Historical remarks and recent conjectures for integer partitions

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Vanderbilt Univ. Number Theory Seminar

August 18, 2020

Outline

1. Ramanujan congruences for $p(n)$, (c)ranks.

2. Rogers-Ramanujan identities and refinements

Ramanujan congruences for $p(n)$

$$p(5n+4) \equiv 0 \mod 5$$

$$p(7n+5) \equiv 0 \mod 7$$

$$p(11n+6) \equiv 0 \mod 11$$

? combinatorial proofs splitting into equal classes ?

Dyson's rank (1944)

$\mathrm{rank}(\lambda) = \lambda_1 - \lambda_1'$

rank () $= 4-3=1$

Atkin and Swinnerton-Dyer proved it works for

5, 7 (1954)

Andrews-Garvan crank

(June 6, 1987)

$$\mathrm{AGcrank}(\lambda) = \begin{cases} \lambda_1 & \lambda \text{ has no 1's} \\ \mu(\lambda) - (\#\text{1's in } \lambda) & \lambda \text{ has} \end{cases}$$

$\mu(\lambda)$ = # parts of λ which are greater than #(1

works for 5, 7, 11

5-core crank (Garvan-Kim-S. 1990)

Uses the 5-residue diagram of λ

$\lambda = 5421$

0	1	2	3	4
4	0	1	2	
3	4			
2				

$r_0 = 2 \quad r_1 = 2 \quad r_2 = 3 \quad r_3 = 2 \quad r_4 = 3$

$$\text{5-core crank}(\lambda) = r_1 + 2r_2 - 2r_3 - r_4$$

works for 5, 7, 11 (7-core, 11-core)

	rank	AGcrank	Scorecrank
4	3	4	1
31	1	0	2
22	0	2	0
211	-1	-2	-2
1111	-3	-4	-1

$\mathrm{rank}_4(z) = z^3 + z^1 + z^0 + z^{-1} + z^{-3}$

$\mathrm{AGcrank}_4(z) = z^4 + z^0 + z^2 + z^{-2} + z^{-4}$

$\mathrm{Scorecrank}_4(z) = z^1 + z^2 + z^0 + z^{-2} + z^{-1}$

$$\mathrm{rank}_n(z) = \sum_{\lambda \in P(n)} z^{\mathrm{rank}(\lambda)}$$

$$\mathrm{AGcrank}_n(z) = \sum_{\lambda \in P(n)} z^{\mathrm{AGcrank}(\lambda)}$$

$$\mathrm{5corecrank}_n(z) = \sum_{\lambda \in P(n)} z^{\mathrm{5corecrank}(\lambda)}$$

Generating functions

$$\sum_{n=0}^{\infty} \mathrm{rank}_n(z)\, q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq, q/z; q)_n}$$

$$\sum_{n=0}^{\infty} \mathrm{AGcrank}_n(z)\, q^n = \frac{(q;q)_\infty}{(zq, q/z; q)_\infty}$$

$$\sum_{n=0}^{\infty} \text{5corecrank}_{5n+4}(z)\, q^{n+1} =$$

$$\frac{1}{(q;q)_\infty^5} \sum_{\substack{\vec{a}\cdot\vec{1}=1 \\ \vec{a}\in\mathbb{Z}^5}} q^{Q(\vec{a})}\, z^{\sum_{i=0}^{4} i a_i}$$

$$Q(\vec{a}) = \sum_{i=0}^{4} a_i^2 - \sum_{i=0}^{4} a_i a_{i+1} \qquad a_5 = a_0.$$

5-fold symmetry $\Rightarrow$ bijection on 5corecrank classes

Symmetry groups of the quadratic forms for $tn+r$

$3n$	$\mathbb{Z}_2$	$4n$	$\mathbb{Z}_2$	$5n$	$\mathbb{D}_6$	$6n$	$\mathbb{Z}_2$	$7n$	$\mathbb{D}_8$
$3n+1$	$\mathbb{Z}_2$	$4n+1$	$\mathbb{Z}_2$	$5n+1$	$\mathbb{D}_4$	$6n+1$	$\mathbb{Z}_2$	$7n+1$	$\mathbb{D}_6$
$3n+2$	$\mathbb{Z}_2$	$4n+2$	$\mathbb{Z}_2$	$5n+2$	$\mathbb{D}_4$	$6n+2$	$\mathbb{Z}_2$	$7n+2$	$\mathbb{D}_8$
		$4n+3$	$\mathbb{Z}_2$	$5n+3$	$\mathbb{D}_6$	$6n+3$	$\mathbb{Z}_2$	$7n+3$	$\mathbb{D}_6$
				$5n+4$	$\mathbb{D}_5$	$6n+4$	$\mathbb{Z}_2$	$7n+4$	$\mathbb{D}_6$
						$6n+5$	$\mathbb{Z}_2$	$7n+5$	$\mathbb{D}_7$
								$7n+6$	$\mathbb{D}_8$

QUESTION: Is there an a priori reason for these groups?

