . \tag{20}$$

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NO Polynomials

$$P_n^{\text{NO}}(z) := \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(1 + \frac{z}{h^2}\right) \tag{21}$$

See also important results of Westbury (2006) and Han (2010).

Theorem: Han 2008

Han proved by applying the Nekrasov-kounkov Hook Length Formula, that let $x \in \mathbb{R}_{>0}$, then $(-1)^n a_n(x^2-1) > 0$ for $n \leq \max\{4, x\}$.

Using our approach of recursively defined polynomials $P_n(x)$ leads actually to a linear condition. Let $z \in \mathbb{C}$ and let $|z| > 10.82(n-1)$. Then $P_n(z) \neq 0$ and $(-1)^n P_n(z) > 0$ if z is real and negative.

Recent Results towards

- We refer to work of Amdeberhan (arXiv) , Keith (2013), and Walsh and Warnaar (2020).
- Heim and Neuhauser (several papers). e.g. Conjecture: $P_n^{\sigma}(x+1)$ is log-concave for all n.
- Significant evidence for the Conjecture by the recent work of Hong and Zhang (published 2021, Research in Number Theory).

Open questions and next steps

- Proof the log-concave conjecture.
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- Extend the result of Heim–Luca–Neuhauser (work in progress).
- Proof that all $P_n(x)$ are Hurwitz polynomials.

A new type of inequality for partitions

Theorem: Bessenrodt-Ono 2016

Let $p(n)$ be the number of partitions. Let a, b integers. Let $a, b > 1$ and $a + b > 9$. Then

$$p(a)\,p(b) > p(a+b). \qquad (BO) \tag{22}$$

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- Condition $a, b > 1$ obvious.
- BO has essentially finitely many exceptions.
- Analytic proof (Lehmer-type estimation).

Related work - Extensions

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- Hou, Jagadeesan (2019): B-O for Dyson partition ranks.
- Heim, Neuhauser (2019): Inequalities of type $p(a)p(b) \geq p(a+b+m-1)$.

Generalisation

Theorem: Chern, Fu, Tang (2018)

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$$p_{-k}(a)\, p_{-k}(b) > p_{-k}(a+b), \tag{23}$$

except for $(a, b, k) = (1, 1, 2), (2, 1, 2), (3, 1, 2), (1, 1, 3)$.

Combinatorial proof.

Polynomization

Let $P_n(x)$ already introduced. Then $p_{-k}(n) = P_n(k)$.

Theorem: Heim, Neuhauser, Troeger (2019)

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Polynomization

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Let $a, b \in \mathbb{N}$, $a + b > 2$ and $x > 2$. Then

$$P_a(x)\, P_b(x) > P_{a+b}(x), \tag{24}$$

The case $x = 2$ is true for $a + b > 4$.

Conjectures

Conjecture CFT: Chern, Fu, Tang (2018)

Let $a > b \geq 1$ and $k \geq 2$, except for $(k, a, b) = (2, 6, 4)$ we have

$$P_{-k}(a-1)\, P_{-k}(b+1) > P_{-k}(a) P_{-k}(b). \qquad (25)$$

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Conjecture HN: Heim, Neuhauser (2019)*

Let $a > b \geq 0$ be integers. Then for all $x \geq 2$:

$$\Delta_{a,b}(x) := P_{a-1}(x)P_{b+1}(x) - P_a(x)P_b(x) \geq 0, \tag{26}$$

except for $b = 0$ and $(a, b) = (6, 4)$. The inequality (26) is still true for $x \geq 3$ for $b = 0$ and for $x \geq x_{6,4}$ for $(a, b) = (6, 4)$. Here $x_{a,b}$ is the largest real root of $\Delta_{a,b}(x)$.

Theorem: Heim, Neuhauser (2020)

The Conjecture HN is true for $b = 1$.

Building on results of Iskander, Jain and Talvola:

Theorem BKRT: Bringmann, Kane, Rolen, Tripp 2020 arXiv

Fix $x \in \mathbb{R}$ with $x \geq 2$, and let $a, b \in \mathbb{N}_{\geq 2}$ with $a > b + 1$. Set $A := a - 1 - \frac{x}{24}$ and $B := b - \frac{x}{24}$, we suppose $B \geq \max\left\{2\, x^{11}, \frac{100}{x-24}\right\}$. Then

$$\begin{aligned} \Delta_{a,b}(x) &= P_{a-1}(x)\, P_{b+1}(x) - P_a(x)\, P_b(x) \\ &= \pi \left(\frac{x}{24}\right)^{\frac{x}{2}+1} (AB)^{-\frac{x}{4}-\frac{5}{4}}\, e^{\pi\sqrt{\frac{2x}{3}}(\sqrt{A}+\sqrt{B})} \left(\sqrt{A} - \sqrt{B}\right) \\ &\quad \left(1 + O_{\leq}\left(\frac{2}{3}\right)\right). \end{aligned}$$

Notation: Let $f(x) = O_{\leq}(g(x))$ mean that $|f(x)| \leq g(x)$ in the relevant domain.

Applications

Corollary: BKRT 2020

For any real number $x \geq 2$ and positive integers

$$b \geq B_0(x) := \max\left\{2\,x^{11} + \frac{x}{24}, \frac{100}{x-24} + \frac{x}{24}\right\}. \qquad (27)$$

Conjecture CFT is true.

Corollary: BKRT 2020

- The CTF Conjecture is true.
- For each x, the HN Conjecture is true for all $b \geq B_0(x)$.

Latest Results

Heim, Neuhauser: Polynomization of the Chern-Fu-Tang Conjecture. Research in Number Theory (published online 22.March 2021).

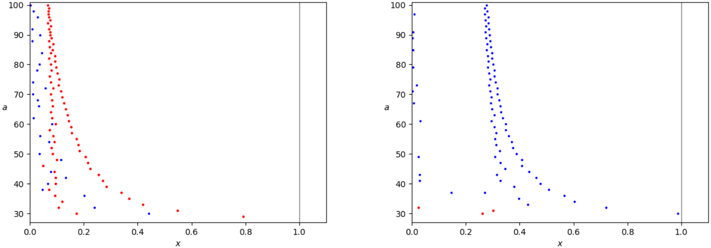
Theorem

Let $a \in \mathbb{N}$, $b \in \{1, 2, 3\}$ and $x \in \mathbb{R}$. For b odd we put $x_0 := 1$ and for b even $x_0 := 2$. Let $a_0 := a_0(b) := b + 2$. Then

$$\Delta_{a,b}(x) > 0 \quad (a \geq a_0, x > 0). \tag{28}$$

Proof method very briefly: Check $\Delta_{a,b}(x_0) \geq 0$ and $\Delta'_{a,b}(x) \geq 0$ for $x \geq x_0$.

General picture



Roots of $\Delta_{a,27}(x)$ and $\Delta_{a,28}(x)$ with positive real part.
Blue $=$ real roots, red $=$ complex roots.

Final Page: Challenges

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- Prove the HN Conjecture, the polynomization of the CFT Conjecture (proven by Bringman et. al.).
- Find a NO type hook length formula for $P_n^{g,h}(x)$ for well chosen g and h.