Comparing and Explaining Examples of Multiple Strategies to Promote Algebra Learning: Instructional Features that Predict Learning

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WHERE DISCOVERIES BEGIN

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Compare & Discuss a "Best Practice" in Mathematics Instruction

- *Share and compare solution strategies* core to reform pedagogy in many countries (Australian Education Ministers, 2006; Brophy, 1999; Kultusministerkonferenz, 2004; NCTM, 2014; Singapore Ministry of Education, 2006; Treffers, 1991)
- Expert teachers use this approach (Lampert, 1990; Richland, Zur & Holyoak, 2007; Shimizu, 1999)

Evidence for Comparing & Discussing Multiple Strategies

- Based on short-term, researcher led studies conducted in classroom, comparing and discussing multiple strategies, rather than discussing strategies one at a time, can improve students'
  - Problem-solving accuracy (procedural knowledge)
  - Flexibility: Knowing multiple strategies and when to use them
  - Understanding of key concepts and strategies (conceptual knowledge)

Rittle-Johnson & Star, 2007, 2009; Rittle-Johnson, Star & Durkin, 2009, 2012; Star & Rittle-Johnson, 2009

Recommended Practice

EDUCATOR'S PRACTICE GUIDE   WHAT WORKS CLEARINGHOUSE

Improving Mathematical Problem Solving in Grades 4 Through 8

Recommendation 4.

Expose students to multiple problem-solving strategies.

1. Provide instruction in multiple strategies.
2. Provide opportunities for students to compare multiple strategies in worked examples.
3. Ask students to generate and share multiple strategies for solving a problem.

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Helping Teachers Use Comparison & Discussion of Multiple Strategies More Frequently and Effectively

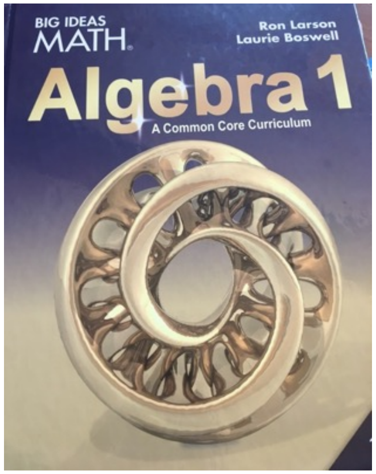
Moving to the “real world”

Focus on Algebra Instruction

- Proficiency in algebra is critical to academic, economic, and life success
  - E.g. Success in algebra is necessary for access to higher mathematics and to many job opportunities (Barnes, Slate, & Rojas-LeBouef, 2010; Hinojosa et al., 2016)
- Students have pervasive difficulties with Algebra
  - E.g., Only 36% of American 8th graders were able to interpret the meaning of a linear equation in a context (National Assessment of Educational Progress, 2017).

The Need for Teacher Support

- Comparing strategies rarely done in textbook lessons on **Algebra**
  - Only 3-4% of examples in 2 U.S. Algebra I textbook included multiple strategies for solving *the same problem*, and comparison was not supported.



Rare example:
Limited support
for comparison

EXAMPLE 3 **Using Structure to Solve a Multi-Step Equation**

Solve $2(1 - x) + 3 = -8$. Check your solution.

SOLUTION

Method 1 One way to solve the equation is by using the Distributive Property.

$2(1 - x) + 3 = -8$	Write the equation.
$2(1) - 2(x) + 3 = -8$	Distributive Property
$2 - 2x + 3 = -8$	Multiply.
$-2x + 5 = -8$	Combine like terms.
$\underline{-5} \quad \underline{-5}$	Subtract 5 from each side.
$-2x = -13$	Simplify.
$\frac{-2x}{-2} = \frac{-13}{-2}$	Divide each side by -2.
$x = 6.5$	Simplify.

▶ The solution is $x = 6.5$.

Check

$2(1 - x) + 3 = -8$

$2(1 - 6.5) + 3 \stackrel{?}{=} -8$

$-8 = -8$ ✓

Method 2 Another way to solve the equation is by interpreting the expression $1 - x$ as a single quantity.

$2(1 - x) + 3 = -8$	Write the equation.
$\underline{-3} \quad \underline{-3}$	Subtract 3 from each side.
$2(1 - x) = -11$	Simplify.
$\frac{2(1 - x)}{2} = \frac{-11}{2}$	Divide each side by 2.
$1 - x = -5.5$	Simplify.

From Big Ideas Algebra I

The Need for Teacher Support

- Need for materials and professional development to help more math teachers use comparison effectively.
  - E.g., High-quality implementation occurred in only 12% of lessons that incorporated multiple strategies (Hill et al., 2014).