5 core crank gives a combinatorial proof of Ramanujan's amazing formula

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\,\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.$$

PROOF: All partitions = (5-quotient) * (5 core)

$$\frac{1}{(q;q)_\infty} = \frac{1}{(q^5;q^5)_\infty^5} \sum_{n=0}^{\infty} a_5(n)\, q^n$$

So

$$\sum_{n=0}^{\infty} p(5n+4)\, q^{5n+4} = \frac{1}{(q^5;q^5)_\infty^5} \sum_{n=0}^{\infty} a_5(5n+4)\, q^{5n+4}$$

But $a_5(5n+4) = 5\, a_5(n)$ (bijection)

$$= \frac{1}{(q^5;q^5)_\infty^5} \cdot 5q^4 \sum_{n=0}^{\infty} a_5(n)\, q^{5n}$$

$$= 5q^4 \frac{1}{(q^5;q^5)_\infty^5} \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty}$$

? Questions? Combinatorial reductions ?

$$\mathrm{rank}_{5n+4}(z) = (1+z+z^2+z^3+z^4) \times \left(\begin{array}{c}\text{positive}\\ \text{polynomial}\end{array}\right)$$

$$\mathrm{rank}_4(z) = (1+z+z^2+z^3+z^4)\left(\frac{1-z+z^2}{z^3}\right)$$

NO

DEFN The modified rank, Mrank,

$$\mathrm{Mrank}_n(z) = \mathrm{rank}_n(z) + \left(z^{n-2} - z^{n-1} + z^{2-n} - z^{1-n}\right)$$

$\lambda = n$ $\qquad n-1 \longrightarrow n-2$

$\lambda = 1^n$ $\qquad 1-n \longrightarrow 2-n$

<u>CONJ</u> $\dfrac{\mathrm{Mrank}_{5n+4}(z)}{1+z+z^2+z^3+z^4}$ is a positive Laurent polynomial

$\dfrac{\mathrm{Mrank}_{7n+5}(z)}{1+z+z^2+z^3+z^4+z^5+z^6}$ also.

Frank Garvan has verified these for

$5n+4 \le 1000$

$7n+5 \le 1000$.

Ramanujan J (2010) 23: 3–15
DOI 10.1007/s11139-008-9143-z

Cranks—really, the final problem

Bruce C. Berndt · Heng Huat Chan ·
Song Heng Chan · Wen-Chin Liaw

Dedicated to our friend George Andrews on his 70th birthday

Received: 29 August 2007 / Accepted: 5 September 2008 / Published online: 1 October 2009

Abstract A survey of Ramanujan's work on cranks in his lost notebook is given. We give evidence that Ramanujan was concentrating on cranks when he died, that is to say, the final problem on which Ramanujan worked was *cranks—not mock theta functions*.

Keywords Crank · Partitions · Theta functions · Ramanujan's lost notebook · Rank

Mathematics Subject Classification (2000) Primary: 11P82 · Secondary: 11P83

B.C. Berndt's research is partially supported by grant H98230-07-1-0088. from the National Security Agency.
H.H. Chan's research is partially supported by National University of Singapore Academic Research Fund, Project Number R146000027112.
W.-C. Liaw's research is partially supported by grant NSC95-2115-M-194-012 from the National Science Council of the Republic of China.

