Counting to Infinity (and Beyond)

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Welcome to the Grand Hilbert Hotel!

The Grand Hilbert Hotel

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Here at the Grand Hilbert Hotel, we have the unique feature of having infinitely-many rooms $r_0, r_1, \ldots, r_n, \ldots$

Finding Rooms in Hilbert's Hotel

Suppose we have...

Finding Rooms in Hilbert's Hotel

Suppose we have. . .

Finding Rooms in Hilbert's Hotel

Suppose we have. . .

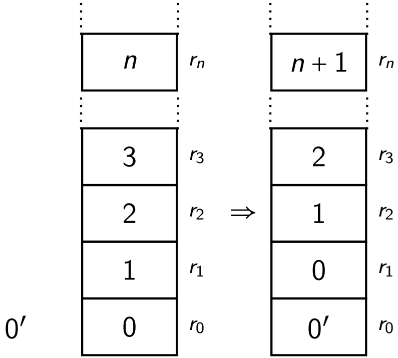
. . . one new guest

Finding Rooms in Hilbert's Hotel

Suppose we have...

...one new guest

Ans. Move everyone up a floor, i.e. send guest n to room r_{n+1}, then put the new guest in room r_0.



Finding Rooms in Hilbert's Hotel

Suppose we have...

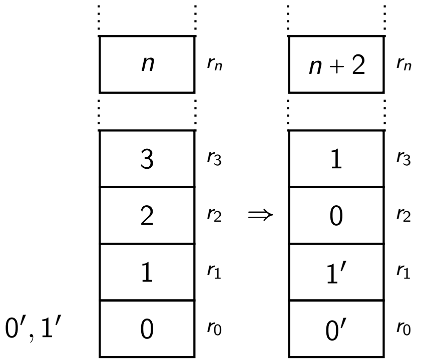
...two new guests

Finding Rooms in Hilbert's Hotel

Suppose we have...

...two new guests

Ans. Move everyone up two floors, i.e. send guest n to room r_{n+2}, then put the two new guests in rooms r_0 and r_1.



Finding Rooms in Hilbert's Hotel

Suppose we have...

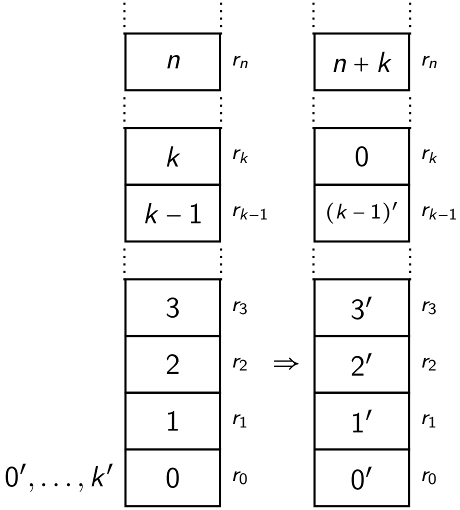
... $k + 1$ new guests

Finding Rooms in Hilbert's Hotel

Suppose we have...

...$k + 1$ new guests

Ans. Move everyone up k floors, i.e. send person n to room r_{n+k}, then put the k new guests in rooms $r_0, r_1, \ldots, r_{k-1}$.



Finding Rooms in Hilbert's Hotel

Suppose we have...

...infinitely-many new guests $0', 1', 2', 3', \ldots$

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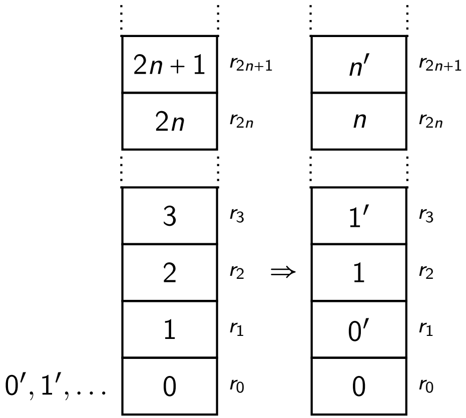
Ans. Move guest n to room r_{2n}, then put the k'-th new guest into room r_{2k+1}.