Use of Compare and Discuss in Typical Algebra Classrooms is Infrequent

Instructional Practice	% of Algebra Lessons
Exposed students to **multiple strategies**	21
Multiple strategies were compared for at least a 1.5-minute continuous block	1
Engaged in **partner/small group work** for at least a 1-minute continuous block	27
Had a **whole-class discussion** for at least a 1.5-minute continuous block	7

From control classrooms in our study

Supplemental Curriculum and Professional Development

- Developed supplemental Algebra I curriculum and professional development for teachers to integrate *Comparison and Discussion of Multiple Strategies* (*CDMS*) in their classrooms.

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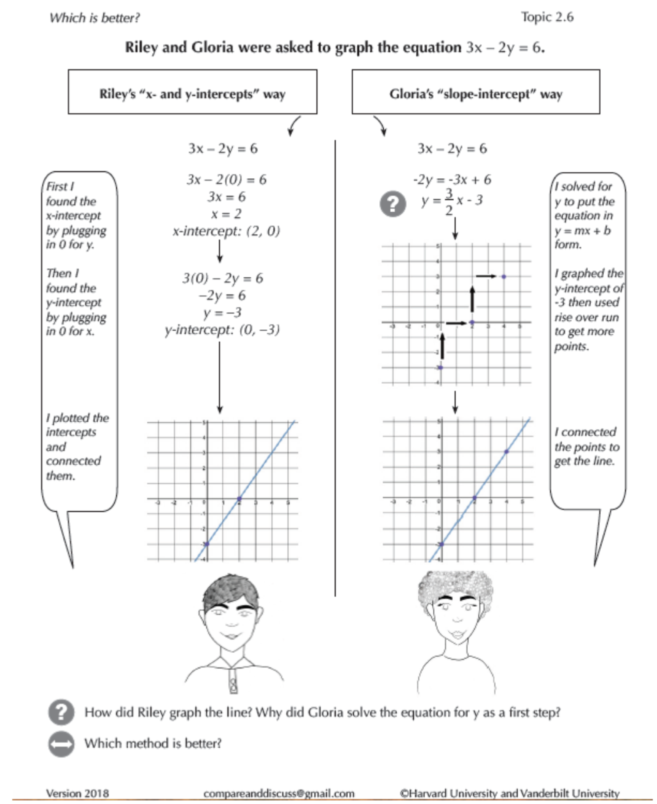
Compare & Discuss: Worked Example Pairs (WEPs)

- Side-by-side comparison of solved problems
- Shows hypothetical students' work and dialogue explaining process
- Includes discussion questions and prompts



Our Supplemental Compare & Discuss Curriculum for Algebra I

- **Accessible online at**
  - **my.vanderbilt.edu/cems**
    - **Resources tab**

- Materials for each lesson:
    - Teacher Guide for planning
    - Worked-example pair
    - Graphic organizer for student discussion
    - Big Idea

- **7-9 lessons per topic. Topics include**
    - Solving linear equations
    - Functions and graphing linear equations
    - Solving systems of equations
    - Polynomials and factoring

Compare & Discuss Problems

Topic 1: Linear Equations

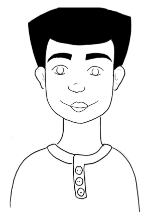
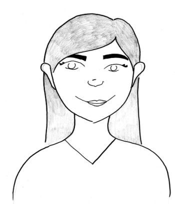
Which is Better?

Compare two correct strategies to learn when it is best to use one or the other

Tim and Emma were asked to solve the linear system

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

Tim's "substitution" way	**Emma's "elimination" way**
$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$	$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$
I solved the second equation for x. ? $x = 3y + 10$	$3x + 2y = 8$ $-3(x - 3y = 10)$ *I multiplied the bottom equation by -3.*
I plugged this into the first equation. $3(3y + 10) + 2y = 8$	
I then solved for y. $9y + 30 + 2y = 8$ $11y + 30 = 8$ $11y = -22$ $y = -2$	$3x + 2y = 8$ $\underline{-3x + 9y = -30}$ $11y = -22$ $y = -2$ *I then used elimination and solved for y.*
I plugged y into the second equation to find x. ? $x - 3(-2) = 10$ $x + 6 = 10$ $x = 4$	$x - 3(-2) = 10$ $x + 6 = 10$ $x = 4$ *I plugged y into the second equation to find x.*
The solution is (4, -2)	*The solution is (4, -2)*

? Why did Tim choose to plug $y = -2$ into the second equation to find x instead of the first equation?

⟷ Which method is better? What are some advantages of Tim's "substitution" way? Of Emma's "elimination" way?

Why Does It Work?

Compare two correct strategies to better understand why the teacher-taught strategy works

Riley and Gloria were given the set of ordered pairs
{(–3, 6), (2, 5), (3, 1), (2, 4), (5, 1)},
and asked to determine if the relation is a function.

