Modular iterated integrals associated with cusp forms

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The context

- $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

- $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

- $\mathfrak{H} = \{x + iy; y > 0\} \qquad z := x + iy$

- $\mathcal{R} := \{\text{real analytic } f : \mathfrak{H} \to \mathbb{C}; f(z) = O(y^C) \text{ as } y \to \infty, \text{ uniformly in } x, \text{ for some } C > 0\}$

- $\mathcal{O} := \{\text{holomorphic } f \in \mathcal{R}\} \qquad \mathcal{O}_c := \{\text{holomorphic } f \in \mathcal{R}_c\}$

The context

- $j(\gamma, z) = c_\gamma z + d_\gamma$ where $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma$
- For $k \in 2\mathbb{N}$, $r, s \in \mathbb{N}$ such that $r \equiv s \mod 2, r + s \geq 4$

$$f \underset{r,s}{|} \gamma(z) = j(\gamma, z)^{-r} j(\gamma, \bar{z})^{-s} f(\gamma z) \qquad \text{for all } f \in \mathcal{R}, \gamma \in \Gamma, z \in \mathfrak{H} .$$

We extend the action to $\mathbb{C}[\Gamma]$ by linearity.

Definition An $f \in \mathcal{R}$ is a *real-analytic modular form of weights* (r, s) *for* Γ *if*

1. for all $\gamma \in \Gamma$ and $z \in \mathfrak{H}$, we have $f \underset{r,s}{|} \gamma = f$, i.e.

$$f(\gamma z) = j(\gamma, z)^r j(\gamma, \bar{z})^s f(z) \qquad \text{for all } z \in \mathfrak{H},$$

2. for some $M \in \mathbb{N}$ and $a_{m,n}^{(j)} \in \mathbb{C}$.

$$f(z) = \sum_{|j| \le M} y^j \left(\sum_{m,n \ge 0} a_{m,n}^{(j)} q^m \bar{q}^n \right) \qquad (q := \exp(2\pi i z))$$

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Notation:

- $\mathcal{M}_{r,s} := \{$real analytic modular forms of weights (r, s) for $\Gamma\}$
- $\mathcal{M} = \bigoplus_{r,s} \mathcal{M}_{r,s}$
- $M_k = \mathcal{M}_{k,0}$ $\qquad S_k = \{$cusp forms of weight $k\}$

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Examples:

- $y =$Im$(z) \in \mathcal{M}_{-1,-1}$.
- If $f \in M_k$, then $f \in \mathcal{M}_{k,0}$ and $\bar{f} \in \mathcal{M}_{0,k}$.
- If $B = \{\pm T^n; n \in \mathbb{Z}\}$, then

$$\mathcal{E}_{r,s}(z) = \sum_{\gamma \in B\backslash\Gamma} \frac{1}{j(\gamma, z)^r j(\gamma, \bar{z})^s} \in \mathcal{M}_{r,s}$$

The context

<u>*Maass operators:*</u>

$\partial_r : \mathcal{M}_{r,s} \to \mathcal{M}_{r+1,s-1}$ and $\bar{\partial}_s : \mathcal{M}_{r,s} \to \mathcal{M}_{r-1,s+1}$ given by

$$\partial_r = 2iy\frac{\partial}{\partial z} + r \qquad \text{and} \quad \bar{\partial}_s = -2iy\frac{\partial}{\partial \bar{z}} + s.$$

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Definition 2. *Modular iterated integrals of length* ℓ Set recursively:

- $\mathcal{MI}_{-1} = 0$
- For integer $\ell \geq 0$, let $\mathcal{MI}_\ell$ be the largest subspace of $\bigoplus_{r,s\geq 0} \mathcal{M}_{r,s}$ satisfying

$$\partial \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + M[y] \times \mathcal{MI}_{\ell-1}$$
$$\bar{\partial} \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + \overline{M}[y] \times \mathcal{MI}_{\ell-1}$$

where $\overline{M}$ is the ring of anti-holomorphic modular forms.

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• Arithmetic significance of the space $\mathcal{MI}_\ell$, e.g.
i. motivic nature evidence by F. Brown,
ii. classical number theoretic invariants, such as L-functions and period polynomials have recently been associated to modular iterated integrals (D., Drewitt)

The questions

Question 1. The elements of $\mathcal{MI}_\ell$ almost exclusively studied so far are those that defined by

$$\partial \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + E[y] \times \mathcal{MI}_{\ell-1} \text{ and } \bar{\partial} \mathcal{MI}_\ell \subset \mathcal{MI}_\ell + \overline{E}[y] \times \mathcal{MI}_{\ell-1}$$

where E is the *subspace of M generated by Eisenstein series.*

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Important because arithmetic information is normally expected to be encapsulated by forms that are cuspidal.

The questions

Question 2. A more concrete characterisation of the space $\mathcal{MI}_\ell$ is given (at least conjecturally) in terms of Γ-*invariant* linear combinations of iterated integrals of modular forms.

Can we construct explicit families of iterated integrals whose invariant pieces are also explicit and belong to $\mathcal{MI}_\ell$?

