Modular forms and Ramsey theory

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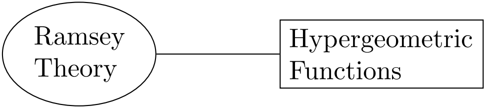
Overview



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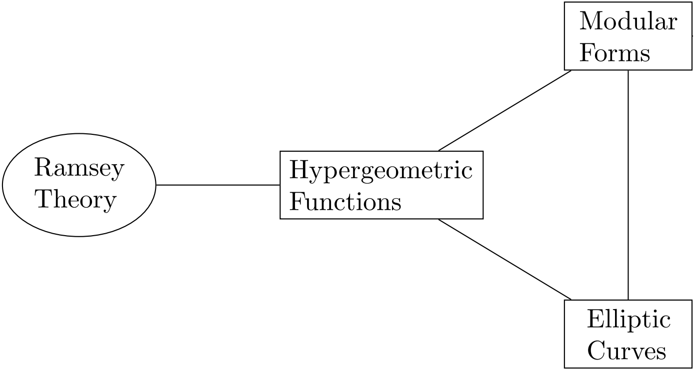
Modular Forms
Ramsey Theory
Hypergeometric Functions
Elliptic Curves

Ramsey Theory

Fundamental Question

Let G be any complete graph with colored edges, and let $r \in \mathbb{Z}^+$.

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Theorem (Ramsey's Theorem)

Let c and $n_1, \ldots, n_c$ be integers. There is a number $R(n_1, \ldots, n_c)$ such that if the edges of a complete graph of order $R(n_1, \ldots, n_c)$ are colored with c different colors, then it must contain a complete subgraph of order n_i whose edges are all color i, for some $1 \leq i \leq c$.

Ramsey Theory

Ramsey Number: $R(m, n)$

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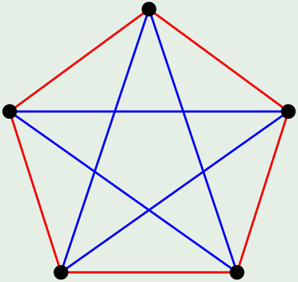
Example



- $R(3, 3) = 6$

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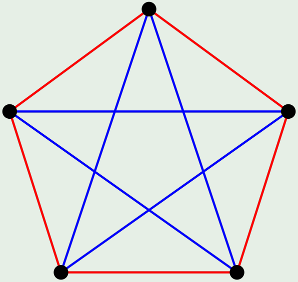

- $R(3,3) = 6$
- Friends and Strangers Theorem: In any party of six people, either at least three of them are pairwise mutual strangers or at least three of them are pairwise mutual acquaintances.

Ramsey Theory

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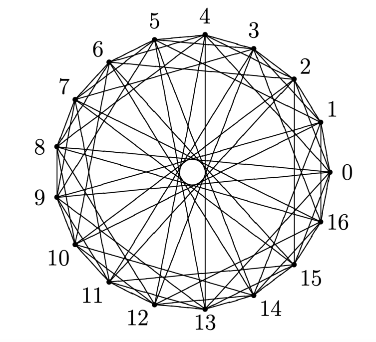

- $R(3,3) = 6$
- $R(4,4) = 18$

Ramsey Theory

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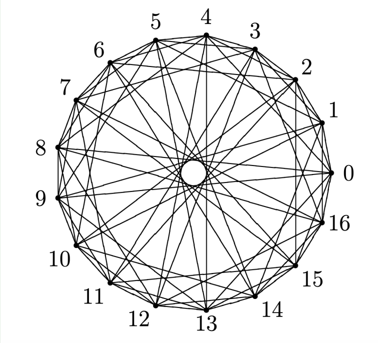

- $R(3,3) = 6$
- $R(4,4) = 18$
- Other known values: $R(2,n)$, $R(3,4)$-$R(3,9)$, $R(4,5)$

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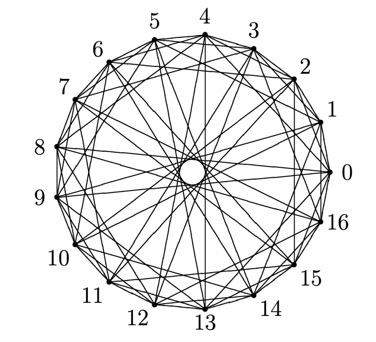