Riley's "make a table" way

I made a table.

x (domain)	y (range)
-3	6
2	5
2	4
3	1
5	1

I saw that 2 in the domain is paired with both a 5 and a 4 in the range.

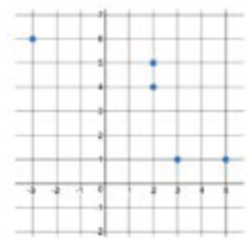
x (domain)	y (range)
-3	6
2 →	5
2 →	4
3	1
5	1

This means the relation is not a function.

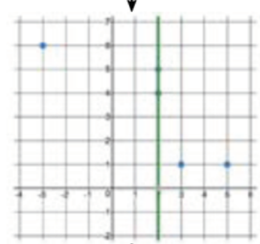
Not a function

Gloria's "graph and vertical line test" way

I graphed the ordered pairs.



I found a vertical line that intersected two of the points.



This means the relation is not a function.

Not a function

How did Riley determine if the relation was a function? How did Gloria determine if the relation was a function?

Why do both methods work? Why does the vertical line test tell us the same thing as the table of values?

Which is correct?

Compare a correct and incorrect strategy to understand why common mistakes are incorrect and to increase use of correct strategies.

Layla and Riley were asked to use factoring to solve the equation $a^2 + 5a - 6 = -12$.

Layla's "set equal to 0" way

First, I set the equation equal to zero by adding 12 to both sides. Then, I factored.

I solved the equations to get my answers.

$a^2 + 5a - 6 = -12$

$a^2 + 5a + 6 = 0$

$(a + 2)(a + 3) = 0$

$a + 2 = 0$ or $a + 3 = 0$

$a = -2$ or $a = -3$

Riley's "factor first" way

$a^2 + 5a - 6 = -12$

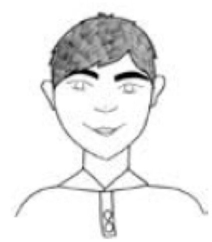
$(a + 6)(a - 1) = -12$

$a + 6 = 6$ or $a - 1 = -2$

$a = 0$ or $a = -1$

First, I factored.

Since 6 times -2 is -12, I set the first part equal to 6 and the second part equal to -2. Then I solved the equations to get my answers.



How could you check to see if Layla or Riley's solutions are correct?

Which method is correct, Layla's "set equal to 0" method or Riley's "factor first" method?

Instructional Routine: 20-minute CDMS cycle

Compare	Discuss
Prepare to Compare • What is the problem asking? • What is happening in the first method? • What is happening in the second method?	**Prepare to Discuss (think, pair)** • How does this comparison help you understand this problem? • How might you apply these methods to a similar problem?

Make Comparisons
• What are the similarities and differences between the two methods?
    ○ Which method is better?
    ○ Which method is correct?
    ○ Why do both methods work?
    ○ How do the problems differ?
	Discuss Connections (share)
	• What ideas would you like to share with the class?
	
	Identify the Big Idea
	• Can you summarize the Big Idea in your own words?

Supporting Comparison in Our Materials

1. Present two different strategies for solving the same problem.

2. Presented as students' solutions to encourage critical reflection.

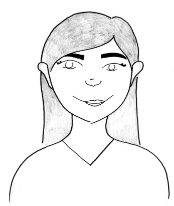
3. Make both examples visible and clear; present side-by-side.

Which is better? Topic 1.7

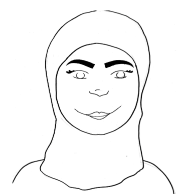
Emma and Layla were asked to solve $2a + 14 = b$ **for *a*, given *b* = 4 and *b* = 8.**

Emma's "solve for *a* first" way	**Layla's "plug in the value first" way**
$2a + 14 = b$	$2a + 14 = b$
$2a + 14 = b$; $-14 \quad -14$	$2a + 14 = 4$; $-14 \quad -14$
$\frac{2a}{2} = \frac{b - 14}{2}$	$\frac{2a}{2} = \frac{-10}{2}$
$a = \frac{b}{2} - 7$	$a = -5$
$a = \frac{4}{2} - 7,\ a = \frac{8}{2} - 7$	$2a + 14 = 8$; $-14 \quad -14$
	$\frac{2a}{2} = \frac{-6}{2}$
$a = -5,\ a = -3$	$a = -3$

Emma: First, I subtracted 14 from both sides. Then I divided by 2. I simplified to solve the equation for a. Then I plugged 4 and 8 in for b. Here are my answers.



Layla: First, I plugged 4 in for b. Then I subtracted 14 from both sides of the equation. Next, I divided both sides by 2 to get my first answer. Then I plugged 8 in for b and solved. Here's my second answer.



How did Emma and Layla solve the equation for *a*?

Which method is better? What is an important difference between Emma's way and Layla's way?