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Such "invariant versions" of iterated integrals of modular forms are important for the theory of real-analytic modular forms, especially for applications to modular graph functions.

The questions

The solution to **Question 2** is based on the answer to

Question 3: *Second-order modular forms of weight k for the group G* are holomorphic functions of moderate growth characterised by the condition

$$f|_k(g-1)(h-1) = f|_k gh - f|_k g - f|_k h + f = 0 \qquad \text{for all } g, h \in G.$$

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The only such forms for $G = \mathrm{SL}_2(\mathbb{Z})$ are the standard modular forms of weight k. Is it possible to adjust the definition so that a non-trivial theory of second-order forms for more general groups, including $\mathrm{SL}_2(\mathbb{Z})$ is available?

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where E is the *subspace of M generated by Eisenstein series.*

What can be said about the remaining modular iterated integrals, i.e. the part originating in *cusp* forms?
We answer this question by constructing an explicit family of modular iterated-type integral originating in cusp forms.

Question 1 (cont.)

Let $f(z)$ be a cusp form of weight k for Γ. Consider its *Eichler integral*

$$F_f(z, X) = \int_{i\infty}^{z} f(w)(w - X)^{k-2} dw$$

(bar means conjugation acting trivially on the polynomial variable X). Set

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$$\phi_{r,s}(f;z,X) := \sum_{\gamma\in B\backslash\Gamma} \frac{F_f(\gamma z,\gamma X)}{j(\gamma,z)^r j(\gamma,\bar{z})^s} j(\gamma,X)^{k-2}.$$

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Let $\phi_{r,s}(f; i, z)$ be defined by

$$\phi_{r,s}(f; z, X) = \sum_{i=0}^{k-2} \phi_{r,s}(f; i, z)(X - z)^i (X - \bar{z})^{k-2-i}.$$

Question 1 (cont.)

Theorem 1 (D.) Assume that $r+s>k$. Then each $\phi_{r,s}(f;i,z)$ ($f \in S_k$, $i=0,\dots,k-2$) is well-defined and a modular iterated integral of length 2 such that

$$\partial \mathcal{MI}_2 \subset \mathcal{MI}_2 + S[y] \times \mathcal{MI}_1 \text{ and } \bar{\partial} \mathcal{MI}_2 \subset \mathcal{MI}_2 + \overline{S}[y] \times \mathcal{MI}_1$$

where S is the *space of cusp forms.*

Question 3

Second-order modular forms of weight k for the group G are holomorphic functions of moderate growth characterised by the condition

$$f|_k(g-1)(h-1) = f|_k gh - f|_k g - f|_k h + f = 0 \qquad \text{for all } g, h \in G.$$

The only such forms for $G = \mathrm{SL}_2(\mathbb{Z})$ are the standard modular forms of weight k. Is it possible to adjust the definition so that a non-trivial theory of second-order forms for more general groups, including $\mathrm{SL}_2(\mathbb{Z})$ is available?

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To allow for genuine second-order forms, we introduce the following construction:

Question 3 (cont)

Let $k_0, k \in 2\mathbb{N}$ and $\mathcal{O}_{k-2}[X]$ be the space of polynomials in X of degree $\leq k-2$ with coefficients in $\mathcal{O}$. We let Γ act on $\mathcal{O}_{k-2}[X]$ by

$$(f.\gamma)(z; X) := f(\gamma z; \gamma X) j(\gamma, z)^{-k_0} j(\gamma, X)^{k-2}$$

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$$(f.\gamma)(z;X) := f(\gamma z; \gamma X) j(\gamma, z)^{-k_0} j(\gamma, X)^{k-2}$$

Definition 3. An $f \in \mathcal{O}_{k-2}[X]$ is called an *extended second-order modular form* if

$$f.(\gamma - 1) \in M_{k_0} \otimes P_{k-2}[X] \qquad \text{and, for all parabolic } \pi \in \Gamma,\ f.\pi = f$$

In contrast to the origninal definition of second-order forms, this gives new objects also for $SL_0(\mathbb{Z})$, *including iterated itegrals of cusp forms:*

$$f_0(z) \int_{i\infty}^{z} f_1(w)(w - X)^{k-2} dw \qquad \text{for } f_0 \in M_{k_0} \text{ and } f_1 \in M_k.$$

Question 3 (cont)

Definition 3 is the basis for a definition applying to *real-analytic functions*: Let $k \in 2\mathbb{N}$, $r, s \in \mathbb{N}^*$ such that $r \equiv s \mod 2$, $r + s > 4$ and $\mathcal{R}_{k-2}[X]$ the space of polynomials in X of degree $\leq k-2$ with coefficients in $\mathcal{R}$. We let Γ act on $\mathcal{R}_{k-2}[X]$ by

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$$(f.\gamma)(z;X) := f(\gamma z; \gamma X) j(\gamma, z)^{-r} j(\gamma, \bar{z})^{-s} j(\gamma, X)^{k-2}$$

Definition 4. An $f \in \mathcal{R}_{k-2}[X]$ is called an *real-analytic iterated integrals* if

$$f.(\gamma - 1) \in \mathcal{E}_{r,s}(z) \otimes P_{k-2}[X] \qquad \text{and, for all parabolic } \pi \in \Gamma,\ f.\pi = f$$

The reason for the name is that a prototype for such functions is

$$\mathcal{E}_{r,s}(z) \int_{i\infty}^{z} f_1(w)(w - X)^{k-2} dw \qquad \text{for } f_1 \in M_k.$$

Question 2

A more concrete characterisation of the space $\mathcal{MI}_\ell$ is given (at least conjecturally) in terms of Γ-*invariant* linear combinations of iterated integrals of modular forms.