- $R(3,3) = 6$
- $R(4,4) = 18$
- Other known values: $R(2,n)$, $R(3,4)$-$R(3,9)$, $R(4,5)$
- $$43 \leq R(5,5) \leq 48$$
  $$\vdots$$
  $$17885 \leq R(19,19) \leq 9075135299$$

Ramsey Theory

Theorem (Greenwood–Gleason, 1955)

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- Ramsey #s can be restated in terms of graphs and complements.
- The Paley graph of order 17 is **self-complementary** and does not contain a complete subgraph of order 4, so $R(4,4) \geq 18$.
- Elementary upper bounds give $R(4,4) \leq 18$. $\square$

Paley Graphs

Definition

For $q \equiv 1 \pmod 4$ a prime power, the **Paley graph** $G(q)$ is the graph with vertex set $\mathbb{F}_q$ where ab is an edge if and only if $\left(\frac{a-b}{q}\right) = 1$.

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  Then $G(q) \cong G(q)^C$ by $x \mapsto xk \pmod q$, where k is a nonresidue.

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  so $g^{\frac{q-1}{2}} = \left(g^{\frac{q-1}{4}}\right)^2 \equiv -1 \pmod q$.

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Theorem (Evans–Pulham–Sheehan, 1981)

If $p \equiv 1 \pmod 4$ is a prime with $p = x^2 + y^2$ and y even, then the number of complete subgraphs of order 4 contained in $G(p)$ is

$$\mathcal{K}_4(G(p)) = \frac{p(p-1)\left((p-9)^2 - 4y^2\right)}{2^9 \cdot 3}.$$

Generalized Paley Graphs

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Definition 1

Let $k \geq 2$ be an integer.

Let q be a prime power such that $q \equiv \begin{cases} 1 \pmod{k}, & \text{if } q \text{ is even} \\ 1 \pmod{2k}, & \text{if } q \text{ is odd.} \end{cases}$

The **generalized Paley graph** $G_k(q)$ is the graph with vertex set $\mathbb{F}_q$ where ab is an edge if and only if $a - b$ is a k-th power residue.

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Generalized Paley graphs are

- **not self-complementary** unless $k = 2$.
- undirected: $\chi_k(-1) = 1$.

Goals

1) Find formulas for $\mathcal{K}_4\left(G_k(q)\right)$ and $\mathcal{K}_3\left(G_k(q)\right)$.

2) Obtain improved bounds for $R_k(4)$ and $R_k(3)$.

3) Identify connections with modular forms and elliptic curves.

Main Results

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Theorem 1 (D.–McCarthy, 2020)

Let $\chi_k \in \widehat{\mathbb{F}_q^*}$ *of order* k. *Then*

$$\mathcal{K}_4\left(G_k(q)\right) = \frac{q^3(q-1)}{24 \cdot k^6} \sum_{(t_1,t_2,t_3,t_4,t_5)\in(\mathbb{Z}_k)^5} {}_3F_2\left(\begin{array}{ccc|c} \chi_k^{t_1}, & \chi_k^{t_2}, & \chi_k^{t_3} & \\ & \chi_k^{t_4}, & \chi_k^{t_5} & \lambda \end{array}\right)_q .$$

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Corollary 1

We have that $\mathcal{K}_4(G(17)) = \mathcal{K}_4\left(G_3(127)\right) = \mathcal{K}_4\left(G_4(457)\right) = 0$, *and so*

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- $128 \leq R_3(4)$.
- $458 \leq R_4(4)$.

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Theorem 2 (D.–McCarthy, 2020)

Let $\chi_k \in \widehat{\mathbb{F}_q^*}$ *of order* k*. Then*

$$\mathcal{K}_3\left(G_k(q)\right) = \frac{q(q-1)}{6 \cdot k^3}\left(q - 3k + 1 + \sum_{\substack{s,t=1 \\ s+t \not\equiv 0 \pmod{k}}}^{k-1} J\left(\chi_k^s, \chi_k^t\right)\right).$$

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We have that $\mathcal{K}_3(G(5)) = \mathcal{K}_3\left(G_3(16)\right) = \mathcal{K}_3\left(G_4(41)\right) = 0$, *and so*

- $6 \leq R(3,3)$.
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Definition

For characters $A_0, A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of $\mathbb{F}_q^*$ and $\lambda \in \mathbb{F}_q$, define the finite field hypergeometric function