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At first glance, there does not appear to be any reasoning behind the choice of subscripts; note that there is no subscript for the second value. However, observe that in each case if we set $a = 1$, then the subscript n is equal to the right-hand side. The reason ρ does not have a subscript is that the value of n in this case would be $3 - 2 = 1$, which has been reserved for the first factor. In the table below, we record the content of page 181.

$$
\begin{aligned}
p(1) &= 1, & \lambda_1 &= \rho_1, \\
p(2) &= 2, & \lambda_2 &= \rho_2, \\
p(3) &= 3, & \lambda_3 &= \rho_3, \\
p(4) &= 5, & \lambda_4 &= \rho_5, \\
p(5) &= 7, & \lambda_5 &= \rho_7 \rho, \\
p(6) &= 11, & \lambda_6 &= \rho_1 \rho_{11}, \\
p(7) &= 15, & \lambda_7 &= \rho_3 \rho_5, \\
p(8) &= 22, & \lambda_8 &= \rho_1 \rho_2 \rho_{11}, \\
p(9) &= 30, & \lambda_9 &= \rho_2 \rho_3 \rho_5, \\
p(10) &= 42, & \lambda_{10} &= \rho \rho_2 \rho_3 \rho_7, \\
p(11) &= 56, & \lambda_{11} &= \rho_4 \rho_7 (a_5 - a_4 + a_2), \\
p(12) &= 77, & \lambda_{12} &= \rho_7 \rho_{11} (a_4 - 2a_3 + 2a_2 - a_1 + 1), \\
p(13) &= 101, & \lambda_{13} &= \rho \rho_1 \left(a_{10} + 2a_9 + 2a_8 + 2a_7 + 3a_6 \right. \\
& & & \qquad \left. + 4a_5 + 6a_4 + 8a_3 + 9a_2 + 9a_1 + 9\right), \\
p(14) &= 135, & \lambda_{14} &= \rho_5 \rho_9 (a_5 - a_3 + a_1 + 1), \\
p(15) &= 176, & \lambda_{15} &= \rho_4 \rho_{11} (a_7 - a_6 + a_4 + a_1), \\
p(16) &= 231, & \lambda_{16} &= \rho_3 \rho_7 \rho_{11} (a_5 - 2a_4 + 2a_3 - 2a_2 + 3a_1 - 3), \\
p(17) &= 297, & \lambda_{17} &= \rho_9 \rho_{11} (a_7 - a_6 + a_3 + a_1 - 1), \\
p(18) &= 385, & \lambda_{18} &= \rho_5 \rho_7 \rho_{11} (a_6 - 2a_5 + a_4 + a_3 - a_2 + 1), \\
p(19) &= 490, & \lambda_{19} &= \rho_1 \rho_2 \rho_5 \rho_7 (a_9 - a_7 + a_4 + 2a_3 + a_2 - 1), \\
p(20) &= 627, & \lambda_{20} &= \rho \rho_3 \rho_{11} (a_{10} + a_6 + a_4 + a_3 + 2a_2 + 2a_1 + 3), \\
p(21) &= 792, & \lambda_{21} &= \rho \rho_3 \rho_4 \rho_{11} (a_8 - a_6 + a_4 + a_1 + 2).
\end{aligned}
$$

These factors lead to the rapid calculation of values for $p(n)$. For example, since $\lambda_{10} = \rho\rho_2\rho_3\rho_7$, then $p(10) = 1 \cdot 2 \cdot 3 \cdot 7 = 42$.

Ramanujan evidently was searching for some general principles or theorems on the factorization of λ_n so that he could not only compute $p(n)$ but make deductions about the divisibility of $p(n)$. No theorems are stated by Ramanujan. Is it possible to determine that certain factors appear in some precisely described infinite family of

Ramanujan factors for

$$\mathrm{AGcrank}_{14}(z) = \left(z^4 + z^2 + 1 + z^{-2} + z^{-4}\right) *$$
$$\rho_9 \left(a_5 - a_3 + a_1 + 1\right)$$

LAST TWO FACTORS =

$$z^{-10} + z^{-7} + z^{-6} + z^{-5} + 2z^{-4} + 2z^{-3} + 2z^{-2} + 2z^{-1}$$
$$+ 3 + 2z + 2z^2 + 2z^3 + 2z^4 + z^5 + z^6 + z^7 + z^{10}$$

CONJ $\dfrac{AG\,\mathrm{crank}_{5n+4}(z)}{\left(z^{4}+z^{2}+1+z^{-2}+z^{-4}\right)}$ is a positive Laurent polynomial

Frank verified this for

$$5n+4 \leq 1000.$$

DEFN The modified $\mathrm{MAGcrank}_n^{(a)}(z)$

$$= \mathrm{AGcrank}_n(z) + \left(z^{n-a} - z^{n} + z^{a-n} - z^{-n}\right)$$

CONJ The following are non-negative Laurent polys

$$\frac{\mathrm{MAG}^{(5)}_{5n+4}(z)}{1+z+z^2+z^3+z^4}, \quad \frac{\mathrm{MAG}^{(7)}_{7n+5}(z)}{1+z+\dots+z^6}, \quad \frac{\mathrm{MAG}^{(11)}_{11n+6}(z)}{1+z+\dots+z^{10}}$$

Frank checked this for $tn+r \leq 1000$.