Finding Rooms in Hilbert's Hotel

Suppose we have...

...two groups of infinitely-many guests $0', 1', 2', 3', \ldots$ and $0'', 1'', 2'', 3'', \ldots$

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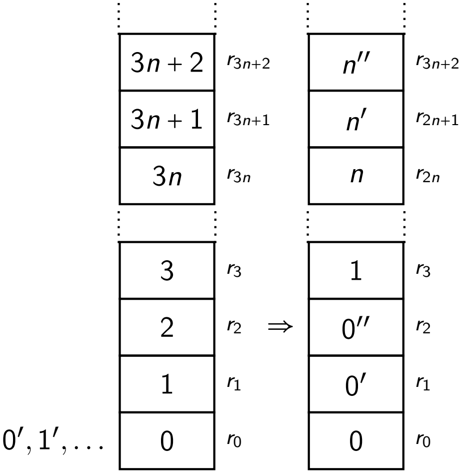
Ans. Move guest n to room r_{3n}, then put the k'-th new guest into room r_{3k+1} and the k''-th new guest into room r_{3k+2}.



Finding Rooms in Hilbert's Hotel

Suppose we have. . .

. . . infinitely-many buses
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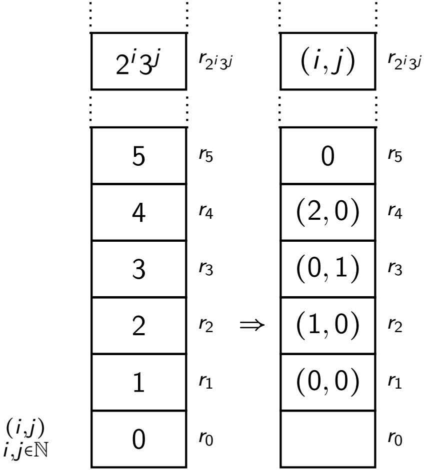
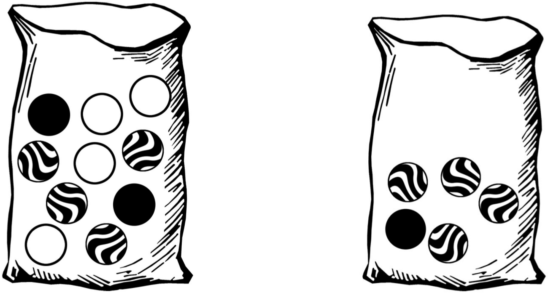
Ans. Move person in room n to room 5^{n+1}, then put the j-th new guest of bus i into room $r_{2^i 3^j}$.



How to Count

Given two bags of marbles, how could we decide which had more marbles?



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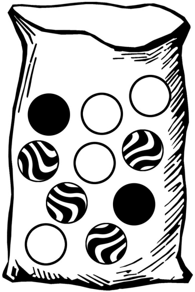

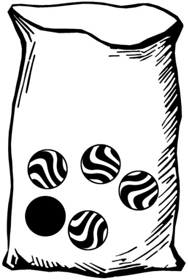

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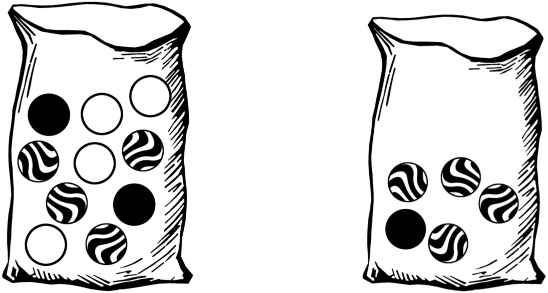
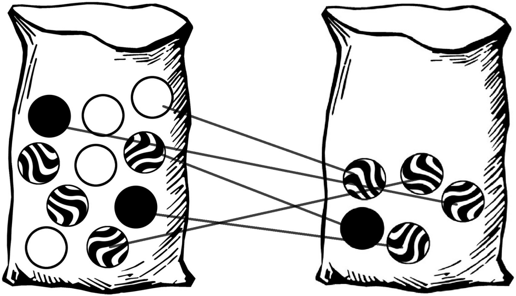


Easy answer: count the number of marbles and compare them.
But what if we didn't have access to the natural numbers $0, 1, 2, 3, \ldots$?