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \;\middle|\; \lambda \right)_q := \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^*}} \binom{A_0\chi}{\chi} \binom{A_1\chi}{B_1\chi} \cdots \binom{A_n\chi}{B_n\chi} \chi(\lambda). $$

Hypergeometric Function Evaluations

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Example

If p is an odd prime, then

$$ {}_2F_1\left(\begin{matrix} \phi_p, & \phi_p \\ & \varepsilon_p \end{matrix} \,\middle|\, 1\right)_p = -\frac{\phi_p(-1)}{p}. $$

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$$ {}_3F_2\left(\begin{matrix} \phi_p, & \phi_p, & \phi_p \\ & \varepsilon_p, & \varepsilon_p \end{matrix} \;\middle|\; 1\right)_p = \begin{cases} 0, & p \equiv 3 \ (4) \\ \frac{4x^2-2p}{p^2}, & p \equiv 1 \ (4),\ p = x^2+y^2,\ x \text{ odd}. \end{cases} $$

Hypergeometric Function Evaluations

Theorem (Greene)

$$
{}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \,\middle|\, x\right)_p = \frac{A_nB_n(-1)}{p}
$$

$$
\times \sum_{y=0}^{p-1} {}_nF_{n-1}\left(\begin{matrix} A_0, & A_1, & \ldots, & A_{n-1} \\ & B_1, & \ldots, & B_{n-1} \end{matrix} \,\middle|\, xy\right)_p \cdot A_n(y)\overline{A_n}B_n(1-y)
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$$

In other words: ${}_{n+1}F_n$ is a "trace" of ${}_nF_{n-1}$.

Connections to Elliptic Curves

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Example

Let $\lambda \in \mathbb{Q} \setminus \{0, 1\}$. Consider the elliptic curve

$$ {}_2E_1(\lambda) : y^2 = x(x-1)(x-\lambda) $$

with Hasse–Weil L-function

$$ L\left({}_2E_1(\lambda, s)\right) = \sum_{n=1}^{\infty} \frac{{}_2a_1(n;\lambda)}{n^s}. $$

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If p is a prime of good reduction, then

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and the number of points on the reduction of ${}_2E_1(\lambda)$ mod p is

$$ N_{{}_2E_1(\lambda)}(p) = p + 1 - {}_2a_1(p;\lambda). $$

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Example

Let $\lambda \in \mathbb{Q} \setminus \{0, 4\}$. Consider the elliptic curve

$$ {}_3E_2(\lambda) : y^2 = x^3 - \lambda^2 x^2 + \left(4\lambda^3 - \lambda^4\right) x + \lambda^6 - 4\lambda^5 $$

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If $\lambda \in \mathbb{Q} \setminus \{0,1\}$ and N_λ is the conductor of ${}_2E_1(\lambda)$, then there is a newform $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_2^{\text{new}}\left(\Gamma_0\left(N_\lambda\right)\right)$ such that

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Example

If $\lambda \in \mathbb{Q} \setminus \{0,4\}$ and N_λ is the conductor of ${}_3E_2(\lambda)$, then there is a newform $f(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_2^{\text{new}}(\Gamma_0(N_\lambda))$ such that

$$b(p)^2 = p + \frac{p^2}{\phi_p(\lambda^2 - 4\lambda)} \cdot {}_3F_2\left(\begin{matrix} \phi_p, & \phi_p, & \phi_p \\ & \varepsilon_p & \varepsilon_p \end{matrix} \,\middle|\, \frac{4}{4-\lambda} \right)_p$$

Further Connections to Modular Forms

Let f be the unique newform in $S_3\left(\Gamma_0(16), \left(\frac{-4}{\cdot}\right)\right)$ with CM by $\mathbb{Q}(i)$:

$$f(z) = \sum_{n=1}^{\infty} \alpha(n) q^n = q \prod_{m=1}^{\infty} \left(1 - q^{4m}\right)^6, \quad q := e^{2\pi i z}.$$

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Corollary 3

Let $p \equiv 1 \pmod 4$ be prime. Then

$$\mathcal{K}_4(G(p)) = \frac{p(p-1)\left((p-9)^2 - 2p + \alpha(p)\right)}{2^9 \cdot 3}.$$