Th (Bringmann-Ono-Rolen-Tripp-Wagner 2019)

The MAG crank conjectures

are true.

CONJ The following are non-negative polynomials

$$\frac{\mathrm{5corecrank}_{5n+4}(z)}{1+z+z^2+z^3+z^4},$$

$$\frac{\mathrm{5corecrank}_{5n+4,j}(z)}{1+z+z^2+z^3+z^4}$$ restricted to BG crank $= j$

Alex Berkovich and Frank Garvan 2006, 2008.

Th (Bringmann-Ono-Rolen, 2019)

The polynomial positivity CONJ for the

$$\frac{5\,\mathrm{corecrank}_{5n+4}(z)}{1+z+z^2+z^3+z^4} \qquad \text{holds.}$$

Alternative AG cranks

DEFN Let $s = \#1\text{'s}$ in λ

$t = \#2\text{'s}$ in λ

$u = \#$ parts in λ greater than $s+t+1$

THM (S. 2019) Another AG crank is $(n \geq 6, n = 4)$

$$\text{crank}(\lambda) = \begin{cases} \lambda_1 & \text{if } s+t=0 \\ \lambda_1 - 1 & \text{if } s=1,\ t=0, \\ \lambda_2 - 1 & \text{if } s=0 \quad t=1 \\ u - s + (s+t+1) * (\#\text{ parts of size } s+t+1) & \\ \qquad \text{if } s+t \geq 2. & \end{cases}$$

PROOF Use AGcrank generating function

$$\frac{(q;q)_\infty}{(qz, q/z; q)_\infty} = \frac{(1-q)(1-q^2)}{(zq;q)_\infty} + \frac{q-q^3}{z} + (q^2+q^5)$$

$$+ \sum_{n=2}^{\infty} q^{n+1} z^{n+1} \left(\frac{1}{(q^3;q)_{n-2}} + \frac{q^{n+1}}{(1-q)(1-q^3)\cdots(1-q^n)} \right)$$

$$+ \sum_{k=2}^{\infty} \frac{q^k (1 + zq + \cdots + z^k q^k)/z^k}{(q^3;q)_{k-2} (1 - z^{k+1} q^{k+1}) (zq^{k+2})_\infty}$$

Crank and mex (Hopkins, Sellers, S., 2020)

DEFN λ partition, j = part of λ,

$\text{mex}_j(\lambda)$ = smallest integer $> j$ which is **NOT** in λ

THM Fix j, n.

partitions of n with crank $\geq j$ = # partitions λ of n such that $\text{mex}_j(\lambda) - j$ is odd.

= # partitions of $n+j$ whose Frobenius symbol has no j in its top row.

Rogers - Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1}{\prod_{n=0}^{\infty}(1-q^{5n+1})(1-q^{5n+4})}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{1}{\prod_{n=0}^{\infty}(1-q^{5n+2})(1-q^{5n+3})}$$

Rogers (1894)
Ramanujan (1917)
Schur (1917-1919)

Math Reviews

917 papers (94 by G. Andrews)

244 2010-2018

11 theses

2018 book by A. Sills

1913 Ramanujan's letter to Hardy

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{\ddots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right)e^{2\pi/5}$$

$$1 - \cfrac{e^{-\pi}}{1+\cfrac{e^{-2\pi}}{1-\cfrac{e^{-3\pi}}{\ddots}}} = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right)e^{\pi/5}$$

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\ddots}}}} = \frac{\prod_{n=0}^{\infty}(1-q^{5n+1})(1-q^{5n+4})}{\prod_{n=0}^{\infty}(1-q^{5n+2})(1-q^{5n+3})}$$

Ramanujan had no proof until 1917.

Rogers papers 1894-5 had RR identities proven.

Rogers had studied expansions of infinite products.

Today "q-Hermite" expansions.

PROOFS

q-series (Watson, Bailey, Ramanujan, Andrews, ...)

Rep theory (Lepowsky-Milne)

Conformal Field Theory (Berkovich, McCoy-Schilling, Lovejoy, Foda, ...)