4. Explanation prompts for students to:
A. Understand each strategy.
B. Compare strategies to identify pros and cons.

(Richland, Zur & Holyoak, 2007)

Helping teachers facilitate comparison

1. Prepare to compare: Take time for students to understand each strategy

2. Make Comparisons

- *1. Ask students to explain similarities and differences.*
  - *1.Mark or list them.*
- *2. Push students to reflect on a key point about the comparison.*

Prepare to Compare

- What is the problem asking?
- What is happening in the first method?
- What is happening in the second method?

Make Comparisons

- What are the similarities and differences between the two methods?
  - Which method is better?

Suggest *When* in lesson to use

Topic 2: Solving Linear Equations- Overview

Section	Table of Contents (Page #)	WEP Type	Suggested Use
2.1	7	Why does it work?	Mid-lesson
2.2	11	Why does it work?	Mid-lesson
2.3	15	How do they differ?	Beginning of lesson
2.4	19	Which is correct?	Mid-lesson
2.5	23	Why does it work?	Mid-lesson
2.6	27	Which is better?	End of lesson
2.7	31	Which is better?	Beginning of lesson
2.8	35	Which is correct?	Mid lesson

Supporting Discussion with Our Materials

Graphic organizer to support
a) **Think – pair – share** routine

b) Students summarizing the big idea in own words

Discuss Connections

Is there a situation where substitution would be better than elimination, or vice versa?

Think, Pair. First, think about the question(s) above independently. Then, get with a partner and discuss your answers. After talking with your partner, what is your answer?

Think	Pair

Share. After reviewing the worksheet as a class, summarize the answer(s) your class agrees on. Was this different from your original response?

Big Idea. When your teacher tells you to do so, write w
example, in your own words.

Teacher Guide includes teacher questions and a potential student answer

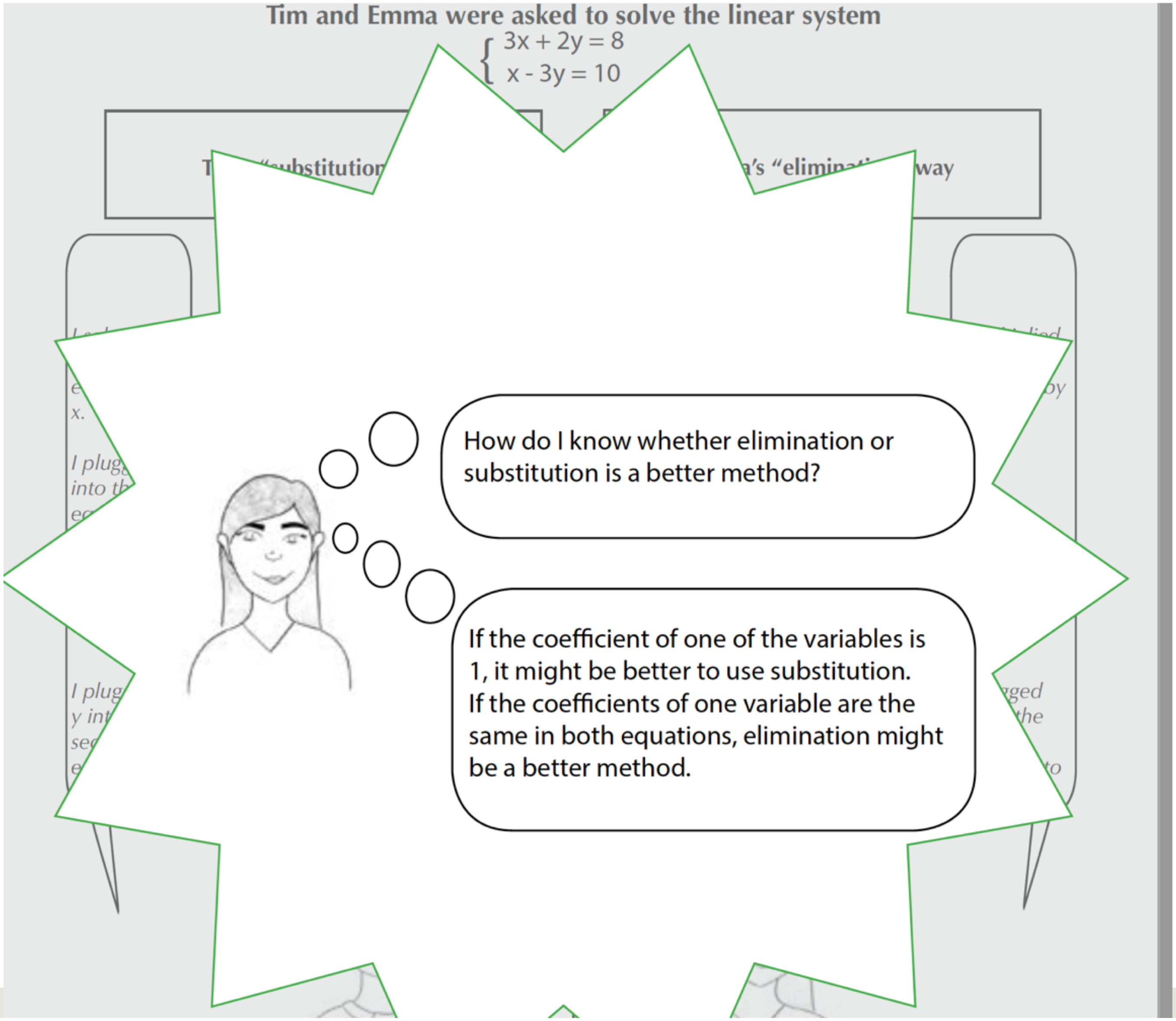
Prepare to Discuss (Think, Pair) **Is there a situation where substitution would be better than elimination, or vice versa?**

