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Building Algebraic Knowledge and Flexibility: A Comparison of Secondary-School Students in the U.S. and Japan

Objective. Success in algebra is necessary for access to higher mathematics and is correlated with positive life outcomes, such as college graduation (Adelman, 2006; National Mathematics Advisory Panel, 2008). Difficulties with algebra, and mathematics learning more generally, are more substantial in the United States than in some other countries, especially East Asian countries such as Japan (e.g., PISA, 2018). The objective of the current study was to better understand differences in algebra knowledge among secondary-school students in the U.S. and Japan.

Perspectives. Competency in algebra requires developing conceptual and procedural knowledge. *Conceptual knowledge* is knowledge of concepts, which are abstract, and general principles (e.g., Byrnes & Wasik, 1991; Canobi, 2009; Rittle-Johnson et al., 2001). *Procedural knowledge* is often defined as knowledge of mathematical strategies (e.g., Baroody et al., 2007; Byrnes & Wasik, 1991; Canobi, 2009; Rittle-Johnson et al., 2001). A strategy is a series of steps, or actions, done to accomplish a goal. Success in algebra also requires flexibility in the use of symbolic procedures. Students need to know multiple strategies for solving problems and be able to select the most appropriate strategy for a given problem (Star & Rittle-Johnson, 2008; Woodward et al., 2012). We use the term *procedural flexibility* to refer to this ability (Star, 2005, 2007; Star & Seifert, 2006; Verschaffel et al., 2009). Learners who develop procedural flexibility are more likely to use or adapt existing strategies when faced with unfamiliar problems and to have greater conceptual knowledge (Blöte et al., 2001; Hiebert et al., 1996). Furthermore, procedural flexibility is a salient characteristic of experts in mathematics (Dowker, 1992; Star & Newton, 2009). Procedural flexibility is distinct from, but related to, conceptual and procedural knowledge of algebra (Schneider et al., 2011).

Recent cross-national research conducted in Finland, Spain and Sweden suggests potential cross-national differences in students' procedural flexibility within algebra (Star et. al., under review). Different emphases in educational practice and prior knowledge might explain these cross-national differences. Comparisons of 8th grade mathematics instruction from the 1990's suggests students in Japan will have greater procedural flexibility than students in six other countries, including the U.S (Hiebert et al., 2003). In Japanese lessons, problems were more likely to have multiple strategies publicly presented and to examine and compare the multiple strategies than problems in lessons from other countries. Overall, prior research suggests that Japanese students should have greater algebra knowledge, including procedural flexibility, than students in the U.S. However, past studies have not directly assessed and compared procedural flexibility of students in the U.S. and Japan.

In the current study, we contrasted the algebra knowledge of students from a school in Japan to the knowledge developed by U.S. students who did or did not participate in an evaluation of an experimental approach to more productively support learning of algebra, including procedural flexibility. The experimental approach centers on *Comparison and Discussion of Multiple Strategies (CDMS*) and is based on converging evidence from cognitive science and mathematics education research on the importance of multiple strategies, comparison, and discussion for mathematics learning. Due to the difficulties many U.S. teachers face in effectively implementing these key instructional practices (Hiebert et al., 2003; Richland et al., 2012), the CDMS instructional approach incorporates worked examples that anticipate common student strategies, selected to capture important differences in strategies, and sequenced and paired to facilitate meaningful comparison. Both strategies are presented visually and simultaneously, with spatial cues and common language to help students align and map the solution steps, facilitating noticing of important similarities and differences in the strategies (Begolli & Richland, 2015; Gentner, 1983; Namy & Gentner, 2002; Richland et al., 2007). A series of increasingly sophisticated discussion prompts are included, which promote specific comparisons and generalizations tailored to the learning goal of the lesson. Professional development also provided training and practice promoting student explanation and discussion. Thus, we explored the algebra knowledge of U.S. students who were and were not exposed to a CDMS approach to the knowledge of a convenience sample of Japanese students.

Method. Almost 800 U.S. students (N = 779) completed the assessment near the end of the school year. It was a convenience sample drawn from primarily suburban schools in Massachusetts and New Hampshire. A group of 316 students served as the control group, as these students had received their instructor’s usual teaching methods from one of 12 teachers. Another group of 463 students had received specialized, supplemental, instruction based on the CDMS materials from one of 15 teachers. Most students were finishing 9th grade; students from one classroom per condition were in an accelerated math sequence and were finishing 8th grade. Assignment to condition occurred on a school-wide basis and was not random. Schools had predominantly white students and relatively few African American or Asian students; the control schools had more Hispanic students than the treatment schools (26% vs 6% of students). Rates of students qualifying for free- or reduced-price lunch was fairly low at treatment schools (17% of students) and moderate at control schools (35%).