Symmetric Functions (Stembridge, Warnaar

Combinatorial (Garsia-Milne)

birth of the involution principle

Computer algebra (Paule, Zeilberger et al)

Probability (Fulman, random matrices over $GF(q)$)

Statistical Mechanics (Baxter, Hard hexagon model)

$$I(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-xt+t^2} e^{-x^2/2}\, dx = e^{t^2}$$

$$I_q(t) = \frac{(q;q)_\infty}{2\pi} \int_0^{\pi} (te^{i\theta}, te^{-i\theta}; q)_\infty (e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta$$

$$= \sum_{N=0}^{\infty} \frac{q^{N^2-N}\, t^{2N}}{(1-q)(1-q^2)\cdots(1-q^N)}$$

q Hermite orthogonality

$t = q, \sqrt{q}$ PRODUCT SIDE

$1^2 + 2^2 = 5$

Fourier orthogonality

MacMahon gave the combinatorial interpretation

THM # of partitions of n into parts with difference ≥ 2

equals

\# of partitions of n into parts $\equiv 1, 4 \bmod 5$.

EX $n = 10$

$$\left\{\begin{array}{l} 10 \\ 9+1 \\ 8+2 \\ 7+3 \\ 6+4 \\ 6+3+1 \end{array}\right. \qquad \left\{\begin{array}{l} 9+1 \\ 6+1+1+1+1 \\ 4+4+1+1 \\ 6+4 \\ 4+1+1+1+1+1+1 \\ 1+1+1+1+1+1+1+1+1+1 \end{array}\right.$$

no 1's

$$\left\{\begin{array}{l} 10 \\ 8+2 \\ 7+3 \\ 6+4 \end{array}\right. \qquad \left\{\begin{array}{l} 8+2 \\ 7+3 \\ 3+3+2+2 \\ 2+2+2+2+2 \end{array}\right.$$

GENERALIZATIONS — Andrews – Gordon, Bressoud

Hikami, Garoufalidis – Li – Zagier (knot polynomials 2015)

NEW R-R type identities — Kanade – Russell (2014–)

Bringmann – Jennings – Schaffer – Mahlburg 2018

Rosengren 2019 (evaluate integral in 2 ways)

<u>TODAY</u> No known q-series proof with only positive terms, and <u>no</u> cancellation.

$$\frac{1}{\prod_{n=0}^{\infty}(1-q^{5n+1})(1-q^{5n+4})} = \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{5n^2+n}{2}}$$

$$= \frac{1}{(q;q)_\infty}(q^2, q^3, q^5; q^5)_\infty$$

New refinements of RR mod 5 (O'Hara-S. 2014, 2018)

+ other partition theorems

Marking a part

Marking a sum of parts

Variation: Shifting parts

<u>Theorem</u> Let $M \equiv 1$ or $4 \bmod 5$.

\# partitions of n into parts $\equiv 1, 4 \bmod 5$ with k M's

= \# partitions of n with difference ≥ 2, λ,

1. λ has one part, $\lfloor n/M \rfloor = k$
2. λ has ≥ 2 parts, $\lfloor (\lambda_1 - \lambda_2 - 2)/M \rfloor = k$.

Weight 2, 3, 7 in 2nd R-R

$$\frac{1}{(1-tq^2)(1-wq^3)(1-vq^7)}\left[1 \ast \sum_{k=1}^{7} \frac{q^{k^2+k}}{(1-q)\cdots(1-q^k)}\right]$$

$$= \frac{1}{(1-q^2)(1-q^3)(1-q^7)} \times \text{(next page)}$$

$$1 + \frac{q^2(t+wq)}{1-tq^2} + q^6\frac{(w^2+vq+q^2)}{(1-tq^2)(1-wq^3)}$$

$$+ q^{12}\frac{(1+q+v^2q^2+vq^3+q^4+q^5+q^6)}{(1-tq^2)(1-wq^3)(1-vq^7)}$$

$$+ q^{20}\frac{(1+q+q^2+q^3+q^4+q^5+q^6)}{(1-tq^2)(1-wq^3)(1-q^4)(1-vq^7)}$$

$$+ q^{30}\frac{(1+q+q^2+q^3+q^4+q^5+q^6)}{(1-tq^2)(1-wq^3)(1-q^4)(1-q^5)(1-vq^7)}$$

$$+ q^{42}\frac{(1+q+q^2+q^3+q^4+q^5+q^6)}{(1-tq^2)(1-wq^3)(1-q^4)(1-q^5)(1-q^6)(1-vq^7)}$$

? Hilbert series, polytopes ?