Discuss Connections (Share) *If one of the equations has a variable with a coefficient of 1, that equation is easy to rearrange, so substitution might be better. If the same variable in both equations has the same or opposite coefficients, then elimination might be better.*

Identify the Big Idea **Can you always use either substitution or elimination? Which is better?**
When solving a system of linear equations, substitution and elimination are both correct methods that will give you the same answer. Substitution might be easier if the variable has a coefficient of 1.

Supporting Discussion with Our Materials

Slide summarizing a Big Idea that should emerge from discussion.



Helping Teachers Leverage Discussion

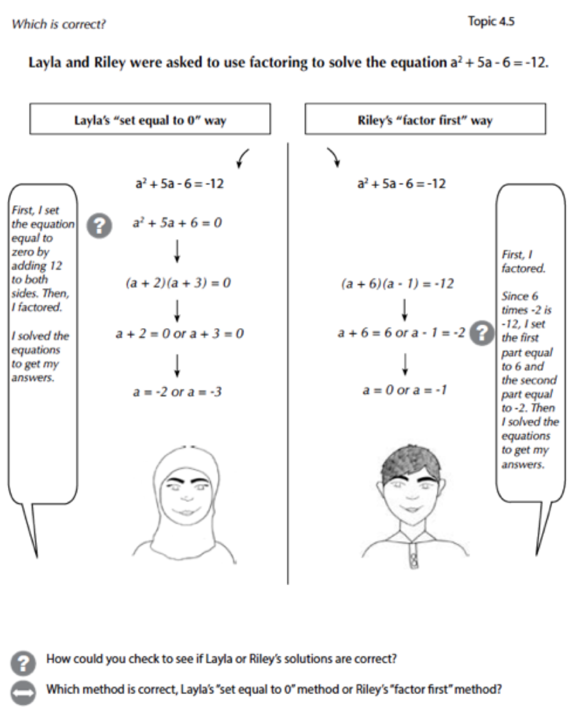
1. Provide professional development on:
   - Asking open-ended questions (e.g., “Why do you think that’s true?”)
   - Re-voicing and summarizing contributions
   - Hearing from many voices
   - Holding participants accountable for listening to others: “Do you agree or disagree with Morgan? Why?”,

Professional Development

- One week (35 hours) during the summer
- After each unit, individual meeting with a researcher
  - Provide personal feedback on videotaped lesson
  - Plan for next unit

Instructional Goals	Concrete Suggestions
+ Call on different students throughout the lesson.	**(The following instructional goals are building on this strength)**
Hear from at least two students whenever you ask a discussion question.	Aim for at least 2 responses per open ended question. Stay on a question longer by asking another student to summarize what they just heard, or if they agree/disagree with another student's response.
Ask follow-up questions in response to student thinking.	Students feel comfortable answering questions in your class, but their responses are brief. Use stems like *Tell me more,* or *Why*
Attend to the sequence of the Implementation Model.	Ensure students compare the methods before moving on to the Discuss phase. Provide students with an opportunity to think independently before they pair. Make sure students still have their Discuss Connections worksheet in hand as they are discussing the Big Idea as a whole group.

