COMPARING SOLUTION STRATEGIES TO PROMOTE ALGEBRA LEARNING AND FLEXIBILITY

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ABSRACT

Comparing solution strategies is an effective way to promote mathematics learning, especially students' procedural flexibility. Indeed, teachers is Asian countries such as Japan often have students compare multiple strategies for solving problems during mathematics instruction (Richland, Zur, & Holyoak, 2007; Shimizu, 1999). To leverage comparison in mathematics instruction more broadly, evidence-based guidelines are needed for how to use comparison effectively. We review our classroom-based research on using comparison to help U.S. students learn algebra, with the goal of developing such evidence-based guidelines.

In short-term experimental, classroom-based studies, comparing different solution strategies for solving the same problem was particularly effective for supporting *procedural flexibility* across students, such as the ability to solve the same problem in multiple ways and to evaluate the strengths and weaknesses of different solution strategies. It was also effective for supporting conceptual and procedural knowledge among students with some prior knowledge of one of the strategies.

We next developed a supplemental Algebra 1 curriculum to foster use of comparison. In our initial evaluation, we conducted a randomized-control trial with teachers and their students in Grades 8 and 9. Teachers used our supplemental materials much less often than expected, and student learning was not greater in classrooms that had been assigned to use our materials. Students' procedural knowledge was positively related to greater implementation of the intervention, suggesting the approach has promise when used sufficiently often.

In on-going research, we are working to better support Algebra teachers in their effective use of comparing solution strategies, with an emphasis on the importance of student explanation and discussion of the comparisons. We increased support for having high-quality class discussions through a classroom routine and additional professional development, and this approach seems to be effective. We also specified which of our materials to use when within their existing curricula and designed unit assessments to reveal the conceptual knowledge and procedural flexibility that our materials are meant to promote. Data collection is on going.

Overall, our research has identified some evidence-based guidelines for how to use comparison effectively, including (a) regular and frequent comparison of multiple strategies, particularly after students have developed some fluency with one initial strategy; (b) judicious selection of strategies and problems to compare; (c) carefully-designed visual presentation of the multiple strategies; and (d) use of small group and whole class discussions around the comparison of multiple strategies, focusing on the similarities, differences, affordances, and constraints of the different strategies. We continue to work on ways to effectively support teachers in their implementation of this approach.

Keywords: comparison; multiple strategies; discussion; procedural flexibility; algebra

We often learn through comparison. For example, we compare new words, objects and ideas to ones we already know, and these comparisons help us recognize what features are important and merit more attention. Indeed, comparison aids learning across a broad array of topics, ranging from babies learning the distinction between dogs and cats (Oakes & Ribar, 2005), to preschoolers learning new words (e.g., Namy & Gentner, 2002), to business-school students learning contract negotiation skills (Gentner, Loewenstein, & Thompson, 2003). A recent meta-analysis confirmed that comparison promotes learning across a range of domains (Alfieri, Nokes-Malach, & Schunn, 2013). As Goldstone and colleagues noted: "Comparison is one of the most integral components of human thought…. Furthermore, research has demonstrated that the simple act of comparing two things can produce important changes in our knowledge" (2010, p. 103).

Comparison is also included as a best practice in mathematics education. Having students share and compare solution strategies for solving a particular problem (e.g., discuss the similarities and differences in the strategies) lies at the core of reform pedagogy in many countries throughout the world (Australian Education Ministers, 2006; Brophy, 1999; Kultusministerkonferenz, 2004; NCTM, 2014; Singapore Ministry of Education, 2006; Treffers, 1991). Its inclusion as a best practice was based on observational research that expert teachers in the U.S. and teachers from high-performing countries such as Japan have students compare multiple strategies for solving problems during mathematics instruction (Ball, 1993; Lampert, 1990; Richland, Zur, & Holyoak, 2007; Shimizu, 1999).

Based on this convergence in the cognitive science and mathematics education literatures, we identified comparison as a promising instructional practice for improving mathematics learning. Evidence-based guidelines are needed for how to use comparison effectively in order to leverage comparison in mathematics instruction more broadly. We review our classroom-based research on using comparison to help U.S. students learn algebra, with the goal of developing such evidence-based guidelines.

Evidence: Comparing Solution Strategies Can Improve Mathematics Learning

Over the past decade, our research team has conducted both small-scale researcher-led experimental studies in controlled classroom settings as well as year-long teacher-led experimental studies. We briefly review each in turn.