Conjectural Connections

Let g_1 be the unique newform in $S_2\left(\Gamma_0(27)\right)$ with CM by $\mathbb{Q}\left(\sqrt{-3}\right)$:

$$g_1(z) = \sum_{n=1}^{\infty} \beta_1(n) q^n = q \prod_{m=1}^{\infty} \left(1 - q^{3m}\right)^2 \left(1 - q^{9m}\right)^2 ,$$

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and let $g_2 \in S_3\left(\Gamma_0(27), \left(\frac{-3}{\cdot}\right)\right)$ be the non-CM newform

$$g_2(z) = \sum_{n=1}^{\infty} \beta_2(n) q^n = q + 3iq^2 - 5q^4 - 3iq^5 + 5q^7 - 3iq^8 + \cdots .$$

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Conjecture 1

Let $p \equiv 1 \pmod 6$ be prime. Then

$$\mathcal{K}_4\left(G_3(p)\right) = \frac{p(p-1)}{2^3 \cdot 3^7} \Big[p^2 - 5p\left(\beta_1(p) + 11\right) + 10\beta_1(p)^2 + 85\beta_1(p) + 316 + 12\beta_2(p) \Big].$$

Conjectural Connections

Define the following newforms:

$$h_1(z) = \sum_{n=1}^{\infty} \gamma_1(n) q^n = q \prod_{m=1}^{\infty} (1-q^{4m})^{-2}(1-q^{8m})^8(1-q^{16m})^{-2} \in S_2(\Gamma_0(64));$$

$$h_2(z) = \sum_{n=1}^{\infty} \gamma_2(n) q^n = q + 2iq^3 - q^9 - 6iq^{11} - 6q^{17} + 2iq^{19} + 5q^{25} \cdots \in S_2(\Gamma_0(64), \Psi_1);$$

$$h_3(z) = \sum_{n=1}^{\infty} \gamma_3(n) q^n = q \prod_{m=1}^{\infty} (1-q^m)^2(1-q^{2m})(1-q^{4m})(1-q^{8m})^2 \in S_3(\Gamma_0(8), (\tfrac{-2}{\cdot})); \text{ and}$$

$$h_4(z) = \sum_{n=1}^{\infty} \gamma_4(n) q^n = q + 4iq^3 + 2q^5 - 8iq^7 - 7q^9 - 4iq^{11} - 14q^{13} + \cdots \in S_3(\Gamma_0(32), (\tfrac{-4}{\cdot})),$$

where

- Ψ_1 is the Dirichlet char. mod 64 sending $(63, 5) \mapsto (1, -1)$,
- h_1 has CM by $\mathbb{Q}(i)$, and
- h_2 and h_3 have CM by $\mathbb{Q}(\sqrt{-2})$.

Conjectural Connections

Define

$$\gamma_5(p) := p^2 \, {}_3F_2\left(\begin{matrix} \chi_4, & \chi_4 & \bar{\chi_4} \\ & \varepsilon, & \varepsilon \end{matrix} \;\middle|\; 1 \right)_p .$$

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Conjecture 2

Let $p \equiv 1 \pmod 8$ be prime. Then

$$\mathcal{K}_4\left(G_4(p)\right) = \frac{p(p-1)}{2^{15} \cdot 3}\Big[p^2 - p\left(15\gamma_1(p) + 142\right) + 76\gamma_1(p)^2 + 465\gamma_1(p) + 801 + 10\gamma_1(p)\gamma_2(p) + 30\gamma_3(p) + 30\gamma_4(p) + 12\gamma_5(p)\Big].$$

Counting Vertices and Edges

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1) $\#V\left(G_k(q)\right) = q$
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Proposition

3) $\#V\left(H_k(q)\right) = \frac{q-1}{k}$

4) $$\#E\left(H_k(q)\right) = \frac{q-1}{2k^3}\left(q - 3k + 1 + \sum_{\substack{s,t=1 \\ s+t \not\equiv 0\,(k)}}^{k-1} J\left(\chi_k^s, \chi_k^t\right)\right)$$

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Proposition

5) $$\#V\left(H_k^1(q)\right) = \frac{1}{k^2}\left(q - 3k + 1 + \sum_{\substack{s,t=1 \\ s+t \not\equiv 0 \pmod{k}}}^{k-1} J\left(\chi_k^s, \chi_k^t\right)\right)$$

6) $$\#E\left(H_k^1(q)\right) = \frac{1}{2k^5} \sum_{t_1,t_2,t_3,t_4,t_5=0}^{k-1} q^2 \, {}_3F_2\left(\begin{matrix} \chi_k^{t_1}, & \chi_k^{t_2}, & \chi_k^{t_3} \\ & \chi_k^{t_4}, & \chi_k^{t_5} \end{matrix} \middle| \, 1\right)_q$$

Character Sums in Graph Theory

Proof of Proposition.