Japanese students (N = 78) from two 9th grade classrooms at one school in a major city in the central district of Japan completed the assessment after covering all algebra units for the year. A vast majority of the population in Japan is ethnically Japanese (98%). It was a publicly-funded school, but it did have an entrance exam, so students' mathematics scores were likely to be higher than average. Although we were not able to directly compare student demographics of the samples, the wealth parity index for academic achievement is substantially higher in Japan than in the United States, meaning that the disparity in academic achievement based on family wealth is much lower in Japan than in the United States (https://www.education-inequalities.org/). Overall, both the U.S. and Japanese students were convenience samples drawn from schools that were likely above average in student performance for their respective countries.

Materials. The U.S. research team created a measure of students' conceptual knowledge, procedural knowledge and procedural flexibility for major algebra topics (e.g., linear equations, systems of equations, graphing quadratics). A team in Japan translated the assessment; they suggested wording changes to improve clarity and those changes were made to both the English and Japanese version. Two conceptual items were cut due to linguistic differences that were too difficult to translate. This resulted in a total of 23 scorable items: 8 conceptual knowledge items (see Figure 1 for sample items), 5 procedural knowledge items, and 10 flexibility items (see Figure 2 for sample items). All of the items were multiple choice except for 2 of the procedural knowledge items, which asked for the numeric solution to a problem. Students received one point per item for providing the correct answer. On the Japanese version, a prompt to explain their reasoning was added to one of the conceptual knowledge items to allow the Japanese team to further explore student thinking about the item.

Results. As shown in Figure 3, the Japanese sample had substantially higher accuracy than both the U.S. treatment and control groups on the overall assessment, including on the conceptual, procedural and flexibility subscales. To better understand knowledge differences, we inspected performance on each item by student group. Out of the 23 items, U.S. students had higher accuracy than Japanese students on only two items; the items were both conceptual knowledge items. We present a breakdown of responses on these two items in Table 1. The first item targeted students' knowledge of a defining property of linear functions; specifically, that each x value can correspond to only one y value (see item 4 in Figure 1). It was the only item where less than half of Japanese students answered correctly, and each incorrect answer was selected by about 20% of students. In contrast, more than half of US students answered the item correctly, and answer choice c was the most common incorrect answer; this was true across treatment and control students. The Japanese students seemed more likely to think some calculation process would make one of the equations (choices b and d) a linear function and/or were less familiar with evaluating whether equations or tables represent linear functions when the two are mixed as alternatives. The other item targeted understanding systems of equations and when there would be no solution (see item 6 in Figure 1). Accuracy was similar for the Japanese students and the U.S. students in the control condition, but was higher among the U.S. students in the treatment condition. One of the CDMS lessons emphasizes understanding when a system of equation has no solution, which may explain the higher performance by the treatment students. Over-selection of answer choice d (i.e., lines did not cross in graph, but would cross eventually) seemed to account for the poorer performance of the Japanese students. To avoid selecting this answer, students must extend the lines beyond the figure and make an inference. Overall, the Japanese students displayed strong algebra knowledge, but these two items suggest some weaknesses in qualitatively looking at relationships and applying concepts across representations. These items may reflect weakness in reconstructing conceptual knowledge beyond situations taught in their mathematics lessons.

Given our interest in procedural flexibility, we also examined the types of errors being made on the procedural flexibility items. On most of these items, students could make correctness errors and/or efficiency errors. Correctness errors involve ignoring a correct strategy or selecting an incorrect strategy, while efficiency errors involve ignoring a more efficient strategy or choosing a less efficient strategy. See Table 2 for a description of each error type and Figure 2 for sample items illustrating each error type. Efficiency errors were more common than correctness errors across all three groups, with the most common error being choosing a correct but less efficient strategy option. Another common error was failing to recognize some correct strategies. The most substantial difference across the three groups was how often students left items blank, with many more U.S. students leaving items blank, especially in the control group, while almost all Japanese students answered all questions.

To better understand efficiency errors, we examined responses on two items that were designed with demonstrated strategies that were considered most efficient, somewhat efficient and least efficient. Japanese students seemed more attuned to procedural efficiency (see Table 3). First, they chose the most efficient strategy most often. Second, they were more likely to choose the somewhat efficient strategy than the least efficient strategy on both items. They recognized that a strategy that leads to fractions when solving systems of equations (Nadia's way on item 26) and a tabular strategy for factoring a trinomial (item 27) are particularly inefficient. In contrast, students in the U.S. control group struggled to distinguish between the efficiency of the three strategies. Students in the U.S. treatment group had more success distinguishing efficiency that than the U.S. control group, but less success than the Japanese students.