How to Count

Instead, we will pair up marbles.



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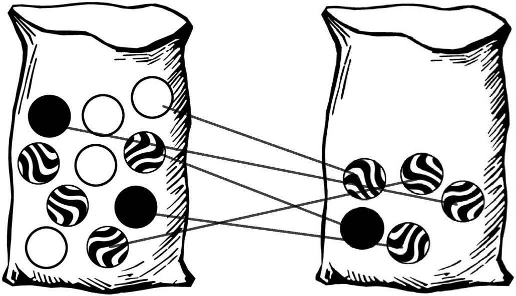

The bag with leftover marbles has larger size.

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Nothing was special about bags of marbles. We can do this to compare the size of *any sets*.

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$\{0, 2, 4, 6, 8, \ldots\}$ is the set of even natural numbers.

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$\{0, 2, 4, 6, 8, \dots\}$ is the set of even natural numbers.
$\left\{\frac{p}{q} \mid p, q \text{ integers}, q \neq 0\right\}$ is the set of rational numbers.

Functions

Definition

A **function** $f : X \to Y$ consists of

- a **domain** X,
- a **codomain** Y, and
- a set of pairs (x, y) of an input $x \in X$ and an output $y = f(x) \in Y$ for each $x \in X$ (the **graph** of f).

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The set $\{(1,2),(2,5),(3,1),(4,0)\}$ is a function from $\{1,2,3,4\} \to \mathbb{N}$.
The set $\{(n, 2n) \mid n \in \mathbb{N}\}$ is the function $f : \mathbb{N} \to \mathbb{N}$ with $f(n) = 2n$.

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The set of all subsets of X is denoted $\mathcal{P}(X)$, the **power set** of X.

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This allows sets to be compared by their "sizes", or **cardinalities**.

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Antisymmetric: (Cantor-Shroeder-Bernstein) $|X| \leq |Y|$ and $|Y| \leq |X|$ implies $|X| = |Y|$.

Finite and Infinite Sets

If $|X| = |\{1, \ldots, n\}|$, then X is **finite**, and we write $|X| = n$. n is the **cardinality** of X.
n is unique – if $|\{1, \ldots, n\}| = |\{1, \ldots, m\}|$, then $n = m$.

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Example

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all infinite.

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$\mathbb{Z}$ – the map $f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ 2(-n) - 1 & \text{if } n < 0 \end{cases}$ gives a bijection $f : \mathbb{Z} \to \mathbb{N}$.

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$\mathbb{N} \times \mathbb{N}$ – the map $f(n, m) = 2^n 3^m$ gives an injection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Since $g(n) = (n, 0)$ gives an injection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, Cantor Schroeder-Bernstein says that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

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$\mathbb{Q}$ – the map $f(p/q) = \begin{cases} 2^p 3^q & \text{for } p \geq 0,\ q > 0,\ p/q \text{ irreducible} \\ 2^{-p} 3^q 5 & \text{for } p < 0,\ q > 0,\ p/q \text{ irreducible} \end{cases}$ gives an injection $f : \mathbb{Q} \to \mathbb{N}$. Since $g(n) = n$ gives an injection $g : \mathbb{Q} \to \mathbb{N}$, Cantor-Schroeder-Bernstein says that $|\mathbb{Q}| = |\mathbb{N}|$.

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If X is infinite but not countably-infinite, then it is **uncountable**.

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There exist familiar uncountable sets...