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$$\chi_k(a) = \chi_k(b) = \chi_k(1-a) = \chi_k(1-b) = \chi_k(a-b) = 1.$$

Character Sums in Graph Theory

Proof of Proposition (continued).

- Greene: For $A, B, C, D, E \in \widehat{\mathbb{F}_q^*}$,

$$q\,{}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix}\,\middle|\,\lambda\right)_q = \sum_{b\in\mathbb{F}_q} A\overline{C}(b)\overline{B}C(1-b)\overline{A}(b-\lambda)$$

Character Sums in Graph Theory

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$$q^2\,{}_3F_2\left(\begin{array}{ccc|c} A, & B, & C & \\ & D, & E & \end{array}\lambda\right)_q$$

$$= \sum_{a,b\in\mathbb{F}_q} A\overline{E}(a)\overline{C}E(1-a)B(b)\overline{B}D(b-1)\overline{A}(a-\lambda b).$$

Character Sums in Graph Theory

Proof of Proposition (continued).

- Char. Orthogonality: If $k \geq 2$, $q \equiv 1 \pmod{k}$ is a prime power,

$$\frac{1}{k}\sum_{t=0}^{k-1}\chi_k^t(b) = \begin{cases} 1, & \text{if } b \text{ is a } k\text{-th power} \\ 0, & \text{if } b \text{ is not a } k\text{-th power.} \end{cases}$$

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- For each vertex a of $H_k(q)$,

$$\deg_{H_k(q)}(a) = \frac{1}{k^2} \sum_{b \in \mathbb{F}_q^* \setminus \{a\}} \sum_{s,t=0}^{k-1} \chi_k^s(b)\chi_k^t(a-b).$$

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- Rewrite in terms of Jacobi sums.

Character Sums in Graph Theory

Proof of Proposition (continued).

- For each vertex a of $H_k^1(q)$,

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- Rewrite in terms of ${}_2F_1$.
- The number of edges in $H_k^1(q)$ is

$$\#E\left(H_k^1(q)\right) = \frac{1}{2} \sum_{a \in V\left(H_k^1(q)\right)} \deg_{H_k^1(q)}(a).$$

$\square$

A Lemma in Graph Theory

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For $n \in \mathbb{Z}^+$, we have that

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$\#\{\text{complete subgraphs of order 4 in } G_k(q)\} \approx \#\{\text{edges in } H_k^1(q)\}$.

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Proof of Lemma.

- The subgraph $(0, a_1, \ldots, a_n)$ of $G_k(q)$ is complete iff $\chi_k\left(a_i\right) = \chi_k\left(a_i - a_j\right) = 1$ for all $1 \leq i < j \leq n$.

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Proof of Lemma (continued).

- The subgraph $(1, a_1, \ldots, a_n)$ of $H_k(q)$ is complete iff $\chi_k\left(a_i\right) = \chi_k\left(a_i - 1\right) = \chi_k\left(a_i - a_j\right) = 1$ for all $1 \leq i < j \leq n$.
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- This is true iff $(a_1, \ldots, a_n)$ is a complete subgraph of $H_k^1(q)$.
- Fix $a \in \left(\mathbb{F}_q^*\right)^k$. Repeat for all $\frac{q-1}{k}$ vertices by aut. $\lambda \mapsto a\lambda$.
- Divide by $n+1$ repeated subgraphs. $\square$

Proofs of Main Theorems

Proofs of Theorems 1 and 2.

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- Proposition $\Longrightarrow \mathcal{K}_2\left(H_k^1(q)\right)$.

Proofs of Main Theorems

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- Proposition $\implies \mathcal{K}_2\left(H_k^1(q)\right)$.
- Lemma $\implies \mathcal{K}_4\left(G_k(q)\right)$. $\square$

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- Reduction formulas for hypergeometric functions, e.g.

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- 10 distinct cases depending on $\vec{t} \in (\mathbb{Z}_k)^5$.

Proofs of Main Theorems

Proofs of Corollaries 1 and 2 (continued).

