What's Yellow and Equivalent to the Axiom of Choice?

(Axiom of Choice)

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Background

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  $\bigcap X := \{x \mid x \in y \text{ for every } y \in X\}$
- $A \smallsetminus B$ — the **difference** of B from A, i.e. the set of elements in A not in B.

The Axiom of Choice

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For any collection X of non-empty sets, there exists a function f (a **choice function**) that assigns to each set x in X an element $f(x)$ of x.

1 Cantor and His Set Theory

2 Paradoxes and a Savior

3 Applications of Choice

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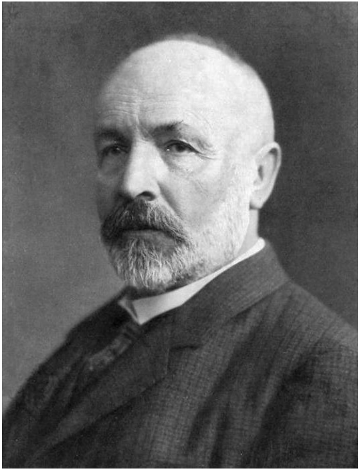
Definition ((Dedekind) Finite and Infinite Sets)

A is **(Dedekind) infinite** if there is $B \subsetneq A$ such that A and B are in bijection.
A is **(Dedekind) finite** if it is not (Dedekind) infinite.

The Major Players – Georg Cantor



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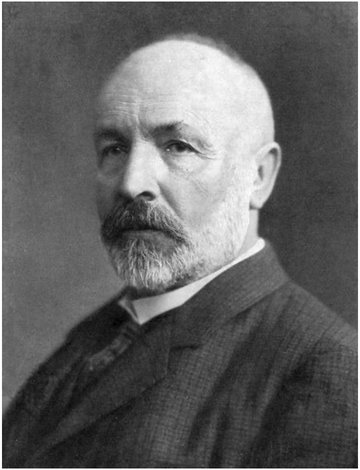

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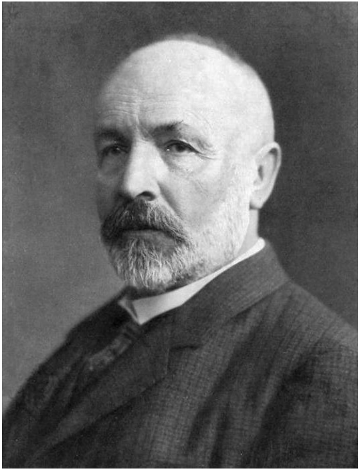

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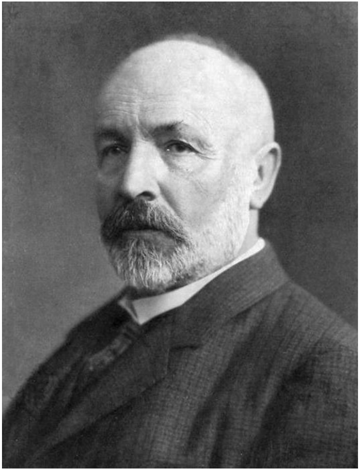

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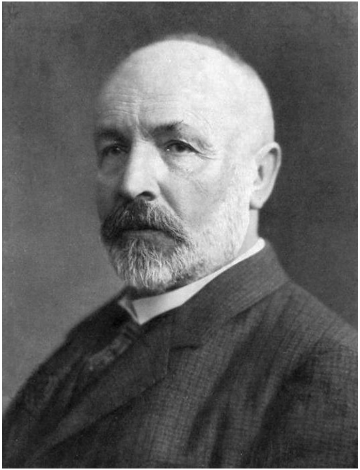
- In 1867 Cantor received his doctorate and shortly began teaching at the University of Halle.



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A number theorist, held a strong position on what mathematics *was*.

"God made the integers, all the rest is the work of man."

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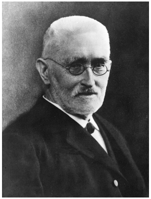
- Richard Dedekind was a German number theorist who communicated often with Cantor and was a large proponent of Cantor's set theory.



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In Cantor's gradual proof involved him asking the following question:

Q: If P is a certain set of exceptional points, then is the claim true if we only assume that Equation $*$ holds for all $x \in (-\pi, \pi) \setminus P$?