Scholarly significance. As a reminder, these cross-national comparisons were meant to prompt reflection on potential differences in knowledge and instruction, not make general claims about performance differences among a representative sample of students from the two countries. The Japanese students had stronger algebra knowledge than the U.S. students that participated in this study, in line with international comparisons based on representative samples (e.g., PISA, 2018). The exception was lower accuracy by the Japanese students on two conceptual knowledge items, suggesting some weaknesses in qualitatively looking at relationships and applying concepts across representations. This may reflect typical Japanese mathematics instruction focusing on giving answers to problems, with less attention to more qualitative analysis of mathematical concepts. The current study extends past cross-country comparisons by considering procedural flexibility. Efficiency errors were the most common error across groups, but the Japanese students were more attuned to the efficiency of strategies, even when compared to U.S. students participating in an intervention that targeted procedural flexibility. This may be because presentation of multiple strategies and examination and comparison of the multiple strategies may be more frequent in Japanese instruction (Hiebert et al., 2003).

Table 1

Error Analysis on Two Items on Which U.S. Students Had Higher Percent Correct than Japanese Students.

	Proportion of students selecting each response option		
	Japan	U.S. Treatment	U.S. Control
Item 4: Property of linear functions			
Correct (choice a)	36.9	57.1	51.1
Choice b	21.4	4.3	7.6
Choice c	20.2	21.4	22.5
Choice d	19.1	10.8	11.4
Blank	2.4	6.3	7.3
Item 6: Graph representing system of equations with no solution			
Correct (choice a)	66.2	79.2	62.0
Choice b	9.1	7.6	11.1
Choice c	0.0	3.3	7.9
Choice d	23.4	8.0	13.0
Blank	1.3	2.0	6.0

Table 2

Error Analysis on Procedural Flexibility Items

Response Type (number of items where error was possible in parentheses)	Example Response See Figure 2 for referenced items	Proportion of Responses (Proportion of Errors for that Group in Parentheses) Japan	U.S. Treatment	U.S. Control
1. Correct Answer		0.68	0.38	0.23
2. Correctness Error		0.13	0.22	0.23
		(0.40)	(0.35)	(0.30)
a. Select incorrect strategy (n = 1)	Select Jamal's way on item 22	0.02 (0.05)	0.04 (0.07)	0.06 (0.07)
b. Ignore a correct strategy (n = 3)	Fail to recognize all 3 strategies are correct on item 21	0.11 (0.35)	0.18 (0.28)	0.17 (0.22)
3. Efficiency Error		0.18	0.31	0.37
		(0.55)	(0.50)	(0.48)
a. Choose correct but less efficient strategy (n = 4)	Select Gabriella's or Nadia's way on item 27	0.16 (0.48)	0.27 (0.43)	0.32 (0.42)
b. Ignore a more efficient strategy (n = 1)	Identify strategy as a very good way on item 23b	0.02 (0.07)	0.04 (0.07)	0.05 (0.06)
4. Blank	Left item blank or wrote "I don't know"	0.01 (0.05)	0.09 (0.15)	0.17 (0.22)

Note: Items 23A and 24A&B were omitted from the error analysis because responses did not fit into these categories.

Table 3

Attunement to Procedural Efficiency: Choosing Strategies of Varying Efficiency as Best Strategy to Use on a Timed Test

	Proportion of students selecting		
	Japan	**U.S. Treatment**	**U.S. Control**
Item 26: Systems of Equations			
Most efficient (choice a)	0.63	0.43	0.34
Somewhat efficient (choice b)	0.28	0.29	0.24
Least efficient (choice c)	0.08	0.17	0.24
Blank	0.01	0.11	0.18
Item 27: Factoring Trinomial			
Most efficient (choice b)	0.88	0.57	0.40
Somewhat efficient (choice c)	0.09	0.25	0.25
Least efficient (choice a)	0.01	0.10	0.20
Blank	0.01	0.08	0.16

Figure 1

Sample Conceptual Knowledge Items

4) Circle the example that could represent a linear function.

a.

x	-3	0	3
y	4	6	8

b. $\frac{5}{x} + y = -7$

c.

x	1	3	5	3
y	4	2	0	-2

d. $x + \frac{2}{y} = 4$

6) Which of the following graphs could represent a system of equations with no solution?