Short-Term, Researcher-Led Classroom Research

In our initial research, we redesigned 2-3 math lessons on a particular topic in several different ways and implemented these lessons during students' mathematics classes. Here, we focus on the most effective: comparing multiple strategies for solving the same problem (*comparing strategies*). In these studies, students in the control condition typically studied the same mathematical content but sequentially (one at a time), without comparison. This allowed us to isolate the effectiveness of comparison. Students worked with partners within the same classroom, and each pair of students was randomly assigned to condition.

Study Design

In these short-term studies, worked examples along with prompts to explain the examples were the core of our instructional materials across conditions. Worked examples present solution strategies step-by-step and are a very effective way to help novices learn new procedures and related concepts (Atkinson, Derry, Renkl, & Wortham, 2000; Sweller & Cooper, 1985). They are commonly used in textbooks, so they are also familiar to students. Worked examples are also an effective way to introduce students to alternative, more efficient strategies (Star & Rittle-Johnson, 2008). To improve learning from worked examples, students should be prompted to generate explanations while studying the examples (see Atkinson et al., 2000). Generating explanations aids comprehension and transfer by promoting integration of new information with prior knowledge (Chi, 2000) and by guiding attention to structural features over surface features of the to-be-learned content (McEldoon, Durkin, & Rittle-Johnson, 2013; Siegler & Chen, 2008).

When designing our comparison materials, we extracted design principles from the comparison literature. First, examples to be compared were presented simultaneously to facilitate comparison (Begolli & Richland, 2015; Gentner, 1983). Second, spatial cues (e.g., side-by-side presentation) and common language were used to help students align and map the solution steps, to facilitate noticing of important similarities and differences in the examples (Namy & Gentner, 2002; Richland et al., 2007). Third, the explanation prompts focused on specific aspects of the examples to compare because this is encouraged by expert mathematics teachers (Fraivillig, Murphy, & Fuson, 1999; Huffred-Ackles, Fuson, & Sherin Gamoran, 2004; Lampert, 1990) and improves learning from comparison relative to generic, open-ended prompts "to compare" (Catrambone & Holyoak, 1989; Gentner et al., 2003). Finally, we provided some direct instruction to supplement learners' comparisons because direct instruction has been found to improve learning from comparison (Gick & Holyoak, 1983; Schwartz & Bransford, 1998; VanderStoep & Seifert, 1993).

Materials for the control condition included the same worked examples presented one at a time without supports for comparison. All students worked with a partner when studying the worked examples because working with a partner provides a familiar context for students to generate explanations, and students who collaborate with a partner tend to learn more than those who work alone (e.g., Johnson & Johnson, 1994; Webb, 1991). All children received the same direct instruction after studying the worked examples.

The instructional materials often focused on multi-step equation solving. Consider the equation $3(x + 2) = 6$. Two possible first steps are to distribute the 3 or to divide both sides by 3. Although the former is almost universally taught as part of the algorithm for solving this type of equation, the latter approach is arguably more efficient because it reduces the number of computations and steps needed to solve the equation. Regrettably, students often memorize rules and do not learn flexible and meaningful strategies for solving equations (Kieran, 1992). They also struggle to understand key algebraic concepts. For example, only 59% of U.S. 8th graders were able to find an equation that is equivalent to $n + 18 = 23$ (National Assessment of Educational Progress, 2011). Thus, improving students' knowledge of multi-step equation solving is greatly needed.

For student outcomes, we focused on three critical components of mathematical competence: procedural knowledge, procedural flexibility, and conceptual knowledge. *Procedural knowledge* is the ability to execute action sequences to solve problems, including the ability to adapt known procedures to unfamiliar problems (Rittle-Johnson, Siegler, & Alibali, 2001). *Procedural flexibility* includes the knowledge of multiple strategies as well as the ability to choose the most appropriate method based on specific problem features (Kilpatrick, Swafford, & Findell, 2001; Star, 2005; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). Procedural flexibility supports efficient problem solving and is also associated with greater accuracy solving novel problems and greater understanding of domain concepts (e.g., Blöte, Van der Burg, & Klein, 2001; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Hiebert et al., 1996). Finally, *conceptual knowledge* is "an integrated and functional grasp of mathematical ideas" (Kilpatrick et al., 2001). This knowledge is flexible and not tied to specific problem types, and is therefore generalizable. However, the knowledge may be implicit and not easily articulated (Alibali & Nathan, 2012; Prather & Alibali, 2009). Mathematics competence rests on students developing all three types of knowledge (Kilpatrick et al., 2001).