Cantor's work on Trigonometric Series

To describe the exceptional sets, Cantor defined

$$P' := \{\text{condensation points of } P\}$$
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This was Cantor's first hint of the eventual theory of ordinal numbers.

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This was the birth of set theory; never before had mathematicians considered that infinite sets could have different 'sizes'.

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Thus, Cantor's 1874 findings can be subsumed as

$$|\{\text{real algebraic numbers}\}| = |\mathbb{N}| < |\mathbb{R}|$$

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> *"Can a surface (say, a square that includes the boundaries) be uniquely referred to a line (say, a straight-line segment that includes the end points) so that for every point of the surface there is a corresponding point of the line and conversely, for every point of the line there is a corresponding point on the surface? Methinks that answering this question would be no easy job, despite the fact that the answer seems so clearly to be 'no' that proof appears almost unnecessary."*

Ultimately, the answer was 'yes': Cantor showed that $|[0,1]| = |[0,1]^2|$, and this extended to higher dimensions as well.

"I see it, but I don't believe it."

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(2) **Cantor's Continuum Hypothesis**: Show that there is no intermediate set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$.

To establish the first, Cantor made use of a result called the **Well-Ordering Theorem**, which states that every set can be *well-ordered*.

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Totality: For any $x, y \in W$, either $x \leq y$ or $y \leq x$.

Well-foundedness: Any non-empty subset V of W has a $\leq$-least element.

Ordinals - Motivation

Cantor found that there was a standard way of enumerating the elements of a well-ordered set W:

w_0

By well-foundedness, W has a least element w_0.

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Cantor found that there was a standard way of enumerating the elements of a well-ordered set W:

w_0, w_1

If $W \setminus \{w_0\}$ is non-empty, then it has a least element w_1.

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w_0, w_1, w_2

If $W \setminus \{w_0, w_1\}$ is non-empty, then it has a least element w_2.

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$w_0, w_1, w_2, \cdots, w_n, \cdots$

Continuing in this way, we may define $w_3, w_4, w_5, \ldots, w_n, \ldots$ corresponding to each natural number.

Ordinals - Motivation

Cantor found that there was a standard way of enumerating the elements of a well-ordered set W:

$$w_0, w_1, w_2, \cdots, w_n, \cdots, w_\omega$$

If $W \setminus \{w_0, w_1, \ldots\}$ is non-empty, it has a least element, which we denote w_ω.

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$$w_0, w_1, w_2, \cdots, w_n, \cdots, w_\omega, w_{\omega+1}$$

But then we may consider $W \setminus \{w_0, w_1, \ldots, w_\omega\}$; if it is non-empty, it has a least element, which we denote $w_{\omega+1}$.

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We can continue in this way, defining $w_{\omega+2}, w_{\omega+3}, w_{\omega+4}, \ldots, w_{\omega+n}, \ldots$.

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$$w_0, w_1, w_2, \cdots, w_n, \cdots, w_\omega, w_{\omega+1}, \cdots, w_{\omega+n}, \cdots, w_{\omega\cdot 2}$$

Then, as before, we may considering the next element bigger than all of these, $w_{\omega+\omega} = w_{\omega\cdot 2}$. As long as possible, we can continue even further, to $w_{\omega\cdot 3}, w_{\omega\cdot 4}, \ldots, w_{\omega\cdot n}, \ldots$.

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Past each of the elements of $w_{\omega\cdot n}$ is the least element above all of them, $w_{\omega\cdot\omega} = w_{\omega^2}$.

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And nothing stops us from continuing from there, at least until we run out of elements.

Ordinals - Cantor's and Modern Definitions

Cantor defined his ordinal numbers by starting with 0 (today, we define $0 := \varnothing$) and defining two generation rules:

Definition

Cantor's Intuitive Definition	Modern Definition

Ordinals - Cantor's and Modern Definitions

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(1)	Given an ordinal α, form its **successor ordinal** $\alpha + 1$.	$\alpha + 1 := \alpha \cup \{\alpha\}$
(2)	Given a set of ordinals S with no maximum, form its **limit ordinal** $\sup S$.	$\sup S := \bigcup S$

Note that in the modern definition, an ordinal is the set of all smaller ordinals.

Cardinals

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Without it, uncountable ordinals need not exist.