Theorem

$|\mathbb{N}| < |[0,1]|$. *In particular,* $\mathbb{R}$ *is uncountable.*

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Proof. Suppose $f : \mathbb{N} \to [0,1]$ is given. Then we would have a list of real numbers written in decimal notation:

$$\begin{aligned}
f(0) &= 0.a_{00}a_{01}a_{02}a_{03}a_{04}a_{05}\ldots \\
f(1) &= 0.a_{10}a_{11}a_{12}a_{13}a_{14}a_{15}\ldots \\
f(2) &= 0.a_{20}a_{21}a_{22}a_{23}a_{24}a_{25}\ldots \\
f(3) &= 0.a_{30}a_{31}a_{32}a_{33}a_{34}a_{35}\ldots \\
f(4) &= 0.a_{40}a_{41}a_{42}a_{43}a_{44}a_{45}\ldots \\
\vdots & \qquad \vdots
\end{aligned}$$

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Proof. Then $s = 0.s_1 s_2 s_3 s_4 s_5 \dots$ where

$$s_n = \begin{cases} 5 & \text{if } a_{nn} \neq 5 \\ 7 & \text{otherwise} \end{cases}$$

does not show up in the above list – by construction it disagrees with $f(n)$ at the n-th decimal place.
This implies f cannot be surjective. □

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For example, given

$$
\begin{aligned}
f(0) &= 0.\textcolor{red}{1}38493\ldots \\
f(1) &= 0.5\textcolor{red}{8}3493\ldots \\
f(2) &= 0.09\textcolor{red}{5}813\ldots \\
f(3) &= 0.028\textcolor{red}{4}83\ldots \\
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Then $s = 0.55757\ldots$ does not show up in the above list.

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If $f(x) = Y$ for some x, then ask the question: "Is $x \in Y$?"

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For any set X, $|X| < |\mathcal{P}(X)|$.

Proof. Given any function $f : X \to \mathcal{P}(X)$, we show that f is not surjective.

We claim

$$Y = \{x \in X \mid x \notin f(x)\}$$

is not in the image of f.

If $f(x) = Y$ for some x, then ask the question: "Is $x \in Y$?"

If not... then by definition $x \in Y$, a contradiction.

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Cantor's Theorem: General Case

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If so... then by definition $x \notin f(x) = Y$, another contradiction.
Thus, there is no such x, so f is not surjective. □

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Both of these questions are answered in the affirmative by the **Axiom of Choice**, which roughly states that you can simultaneously make many choices at once.

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- Reflexive: $x \leq x$ for every $x \in X$.
- Antisymmetric: $x \leq y$ and $y \leq x$ implies $x = y$.
- Transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$.
- Total: for any $x, y \in X$, either $x \leq y$ or $y \leq x$.
- Well-Founded: A non-empty subset of X has a least element.

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Example

$\mathbb{N}$ is well-ordered by the standard ordering.

Comparing Well-Ordered Sets

Definition

If X and Y are ordered by $\leq_X$ and $\leq_Y$, then a **order-isomorphism** is a bijection $f : X \to Y$ such that

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Write

$$(X, \leq_X) \leq_{wo} (Y, \leq_Y)$$

if $(X, \leq_X)$ is order-isomorphic to an initial segment of $(Y, \leq_Y)$.

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There are well-orderings $\leq_X$ and $\leq_Y$ of X and Y, respectively.
Then either $(X, \leq_X) \leq_{wo} (Y, \leq_Y)$ or $(Y, \leq_Y) \leq_{w0} (X, \leq_X)$. In particular, $|X| \leq |Y|$ or $|Y| \leq |X|$. □

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Corollary

A set is infinite if and only if it is in bijection with a proper subset.

Ordinals and Cardinals

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Since any set X can be well-ordered, to each set there is a unique cardinal in bijection with X. Hence, the cardinals give canonical representative of the cardinalities.

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There is a *beyond infinite* number of cardinalities.