The majority of our studies focused on comparing multiple strategies for solving the same problem. Expert mathematics teachers in the U.S. (e.g. Ball, 1993; Lampert, 1990) and teachers in high performing countries such as Japan (Richland et al., 2007; Shimizu, 1999) often use this approach. All of the strategies presented were correct, but they varied in terms of which method most appropriate and efficient for solving a particular problem. Students studied pairs of worked examples and were prompted to compare them (*compare-strategies condition)* or studied the same examples one at a time and were prompted to reflect on them individually (*sequential condition*). Our comparison prompts focused student attention on recognizing that both strategies adhered to domain principles, but that a particular method was more efficient for solving a particular problem.

Results

In Rittle-Johnson and Star (2007), seventh-grade students (N = 70) in pre-algebra classes learned about solving multi-step linear equations during 3 class periods. All students first took a pretest focused on our three outcome measures, then they completed a packet of worked examples with their partner, and finally they completed a posttest similar to the pretest. As predicted, those who compared strategies gained greater procedural flexibility than those who studied examples sequentially. They also had greater procedural knowledge (i.e., success solving equations). The two groups did not differ in conceptual knowledge, although the reliability of the measure was poor and was not closely aligned with the concepts students were likely to learn from the comparisons.

Students' explanations during the intervention confirmed that those who compared strategies often compared the similarities and differences in solution steps across examples and evaluated their efficiency and accuracy; these students were also more likely to use alternative strategies when solving practice problems during the intervention. In turn, frequency of making explicit comparisons during the intervention and frequency of using alternative strategies on the practice problems were each predictive of learning outcomes. Overall, comparing strategies helped students differentiate important characteristics of examples (e.g., efficiency) and consider multiple strategies. We found parallel results for 157 fifth- and sixth-grade students learning about estimating answers to multiplication problems (e.g., About how much is 37 x 29?) (Star & Rittle-Johnson, 2009).

Our subsequent research explored the importance of prior knowledge for learning from comparing strategies. We worked with 236 seventh- and eighth-grade students whose schools did not use a pre-algebra curriculum, and thus had had limited experience solving equations (Rittle-Johnson, Star, & Durkin, 2009). Students who did not attempt algebraic strategies at pretest (i.e., novices) benefited most from studying examples sequentially, rather than from comparing strategies. The novices in the compare-strategies condition seemed overwhelmed during the intervention – they completed less of the intervention materials and were less successful implementing non-standard strategies when prompted. In contrast, students who attempted algebraic strategies at pretest learned more from comparing strategies. Further, general mathematics achievement did not influence the effectiveness of comparison; rather, it was prior knowledge of one of the demonstrated strategies that influenced the effectiveness of comparison. However, a follow-up study suggested that slowing the pace of instruction allowed novices to learn from comparing strategies (Rittle-Johnson, Star, & Durkin, 2012). When the materials covered less content in more time, regardless of students' prior knowledge, comparing strategies supported more flexible use of procedures than sequential study, including on a one-month retention test. On other outcome measures, the compare-strategies and sequential conditions learned a comparable amount.

In sum, our short-term researcher-led classroom studies provided strong evidence in support of the power of comparison for improving mathematics learning, especially comparing and explaining multiple solution strategies. In large part because of our research, a recent Practice Guide from the U.S. Department of Education identified comparing multiple solution strategies as one of five recommendations for improving mathematical problem solving in the middle grades (Woodward et al., 2012). The success of these short-term experiments prompted us to design and implement two large year-long teacher-led experimental studies.

Teacher-Led Classroom Studies

Given the promise of using comparison to promote algebra learning, we sought to provide teachers with support and structured materials to encourage their use of *Comparison and Explanation of Multiple Strategies* (*CEMS*) in the classroom. Comparison is a reasonable adaptation of current teaching practices (Star, 2016), but there is a need for materials and training to help teachers use comparison in mathematics instruction. In response to this need, we created a supplemental curriculum and associated professional development to provide teachers with an implementable and effective way to incorporate a CEMS approach into their practice. In Study 1, we conducted an initial evaluation of teachers' effective use of our CEMS approach. In Study 2, we modified the approach to better support teachers in their use of the approach.