Summary of how and when we use worked examples



- HOW:
  - Side-by-side presentation of 2 worked examples, with reflection questions, to scaffold comparison and discussion of the examples
- WHEN:
  - Can be used to introduce, expand or review ideas (e.g., beginning, middle or end of lesson)

Teacher Implementation Studies

- Teachers:
  - Asked to use our materials several times a week, during 5 units of instruction.
  - No researcher present during instruction.
  - After each unit, met individually with a researcher for feedback and to plan

Data Collected

- Student knowledge
  - Overall researcher-designed assessment at beginning and end of school year
  - Researcher-designed unit assessments at beginning and end of 5 units (pre & post)
- Instructional quality: Videos of instruction (target 2-3 videos per unit)
- Dosage: Teacher logs and completed classroom handouts used to document proportion of our materials that were used

Pilot Year

- **Pilot year (AY 2017-2018)**: 9 treatment teachers used our CDMS approach with 348 students.
  - Schools were in suburban and rural MA and NH and served predominantly white, middle-class students.
  - Most students were in 9^{th} grade, with one 8^{th} grade classroom. One 9^{th} grade class was a remedial class.
  - Today, focus on 9^{th} grade students in regular pace course, taught by 7 different teachers (n = 315)
- Goal: Gather evidence for *promise* of the intervention and identify needed revisions

Sample Student Knowledge Assessment Items

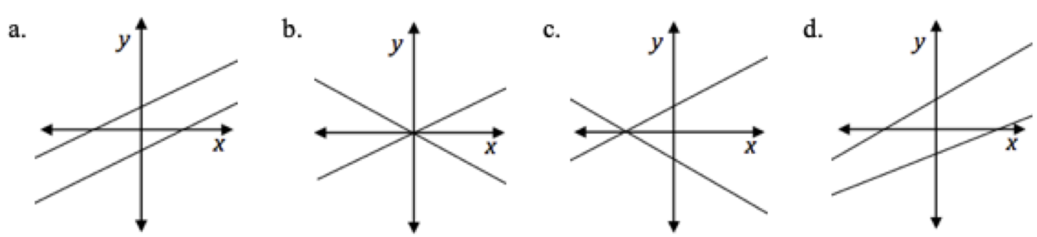
- **Procedural knowledge**
  - Solve the equation below for *y*. *Show all of your work.*

    $$5(y-2)=-3(y-2)+4$$

  - The points shown in the table lie on a line. What is the slope of the line?

- **Conceptual knowledge** (e.g., equivalent equations, like terms, graph-equation relations)

) Which of the following graphs could represent a system of equations with no solution?



Which of the following is a like term to (could be combined with) **7(*j* + 4)**?

a. $7(j+10)$
b. $7(p+4)$
c. $2(j+4)$
d. both a and c

Procedural Flexibility Items

) Below is the beginning of Gabriella's, Jamal's, and Nadia's work in solving the equation $x + 7 - 3 = 12 - 2x$. To start solving this problem, which way(s) may be used?

Gabriella's way:	Jamal's way:	Nadia's way:
Subtract 3 from 7: $x+4=12-2x$	Add $2x$ to both sides: $3x+7-3=12$	Subtract (7 – 3) from both sides: $x=8-2x$

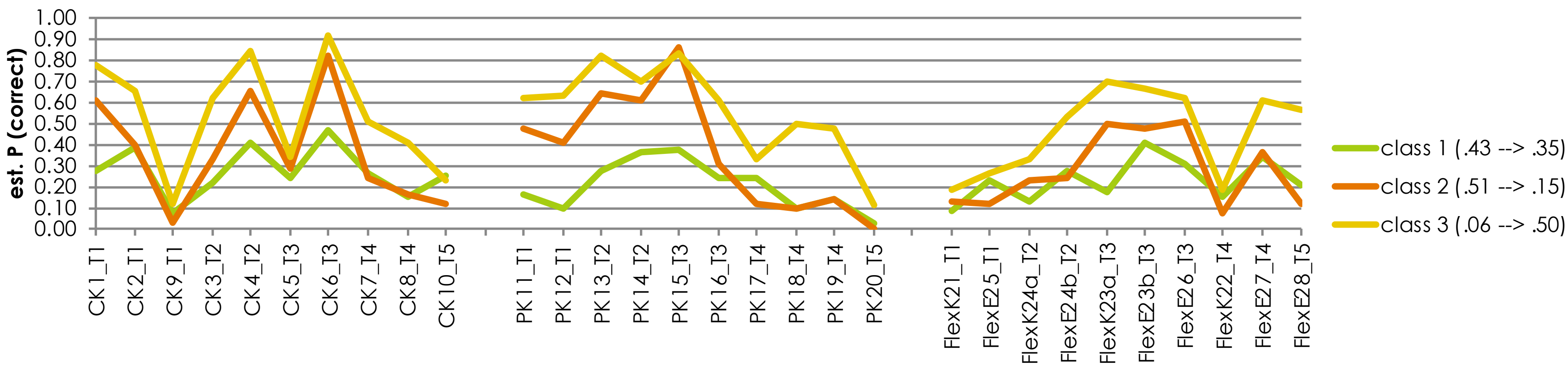
On a timed test, which would be the BEST way to start solving this system of equations? (Circle the letter for the best way.)

$$\begin{cases} 4x-3y=11 \\ 5x+y=19 \end{cases}$$

a. Gabriella's way:	b. Jamal's way:	c. Nadia's way:
$5x+y=19$ $y=19-5x$ …	$5\cdot(4x-3y=11)$ $-4\cdot(5x+y=19)$ ――――― …	$4x-3y=11$ $x=\frac{3y+11}{4}$ …