Theorem Let $\mathbb{A} = \{2, 3, 6, 9, 11, 14, 16, \dots\}$

\# partitions of n into parts from $\mathbb{A}$

= \# partitions of n with difference ≥ 2, λ,

① if λ has a single part, then $n \not\equiv 1 \mod 3$

② if λ has at least two parts, then

$$\lambda_1 - \lambda_2 - 2 \not\equiv 1 \mod 3.$$

FACT # partitions of n into parts $\equiv 1,4 \bmod 5$

$\geq$ # partitions of n into parts $\equiv 2,3 \bmod 5$.

DEFN Fix Θ partition with parts from $\mathbb{A}$.

$E_{\Theta}^{\mathbb{A}}(n,k) =$ # partitions of n into parts from $\mathbb{A}$ containing Θ k times.

THM $\mathbb{A} = \{1,4,6,9,\ldots\}$, $\mathbb{B} = \{2,3,7,8,\ldots\}$

$\lambda \vdash M$ from $\mathbb{A}$ $\qquad \Theta \vdash M$ from $\mathbb{B}$

$E_{\Theta}^{\mathbb{B}}(n,k) \leq E_{\lambda}^{\mathbb{A}}(n,k)$ if $M \geq 3$.

Questions and Speculations

(1) Quintic transformation

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(tq)^{2n}}{(1-q)\cdots(1-q^n)} = \frac{(t^4q^9, t^2q^5, t^4q^6; q^5)_\infty}{(t^2q^3; q)_\infty}$$

$$\times {}_3\phi_2\left(\begin{matrix} t^2q^2, t^2q^3, t^2q^5 \\ t^4q^9, t^4q^6 \end{matrix} \middle| q^5; t^2q^5\right)$$

ONLY KNOWN PROOF

USES ORTHOGONAL

POLYNOMIALS

② Polynomial versions (Bressoud, Schur, Sills)

Ⓐ $$\sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} 2n \\ n+2j \end{bmatrix}_q$$

LHS = partitions with difference ≥ 2, rank ≤ $n-1$

RHS = ??
mod 1,4

Ⓑ $$\sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n+1-j \\ j \end{bmatrix}_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n+1 \\ \left\lfloor \frac{n+5j+2}{2} \right\rfloor \end{bmatrix}_q$$

LHS = partitions with difference ≥ 2, $\lambda_1 \le n$

(q-Fibonacci)

RHS = ??
mod 1,4

③ Rewrite $\dfrac{1}{\prod_{n=0}^{\infty}(1-q^{5n+1})(1-q^{5n+4})} = \sum_{m=0}^{\infty} \dfrac{q^{m^2}\, p_m(q)}{(1-q^5)\cdots(1-q^{5m})}$

where $p_m(1) = 5^m$, $p_m(q) \geq 0$. What are these

5^m partitions with difference ≥ 2 ?

<u>CONJ</u> (2020)

$$\prod_{n=0}^{\infty} (1-xq^{5n+1})^{-1}(1-yq^{5n+4})^{-1} = \sum_{m=0}^{\infty} q^{m^2} \frac{p_m(q,x,y)}{(1-q^5)\cdots(1-q^{5m})}$$

where $p_m(q,x,y) \geq 0$, $p_m(q,1,1) = \prod_{k=1}^{m} (1+q^k+q^{2k}+q^{3k}+q^{4k})$

Is there an x,y-weighting on these 5^m partitions?

4 Let $1, 4, 6, 9, 11, 14, \ldots = a_1, a_2, a_3, \ldots$

$1, 4 \bmod 5$

<u>FACT</u> $\dfrac{a_1 a_2 \cdots a_m}{m!} = \dfrac{(\text{an integer})}{5^E}.$

<u>Th</u> Let $a_1, a_2, \ldots$ be the integers $\pm i \bmod p$, p is a prime. Then

$$\frac{a_1 a_2 \cdots a_m}{m!} = \frac{(\text{an integer})}{p^E}.$$

Idea

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} = \frac{\text{positive polynomial in } q}{(1-q^{a_1})\cdots(1-q^{a_m})}$$

$$\frac{1}{[1][2][3][4]} = \frac{1+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^{10}}{[1][4][6][9]}$$

$$\frac{1}{[1][2][3][4][5]} = \frac{\text{poly of degree } 60}{[1][4][6][9][55]}$$

Many of these questions appear on my web page under "Some problems"

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