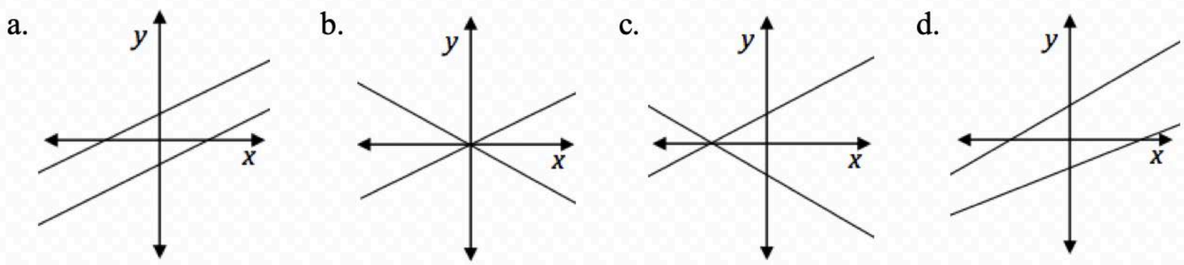


Figure 2

Sample Procedural Flexibility Items. *Correct answer in italics.*

21) Below is the beginning of Gabriella's, Jamal's, and Nadia's work in solving the equation $x + 7 - 3 = 12 - 2x$. To start solving this problem, which way(s) may be used?

a. Gabriella's way

b. Jamal's way

c. Nadia's way

d. Jamal's and Nadia's ways

e. Gabriella's, Jamal's, and Nadia's ways

Gabriella's way:	Jamal's way:	Nadia's way:
Subtract 3 from 7: $x + 4 = 12 - 2x$	Add $2x$ to both sides: $3x + 7 - 3 = 12$	Subtract (7 – 3) from both sides: $x = 8 - 2x$

22) Below is the beginning of Gabriella's, Jamal's, and Nadia's work in simplifying the expression $(7x^3 + 4x) - (8x^2 + 5x - 3)$. To start solving this problem, which way(s) may be used?

a. Gabriella's way

b. Jamal's way

c. Nadia's way

d. Gabriella's and Jamal's ways

e. Gabriella's and Nadia's ways

Gabriella's way:	Jamal's way:	Nadia's way:
$7x^3 - 8x^2 + 4x - 5x + 3$ $7x^3 - 8x^2 + (4-5)x + 3$	$\begin{array}{r} 7x^3 + 4x \\ \underline{-8x^2 - 5x + 3} \end{array}$ $(7-8)x^{3+2} + (4-5)x + 3$	$\begin{array}{r} 7x^3 + 0x^2 + 4x + 0 \\ \underline{-0x^3 - 8x^2 - 5x + 3} \end{array}$ $(7-0)x^3 + (0-8)x^2 + (4-5)x + 3$

23) Jamal solved the following problem:

$$\begin{cases} x + 2y = 9 \\ 3x - 2y = 11 \end{cases}$$

This is how Jamal started the problem:

$$x + 2y = 9$$
$$2y = 9 - x$$
$$y = \frac{9 - x}{2}$$

23b. Do you think this is a good way to start this problem? **Circle one:**

a. Very good way

b. May be used, but not a very good way

c. May not be used

26) On a timed test, which would be the BEST way to start solving this system of equations? (Circle the letter for the best way.)

$$\begin{cases} 4x - 3y = 11 \\ 5x + y = 19 \end{cases}$$

a. Gabriella's way:	b. Jamal's way:	c. Nadia's way:
$5x + y = 19$ $y = 19 - 5x$ …	$5 \cdot (4x - 3y = 11)$ $-4 \cdot (5x + y = 19)$ ――――――― …	$4x - 3y = 11$ $x = \frac{3y + 11}{4}$ …

27) On a timed test, which would be the BEST way to start factoring the trinomial below?

(Circle the letter for the best way.)

$$12x^2 + 24x - 36$$

<table>
<tr><td>a. Gabriella's way:

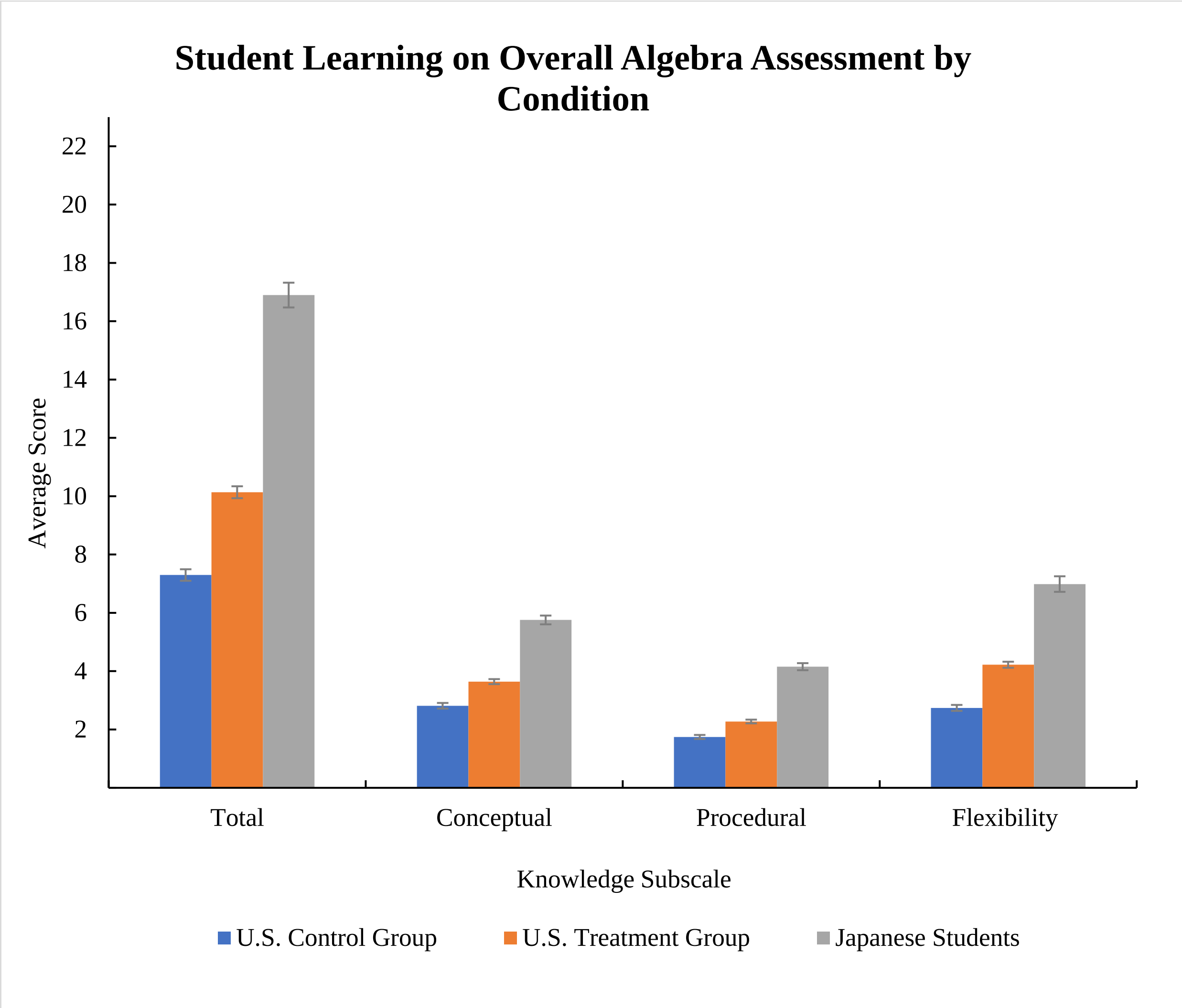
<table>
<tr><th>Factors of 12</th><th>Factors of -36</th><th>Factorization</th></tr>
<tr><td>2, 6</td><td>2, -18</td><td>$(2x + 2)(6x - 18)$</td></tr>
<tr><td>2, 6</td><td>18, -2</td><td>$(2x + 18)(6x - 2)$</td></tr>
<tr><td>6, 2</td><td>2, -18</td><td>$(6x + 2)(2x - 18)$</td></tr>
<tr><td>…</td><td></td><td></td></tr>
</table>
</td>
<td>b. Jamal's way:

$12x^2 + 24x - 36$
$12(x^2 + 2x - 3)$</td>
<td>c. Nadia's way:

$12x^2 + 24x - 36$
$2(6x^2 + 12x - 18)$</td></tr>
</table>

Figure 3

Number Correct on Overall Algebra Assessment and on the Three Subscales for the Japanese and Two U.S. Samples



Notes: The Japanese students had higher scores than either U.S. group on all measures, $p < .001$. Note that internal consistency was satisfactory on the overall assessment (Cronbach's alpha = 0.747 and 0.764 for the Japanese and U.S. samples, respectively) and on the flexibility subscale (0.712 and 0.614 for the Japanese and U.S. samples respectively). However, internal consistency was poor for procedural knowledge (0.585 and 0.521, respectively) and poor to very poor for conceptual knowledge (0.229 and 0.552, respectively), so these results must be interpreted cautiously.

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