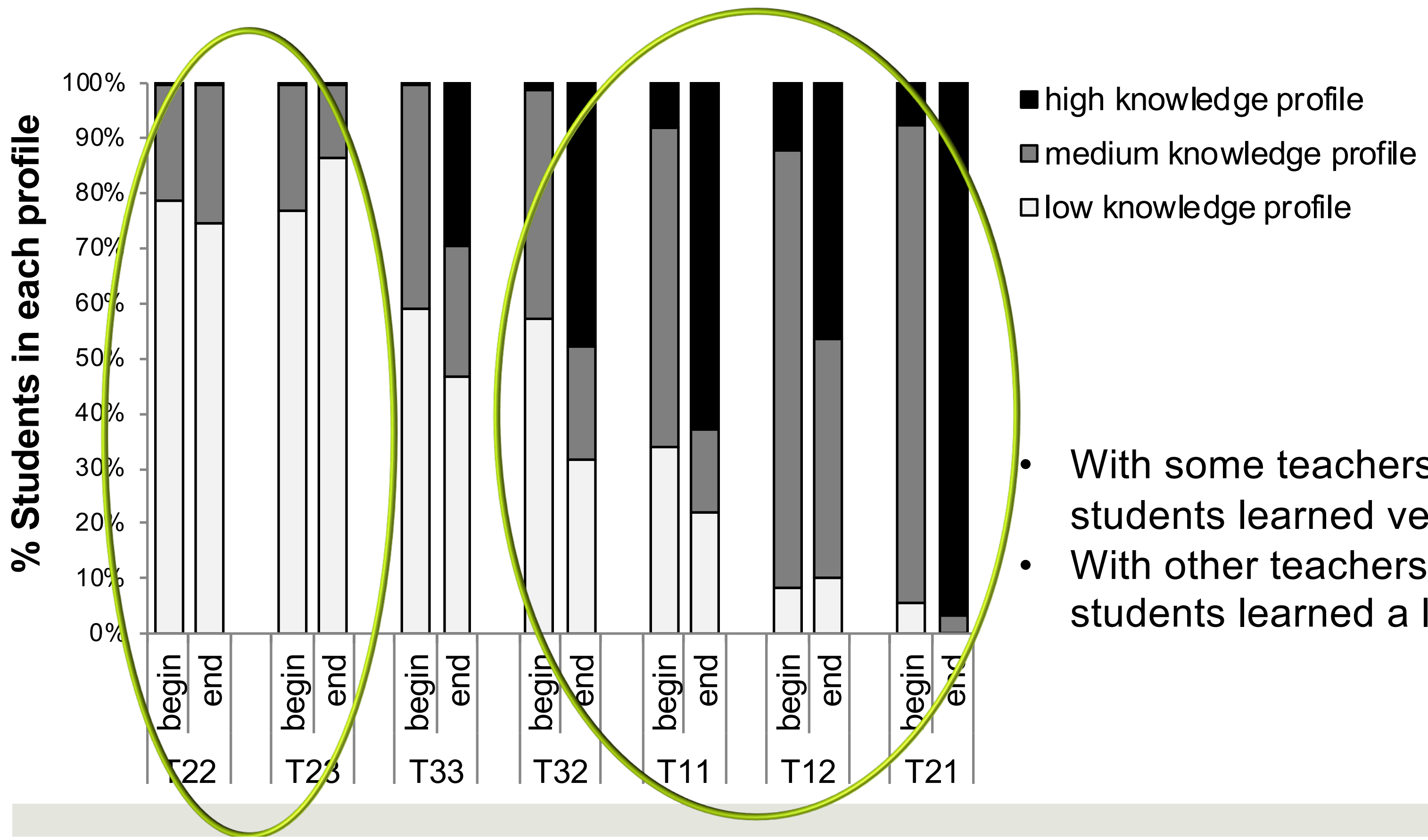
Pilot Year Results: 3 Knowledge Profiles

- Latent transition analysis used to identify different knowledge profiles (classes) on assessment at beginning of school year, and change from beginning to end of year.
  - Best fit was 3 knowledge profiles, without distinction by knowledge type
    - Low knowledge profile (class 1)
    - Medium knowledge profile (class 2)
    - High knowledge profile (class 3; rare at pretest)



Results: Large variability in student knowledge change

- Percentage of students in each profile at beginning and end of school year, by teacher.



- With some teachers, students learned very little
- With other teachers, students learned a lot

Results: Predictors of Change

- What predicts whether a student transitions to a new knowledge profile at posttest?
  - Frequency of use of our materials?
    - Proportion of CDMS Materials Used (out of 40)
  - Quality of implementation?