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The cardinal numbers form a linear hierarchy of sizes

$$0 < 1 < 2 < \cdots < \aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_n < \cdots < \aleph_\omega < \aleph_{\omega+1} < \cdots < \aleph_{\omega^2} < \cdots\cdots$$

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Theorem (Well-Ordering Theorem)

Suppose S is a set. Then there exists a binary relation $\leq$ which is a well-ordering of S.

Cantor, unfortunately, failed to find a proof.

Trichotomy

Lemma

For every ordinal β, there exists an ordinal α such that β and $\aleph_\alpha$ are in one-to-one correspondence.

Proof.

Assume without loss of generality that β does not inject into any $\gamma < \beta$. (Such an ordinal is an *initial ordinal*.)
By construction of the alephs, $\beta \le \aleph_\beta$. Let α be the least ordinal such that β injects into $\aleph_\alpha$. Two observations:

- β is initial, so $\beta \le \aleph_\alpha$.
- β does not inject into $\aleph_\gamma$ for any $\gamma < \alpha$, so $\aleph_\gamma < \beta$ for all $\gamma < \alpha$.

Then $\aleph_\alpha = \bigcup_{\gamma<\alpha} \aleph_\gamma \le \beta \le \aleph_\alpha$, so $\beta = \aleph_\alpha$. □

Trichotomy

Corollary (Trichotomy Theorem)

Suppose S is an infinite set. Then there exists an ordinal α such that S is in one-to-one correspondence with $\aleph_\alpha$.

Proof.

Well-order S, so that $|S| = |\beta|$ for some ordinal β. Then $|S| = |\beta| = |\aleph_\alpha|$ for some ordinal α. □

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Then $\sup S = \bigcup C$ is also a cardinal – a *largest* cardinal. $\sup S = \bigcup C = \aleph_\alpha$ for some ordinal α, but then $\aleph_{\alpha+1}$ is larger, a contradiction.

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Russell's Paradox: Let $S := \{x \mid x \notin x\}$. Does S contain itself?

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In 1904, he formulated the Axiom of Choice and produced a proof of the Well-Ordering Theorem.

Proof of the Well-Ordering Theorem

Lemma (Principle of General Recursion)

For every set S, ordinal α, and function $F : \{h : \beta \to S \mid \beta < \alpha\} \to S$, there is a function $f : \alpha \to S$ such that

$$f(\beta) = F(\textit{restriction of } f \textit{ to } \beta)$$

Proof of Well-Ordering Theorem.

Let γ be an ordinal which does not inject into S, and let f be a choice function for $\mathcal{P}(S) \setminus \{\varnothing\}$. Define a bijection of an initial segment of γ onto S as follows:

Stage 0: Associate 0 with $f(S)$.

Stage α: Associate α with $f(S \setminus \{s_\beta \mid \beta < \alpha\})$ if that set is non-empty.

At some point in this process, we must run out of elements of S (otherwise instead run out of elements of γ and have an injection of γ into S, a contradiction).

The procedure well-orders the elements of S. □

Equivalence of Well-Ordering and Choice

Theorem

The Well-Ordering Theorem implies the Axiom of Choice.

Proof.

Suppose X is a set not containing $\varnothing$.
Let $S = \bigcup X$, and well-order S.
Then define $f : X \to S$ by the rule

$$f(x) := \text{least element of } x \subseteq S$$

□

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Axiom VII: Axiom of Infinity. There exists in the domain at least Z that contains $\varnothing$ and for every $x \in Z$ also contains $\{x\}$.

Developments in the Axiomatization of Set Theory

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> *Despite this, however, many mathematicians continued to use it as mathematics expanded in the succeeding decades. A conflict continued to rage among mathematics about whether it was legitimate, acceptable mathematics. It became the most discussed axiom next to Euclid's parallel axiom. As Lebesgue remarked, the opponents could do no better than insult each other because there was no agreement. He himself, despite his negative and distrustful attitude toward the axiom, employed it, as he put it, audaciously and cautiously. He maintained that future developments would help us decide.*

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Meanwhile, Zermelo's theory evolved. Abraham Fraenkel found short-comings in the theory, and added his *Axiom of Replacement*. With some additional minor modifications, the resulting theory became known as Zermelo-Fraenkel Set Theory, or ZF. (Which did *not* include Choice.)

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His Constructible Universe showed that if ZF was consistent, ZFC (ZF plus the Axiom of Choice) is also consistent.

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Questions?