- To simplify notation:

$$\mathbb{R}_k(q) := \sum_{\substack{s,t=1 \\ s+t \not\equiv 0\ (k)}}^{k-1} J\left(\chi_k^s, \chi_k^t\right),$$

$$\mathbb{S}_k(q) := \sum_{\substack{s,t,v=1 \\ s+t, v+t, v-s \not\equiv 0\ (k)}}^{k-1} J\left(\chi_k^s, \chi_k^t\right) J\left(\overline{\chi_k}^s, \chi_k^v\right),$$

Proofs of Main Theorems

Proofs of Corollaries 1 and 2 (continued).

- To simplify notation:

$$\mathbb{R}_k(q) := \sum_{\substack{s,t=1 \\ s+t\not\equiv 0\ (k)}}^{k-1} J\left(\chi_k^s, \chi_k^t\right),$$

$$\mathbb{S}_k(q) := \sum_{\substack{s,t,v=1 \\ s+t,v+t,v-s\not\equiv 0\ (k)}}^{k-1} J\left(\chi_k^s, \chi_k^t\right) J\left(\overline{\chi_k}^s, \chi_k^v\right),$$

$$X_k := \left\{(t_1, \ldots, t_5) \in (\mathbb{Z}_5)^5 \mid t_1, t_2, t_3 \neq 0, t_4, t_5; t_1 + t_2 + t_3 \neq t_4 + t_5\right\}$$

Proofs of Main Theorems

Theorem 3

$$\mathcal{K}_4\left(G_k(q)\right) = \frac{q(q-1)}{24\cdot k^6}\Big[10\mathbb{R}_k(q)^2 + 5\left(q-2k^2+1\right)\mathbb{R}_k(q) - 15\mathbb{S}_k(q)$$
$$+ q^2 - 5\left(2k^2-3k+2\right)q + 15k^3 - 10k^2 + 1 + q^2\sum_{\vec{t}\in X_k} {}_3F_2\left(\vec{t}\mid 1\right)_{q,k}\Big]$$

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- Use Theorem 3 for each orbit representative with $k = 2, 3, 4$.
- If $\mathcal{K}_\ell\left(G_k(q)\right) = 0$ for some q, then $q < R_k(\ell)$. $\square$

Summary

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Theorem 1

$$\mathcal{K}_4\left(G_k(q)\right) = \frac{q^3(q-1)}{24\cdot k^6} \sum_{(t_1,t_2,t_3,t_4,t_5)\in(\mathbb{Z}_k)^5} {}_3F_2\left(\begin{matrix} \chi_k^{t_1}, & \chi_k^{t_2}, & \chi_k^{t_3} \\ & \chi_k^{t_4}, & \chi_k^{t_5} \end{matrix} \middle| \lambda\right)_q$$

Theorem 2

$$\mathcal{K}_3\left(G_k(q)\right) = \frac{q(q-1)}{6\cdot k^3}\left(q-3k+1+\sum_{\substack{s,t=1\\ s+t\not\equiv 0 \pmod{k}}}^{k-1} J\left(\chi_k^s,\chi_k^t\right)\right)$$

Summary

Corollary 1

- $18 \leq R(4,4)$
- $128 \leq R_3(4)$
- $458 \leq R_4(4)$

Corollary 2

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Corollary 3

Let $p \equiv 1 \pmod 4$ *be prime. Then*

$$\mathcal{K}_4(G(p)) = \frac{p(p-1)\left((p-9)^2 - 2p + \alpha(p)\right)}{2^9 \cdot 3}.$$

Summary

Conjecture 1

Let $p \equiv 1 \pmod 6$ be prime. Then

$$\mathcal{K}_4\left(G_3(p)\right) = \frac{p(p-1)}{2^3 \cdot 3^7}\Big[p^2 - 5p\left(\beta_1(p) + 11\right) + 10\beta_1(p)^2 + 85\beta_1(p) + 316 + 12\beta_2(p)\Big].$$

Conjecture 2

Let $p \equiv 1 \pmod 8$ be prime. Then

$$\mathcal{K}_4\left(G_4(p)\right) = \frac{p(p-1)}{2^{15} \cdot 3}\Big[p^2 - p\left(15\gamma_1(p) + 142\right) + 76\gamma_1(p)^2 + 465\gamma_1(p) + 801 + 10\gamma_1(p)\gamma_2(p) + 30\gamma_3(p) + 30\gamma_4(p) + 12\gamma_5(p)\Big].$$