Instructional Quality Coding

- Procedure: Coded available lessons, with 6-9+ lesson per teacher.
  - Each 7.5 min. video segment coded on 4 pt. scale, with 1 indicating low quality and 4 indicating high quality, for several dimensions. (adapted from Litke, 2019)

- Teacher questioning: Highest-level observed, from simple questions (yes/no or calculations) to "why" and open-ended questions
  - E.g., "What is the answer?" vs. "Can you generate another problem where Riley's method could not be used?"

- Student interaction quality: Highest level of interaction either between the teacher and students or amongst students observed. We defined interaction as the opportunity to verbally share ideas regarding mathematical procedures and/or content within each lesson segment.
  - E.g., High quality examples: "Share with a partner and see if you agree/ disagree and add something that your partner next to you said." Multiple students responding to the same why question.

Implementation Results

Teacher ID	Proportion of CDMS Materials Used	Teacher Questioning Quality Rating	Student Interaction Quality Rating
T11	0.41	2.38	3.10
T12	0.63	3.18	2.98
T21	0.78	3.41	3.37
T22	0.67	3.17	2.93
T23	0.54	3.19	2.84
T32	0.65	3.17	2.88
T33	0.54	3.06	3.11
Aver.	**0.61**	**3.11**	**3.04**

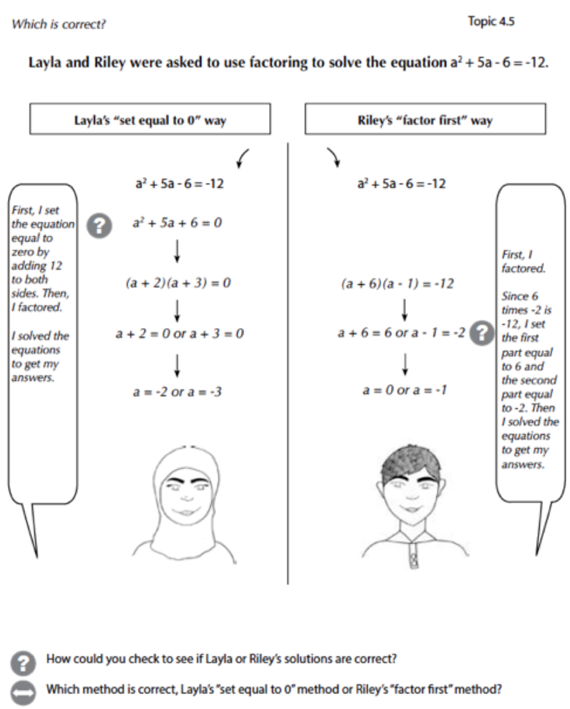
Results: Predictors of Knowledge Change

- The higher teachers' **use of our materials** and the more **teachers facilitated high-quality discussion**, the more likely their students were to transition to a higher-knowledge profile at the end of the school year
  - Latent Transition Analysis (χ^2 (2) = 6.20, p = .045 and χ^2 (2) = 18.77, p < .001, respectively).
- Caveat: These two instructional features were more likely if more of their students had a higher knowledge profile at the beginning of the school year.
  - Suggests higher quality instruction and implementation with more advanced students.

Discussion

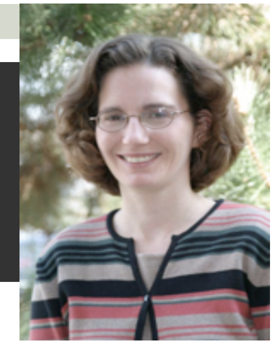
- Highlights a feature of high-quality instruction: Supporting high-quality student interaction, with students explaining ideas with classmates, associated with greater knowledge change.
  - Increase attention in individual professional development on this feature
- Promising, preliminary support for our approach: Greater use of our CDMS approach related to greater knowledge change.
    - Providing materials and routines is key, as is professional development, including feedback.
  - However, frequency and quality of use of our materials was limited by some teachers, especially those with many students with low initial knowledge.
    - Led to revision of some of our materials.

Using worked examples to improve mathematics learning



- HOW:
  - Side-by-side presentation of 2 worked examples, with reflection questions, to scaffold comparison and discussion of the examples
- WHEN:
  - Can be used to introduce, expand or review ideas (e.g., beginning, middle or end of lesson)
- WITH WHOM:
  - Students with more prior knowledge more easily and reliably benefit from comparing and discussing multiple strategies (see also Rittle-Johnson, Star & Durkin, 2009).
  - Students with little prior knowledge need extra support (see also Rittle-Johnson, Star & Durkin, 2012). Still an area of needed attention.

Acknowledgements



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- Visit our CEMS Website at https://my.vanderbilt.edu/cems/ for study materials, presentations and more
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  - Opinions expressed are those of the authors only!

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