<u>Study 1 Instructional Materials and Teacher Professional Development</u>

A team of mathematics education experts, including researchers, mathematicians, and Algebra I teachers, developed supplementary CEMS materials by going through a typical Algebra I course syllabus, identifying core concepts, common student difficulties, and key misconceptions, and then creating comparison materials to attempt to address them. This led to a set of 141 *worked example pairs (WEPs*). A sample WEP is shown in Figure 1. Each WEP showed the mathematical work and dialogue of two hypothetical students, Alex and Morgan, as they attempted to solve one or more algebra problems. The curriculum contained four types of WEPs, with the types varying in what was being compared and the instructional goal of the comparison. As shown in Figure 1, *Which is better?* WEPs showed the same problem solved using two different, correct strategies, with the goal of understanding when and why one method was more efficient or easier than another method for a given problem, as we did in our short-term research on comparing strategies (e.g., Rittle-Johnson & Star, 2007). *Which is correct?* WEPs showed the same problem solved with a correct and incorrect method, with the goal of understanding and avoiding common errors. Comparing correct and incorrect strategies supports gains in procedural knowledge, retention of conceptual knowledge, and a reduction in misconceptions (Durkin & Rittle-Johnson, 2012).

Two new comparison types were also included. *Why does it work?* WEPs showed the same problem solved with two different correct solution strategies with the goal of illuminating the conceptual rationale in one method that is less apparent in the other method. This is in contrast to the *Which is better?* comparisons, where the goal was to learn when and why one method was better for solving particular types of problems. *How do they differ?* WEPs showed two different problems solved in related ways, with an interest in illustrating what the relationship between problems and answers of the two problems revealed about an underlying mathematical concept.

The WEPs were designed to maximize their potential impact based on previous research. As before, the two worked examples were presented side-by-side. To facilitate processing of the examples, we included thought bubbles where two students (Alex and Morgan) described their solution strategies. We used common language in these descriptions as much as possible to help facilitate alignment of the examples.

We formalized an instructional routine to help improve the effectiveness of using our comparison materials, by incorporating reflection prompts into each WEP that sought to engage students in explanation of the to-be-compared multiple strategies. Each WEP had three types of reflection prompts (understand, compare, and make connections) meant to culminate in a discussion of the learning goal for the WEP. First, *Understand* prompts, such as, "How did Morgan solve the equation?" were intended to provide students the opportunity to understand each worked example individually, prior to comparing them. Second, *Compare* prompts, such as "What are some similarities and differences between Alex's and Morgan's ways?", were meant to encourage comparison of the two worked examples. *Understand* and *Compare* prompts were very similar across WEP types and were intended to prepare students to engage in productive reflection on the final, *Make Connections* prompts, such as "On a timed test, would you rather use Alex's way or Morgan's way? Why?" and "Even though Alex and Morgan did different first steps, why did they both get the same answer?"

Teachers were asked to use our materials once or twice a week as a supplement to their usual curriculum. Teachers decided which WEPs to use, when to use them, and for how long to use them.

We also designed a one-week, 35-hour professional development institute to familiarize teachers with the materials and approach (Newton & Star, 2013). Teachers read through and discussed the supplemental curriculum materials and viewed videotaped exemplars of other teachers using the curriculum. In addition, teachers worked in groups to plan and teach sample lessons to their peers using the materials, which were implemented and then debriefed by the group.

Implementation and Results of Study 1

After piloting our materials with 13 Algebra I teachers, we conducted a year-long randomized controlled trial that explored the feasibility of implementation of our Algebra I supplemental curriculum and its impact on teachers' instruction and students' mathematical knowledge (Star et al., 2015). Teachers were randomly assigned to either implement the comparison curriculum as a supplement to their regular curriculum (*treatment* condition; 39 teachers with 781 students) or to continue using their existing curriculum and methods (*'business as usual' control* condition; 29 teachers with 586 students).

Before the school year began, treatment teachers completed the summer professional development institute. During the school year, treatment teachers implemented our CEMS materials on their own, without researcher involvement. Each time teachers used our materials, they were asked to call and report on what WEP they had used. Teachers were also asked to videotape their use of our materials every other week, and teachers in both conditions were asked to videotape their instruction (without use of our materials) once a month.

Analysis of the videotapes indicated that treatment teachers were able to implement our CEMS materials as mostly as intended, although with much more teacher-provided explanations than student-generated explanations (Star, et al., 2015). Further, control teachers did not frequently use specific instructional practices that were integral to the intervention; they rarely presented multiple strategies side-by-side or explicitly compared examples. However, there was large variation in how frequently treatment teachers used our curriculum, and many treatment teachers used the our materials much less frequently than intended. Our materials were used in an average of 19 class periods during the school year (range : 0 to 56 days, with 180 days in the school year). In fact, 18% of participating treatment teachers did not report using the materials even once; 30% of them reported using them 5 times or fewer. Overall, treatment teachers were using our CEMS materials much less often than intended (i.e., low degree of implementation), but when they did use the materials, they used them with high fidelity (i.e., implemented as intended).