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Our approach is based on the real-analytic iterated integrals defined in our answer of our Question 3 above.

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For each $h \in S_k$ we consider

$$r_h^+(\gamma; X) = \int_{\gamma^{-1} i\infty}^{i\infty} h(w)(w-X)^{k-2} dw$$

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and set

$$\psi^+_{h;r,s}(z, X) := \sum_{\gamma \in B\backslash\Gamma} \frac{r_h(\gamma; X)}{j(\gamma, z)^r j(\gamma, \bar{z})^s}$$

and

$$\psi^-_{h;r,s}(z, X) := \sum_{\gamma \in B\backslash\Gamma} \frac{\overline{r_h(\gamma; X)}}{j(\gamma, z)^r j(\gamma, \bar{z})^s}$$

Theorem 2.(D.) Let $r+s>k$.

i. Each $\psi^{\pm}_{h;r,s}$ belongs to the space $\tilde{M}_c^{(2)}(\mathcal{R})$ of real-analytic iterated integrals.

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i. Each $\psi^{\pm}_{h;r,s}$ belongs to the space $\tilde{M}^{(2)}_c(\mathcal{R})$ of real-analytic iterated integrals.

ii. If $\tilde{\mathcal{M}}_{r,s}$ denotes the subspace of $\mathcal{M}_{r,s}$ generated by $\mathcal{E}_{r,s}$, then there is a surjective map

$$\tilde{M}^{(2)}_c(\mathcal{R}) \longrightarrow \tilde{\mathcal{M}}_{r,s} \otimes (S_k \oplus \bar{S}_k)$$

such that $\tilde{\mathcal{M}}_{r,s} \otimes (S_{k_1} \oplus \bar{S}_{k_1})$ is generated by the images of $\psi^{\pm}_{h;r,s}$.

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iii. There is a well-defined linear map from the subspace of $\mathcal{M}^{(2)}_c(\mathcal{R})$ generated by the family $\{\psi^{\pm}_{h;r,s}\}$ to $\oplus_{i=0}^{k-2}\mathcal{M}\mathcal{I}_2$.

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Sketch of proof of part ii.

To construct the map we first define a map

$$\psi : \tilde{M}_c^{(2)}(\mathcal{R}) \longrightarrow \tilde{\mathcal{M}}_{r,s} \otimes Z^1(\Gamma, P_{k-2}[X]).$$

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By definition, if $f \in \tilde{M}_c^{(2)}(\mathcal{R})$ then, for $\gamma \in \Gamma$, there is $v(\gamma) \in P_{k-2}[X]$ s.t.

$$f.(\gamma - 1) = v(\gamma)\,\mathcal{E}_{r,s}.$$

The map v is a 1-cocycle. We set $\psi(f) = v \otimes \mathcal{E}_{r,s}$.

Sketch of proof of part ii. (cont.)

The natural projection $Z^1(\Gamma, P_{k-2}[X]) \to H^1(\Gamma, P_{k-2}[X])$ induces

$$\bar{\psi} : \tilde{M}_c^{(2)}(\mathcal{R}) \longrightarrow \tilde{\mathcal{M}}_{r,s} \otimes H^1(\Gamma, P_{k-2}[X]) \cong \tilde{\mathcal{M}}_{r,s} \otimes (M_k \oplus \bar{S}_k)$$

The last isomorphism is induced by the Eichler-Shimura isomorphism. It can be shown that $\bar{\psi}$ maps $\tilde{M}_c^{(2)}(\mathcal{R})$ to $\tilde{\mathcal{M}}_{r,s} \otimes (S_k \oplus \bar{S}_k)$. With the surjectivity of the Eichler-Shimura map it is deduced that

$$\bar{\psi}(\langle \psi^{\pm}_{h;r,s} \rangle) = \tilde{\mathcal{M}}_{r,s} \otimes (S_k \oplus \bar{S}_k).$$

Question 2 (cont)

Sketch of proof of part iii.
The map is induced by the decomposition

$$\psi^{+}_{h;r,s}(z,X) := \phi_{r,s}(h;z,X) - F_h(z,X)\mathcal{E}_{r,s}(z)$$

and its counterpart for $\psi^{-}_{h;r,s}$.
By Th. 1, the coefficients $\phi_{r,s}(h;i,z)$ $(i=0,\dots,k-2)$ of $\phi_{r,s}(h;z,X)$ are modular iterated integrals of length 2.