To assess student learning, teachers administered a standardized commercial algebra test as well as a researcher-designed assessment of algebra knowledge to their students at the beginning and end of the school year. At the end of the school year, students' algebra knowledge was not higher in classrooms in which our materials were available (Star, et al., 2015). This was not surprising given how infrequently our supplemental curriculum was used in many classrooms.

Given the large variability in use of our curriculum, we evaluated whether increased use of our materials was associated with increased algebra knowledge in treatment classrooms. We calculated the *"dosage"* of curriculum given by each treatment teacher by multiplying the number of reported days each teacher used our materials by how long on average that teacher spent using our materials in a single lesson (based on the videos). Dosage ranged from 0 to 864 minutes ($M = 140$). We found that increased dosage was predictive of greater procedural knowledge of algebra at the end of the school year (Star, et al., 2015). This included solving equations, graphing equations, and factoring expressions. Dosage did not have a significant effect on any other student outcome. This may be because all comparison materials focused attention on procedural knowledge, whereas attention to procedural flexibility and conceptual knowledge varied by comparison type. We also explored whether frequency of using particular comparison types was associated with increased algebra knowledge. Frequency of using *Why does it work?* comparisons was positively correlated with the amount of gain in students' algebra knowledge; correlations for the other types of comparison were positive but not significant.

Overall, providing our supplemental CEMS materials to teachers did not significantly affect students' algebra knowledge. However, many teachers used our materials infrequently. Increased use of our materials had a positive effect on procedural knowledge, although not on other student knowledge outcomes. These findings suggest a potential role for comparison in supporting aspects of algebra learning, but also point to the challenges of supporting teachers' integration of this approach into the curriculum.

<u>Study 2: Revised Instructional Materials and Professional Development</u>

The results of Study 1 indicated that teachers needed more support in their implementation of our CEMS approach. In Study 2, we are working to better support Algebra teachers in their frequent and effective use of CEMS.

First, we specified when teachers should use our materials in conjunction with their existing curriculum. It is time consuming and potentially challenging to select, plan, and integrate our materials. We selected 5 Algebra I units, such as Solving Systems of Linear Equations, and specified 8 – 12 WEPs to use for each unit, linked to particular lessons.

Second, we revised the instructional routine to better promote student explanation and discussion of the comparisons. The development of mathematics knowledge is believed to be enhanced by classroom discussions in which students generate explanations and teachers facilitate a discussion of different student responses (Lampert, 1990; Stein, Engle, Smith, & Hughes, 2008). Indeed, more frequent engagement with other students' strategies and ideas during discussion is related to greater success on a mathematics assessment (Webb et al., 2014). Thus, we increased support for having high-quality class discussions through a classroom routine and additional professional development.

To facilitate classroom discussions, we included a think-pair-share routine, supported by a worksheet for students to complete (see Figure 2). First, students are asked to think on their own about the WEP for a few minutes and then to pair with another student and discuss their ideas. Finally, students share out their ideas with the whole class. After the class discussion, students are asked to write the big idea from the discussion on their worksheet. Classroom observations suggest that this routine has been effective in promoting student explanation and whole class discussion when using our materials.

This year, 9 teachers from 3 school districts are implementing our materials in their Algebra I course, and 10 teachers from another school district are teaching their Algebra I course as they typically do. We have designed unit assessments to reveal the conceptual knowledge and procedural flexibility that our materials are meant to promote. Data collection is on-going, so whether use of our revised CEMS approach improves student learning relative to typical classroom instruction is an open question.

Evidence-Based Guidelines for Using Comparison

Overall, our research has identified some evidence-based guidelines for how to use comparison effectively. This includes:

(a) Regular and frequent comparison of multiple strategies, particularly after students have developed some fluency with one initial strategy;

(b) Judicious selection of strategies and problems to compare;

(c) Carefully-designed visual presentation of the multiple strategies; and

(d) Use of small group and whole class discussions around the comparison of multiple strategies focusing on the similarities, differences, affordances, and constraints of the different strategies.

Comparison and explanation of multiple strategies is a promising method for improving algebra learning and flexibility. We continue to work on ways to effectively support teachers in their implementation of this approach.

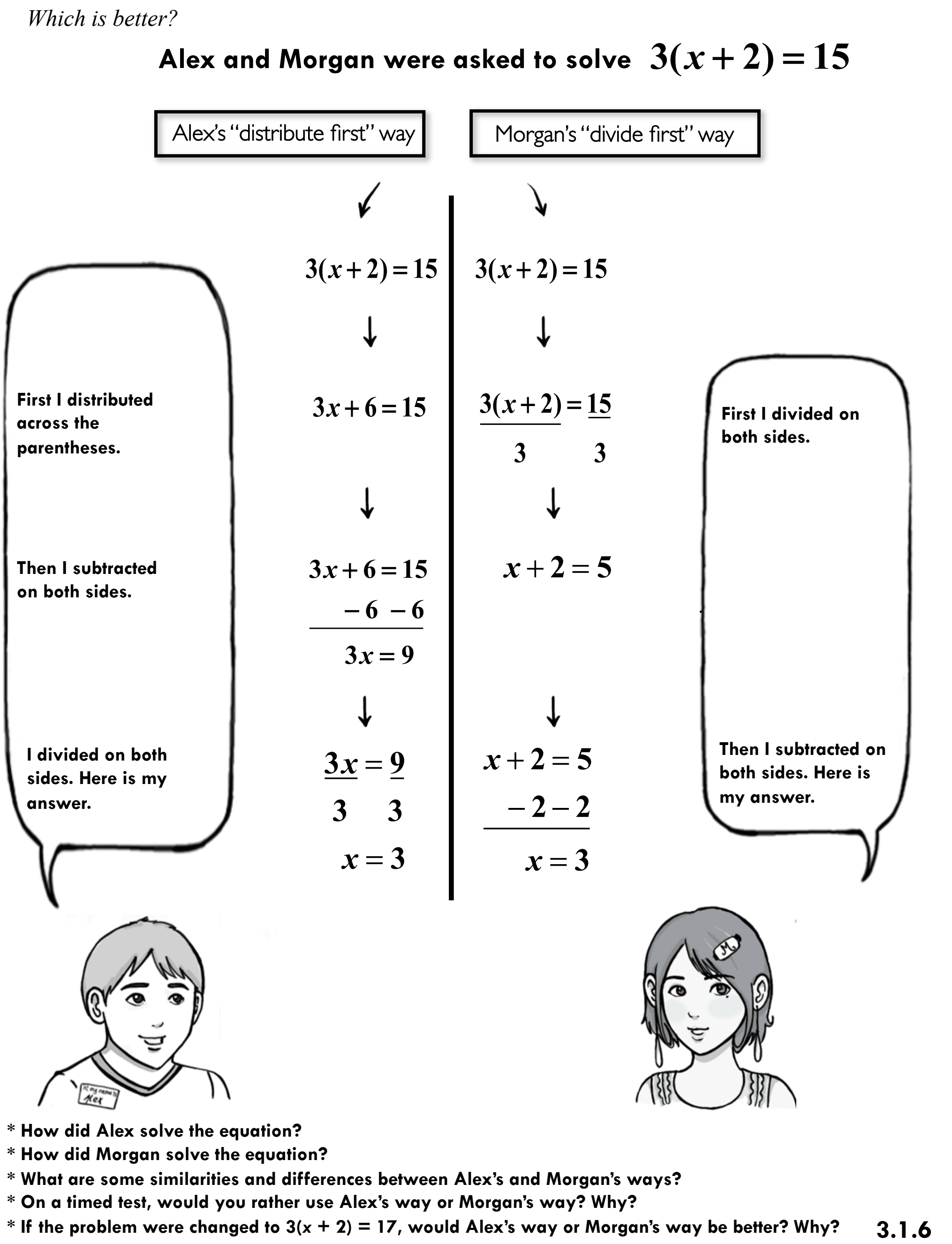

Figure 1. Sample Which is Better? Worked Example Pair (WEP)

Which is better? Ch 1.2.2

Discuss Connections

If solving 5(x + 2) + 7 = 12, which method would be better? Why?

Think, Pair. First, think about the question(s) above independently. Then, get with a partner and discuss your answers. After talking with your partner, what is your answer?

Share. After reviewing the worksheet as a class, summarize the answer(s) your class agrees on. Was this different from your original response?

Big Idea. When your teacher tells you to do so, write what you think is the big idea of this example, in your own words:

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Figure 2. Think-Pair-Share Handout to Promote Student Explanation and Discussion of WEP

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