Weird Models

When Things Stop Being Standard

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Of course, this doesn't actually happen. But where is our intuition wrong?

First Order Logic

Our setting is *First Order Logic* or *predicate logic*.
Roughly speaking, this is the logic that makes sense of well-formed formulas like

$$\forall x \forall y \forall z (f(x, f(y,z)) = f(f(x,y),z))$$

i.e. formulas in which variables are allowed to be quantified over.

- *Sentences* are well-formed formulas in which every variable is bound by a quantifier.
- A *First-Order Theory* is a set of sentences (taken to be *non-logical axioms* of the theory).
- *Provability* is defined in terms of logical axioms, hypotheses, and rules of inference.
- *Satisfiability* is defined in terms of structures that instantiate the functions and relations.

First Order Logic - Language

To make the notions of provability precise, we need to define the syntactics of first order logic.

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(We will assume a signature $\sigma = (\mathscr{F}, \mathscr{R}, \mathrm{ar})$ is fixed unless specified otherwise.)

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Note that there is *unique readability*; a wff is built up from the above rules in exactly one way.

First Order Logic - Logical Axioms and Rules of Inference

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4. $(((\alpha \to \bot) \to \bot) \to \alpha)$
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Modus Ponens: From $(\alpha \to \beta)$ and α conclude β.

Generalization: From φ conclude $\forall x\varphi$.

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Sentence: a wff φ with no free variables.
Let Φ be a set of sentences and φ a sentence. Φ **proves** φ

$$\Phi \vdash \varphi$$

if there is a sequence $\psi_1, \ldots, \psi_n$ such that

1. ψ_i is either
   - an axiom,
   - an element of Φ (*hypothesis*), or
   - the result of Modus Ponens or Generalization applied to earlier ψ_j's.
2. $\psi_n = \varphi$.

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Consistent	if $\Phi \nvdash \bot$.
Inconsistent	if it is not consistent.
Complete	if for any sentence φ, either $\Phi \vdash \varphi$ or $\Phi \vdash \neg\varphi$.

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Consistent	if $\Phi \nvdash \bot$.
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Complete	if for any sentence φ, either $\Phi \vdash \varphi$ or $\Phi \vdash \neg\varphi$.
Incomplete	if it is not conplete.

First Order Theories

First Order Theory Φ is a set of sentences.
φ is a **theorem** of Φ if $\Phi \vdash \varphi$.

Φ is

Consistent	if $\Phi \nvdash \bot$.
Inconsistent	if it is not consistent.
Complete	if for any sentence φ, either $\Phi \vdash \varphi$ or $\Phi \vdash \neg\varphi$.
Incomplete	if it is not conplete.
Recursively-Enumerable	(roughly) if there is an algorithm which enumerates the elements of Φ.

Example: Peano Arithmetic

Peano Arithmetic is the first order theory in the signature $(\{0, S, +, \cdot\}, \varnothing)$ consisting of the sentences:

1. $\neg \exists x (S(x) = 0)$
2. $\forall x \forall y ((S(x) = S(y)) \to (x = y))$
3. $\forall x (x + 0 = x)$
4. $\forall x \forall y (x + S(y) = S(x + y))$
5. $\forall x (x \cdot 0 = 0)$
6. $\forall x \forall y (x \cdot S(y) = x \cdot y + x)$
7. For each wff φ with free variables $x, y_1, \ldots, y_n$, the sentence

$$\forall y_1 \cdots \forall y_n \Big(\big(\varphi(0, y_1, \ldots, y_n) \wedge \forall x (\varphi(x, y_1, \ldots, y_n) \to \varphi(S(x), y_1, \ldots, y_n)) \big) \to \forall x \varphi(x, y_1, \ldots, y_n) \Big)$$

Example: Ordered Fields

The first order theory of ordered fields has signature $(\{0, 1, +, \cdot\}, \{\leq\})$ and axioms

1. $\neg(0 = 1)$
2. $\forall x \forall y \forall z((x + y) + z = x + (y + z))$
3. $\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$
4. $\forall x \forall y(x + y = y + x)$
5. $\forall x \forall y(x \cdot y = y \cdot x)$
6. $\forall x(x + 0 = x)$
7. $\forall x(x \cdot 1 = x)$
8. $\forall x \exists y(x + y = 0)$
9. $\forall x(\neg(x = 0) \rightarrow \exists y(x \cdot y = 1))$
10. $\forall x \forall y \forall z(x \cdot (y + z) = x \cdot y + x \cdot z)$
11. $\forall x \forall y \forall z(x \leq y \rightarrow x + z \leq y + z)$
12. $\forall x \forall y \forall z(0 \leq z \rightarrow (x \leq y \rightarrow x \cdot z \leq y \cdot z))$

Aside - Godel's Incompleteness Theorems

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Φ *is incomplete.*

Theorem (Godel's Second Incompleteness Theorem)

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If φ is a sentence, then we write $\mathbb{A} \vDash \varphi$ if $\mathbb{A} \vDash_v \varphi$ for every assignment v (equivalently, any one of them).

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Examples of Models

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$\mathbb{Q}, \mathbb{Q}^{\mathrm{alg}} \cap \mathbb{R}, \mathbb{R}, {}^*\mathbb{R}$ are all models of the theory of ordered fields with $0, 1, +, \cdot, \leq$ given their standard definitions.

Godel's Completeness Theorem

Theorem (Completeness Theorem for First Order Logic)

If Φ is a set of sentences, then Φ is consistent if and only if Φ is satisfiable. Equivalently, $\Phi \vdash \varphi$ if and only if $\mathbb{A} \models \varphi$ for every model $\mathbb{A}$ of Φ.

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This gives some insight into why provability of something one way or the other can be difficult: a theory can have *nonstandard* models.

Infinite Models of all Sizes

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4. for every wff $\varphi(x_1, \ldots, x_n)$ and $b_1, \ldots, b_n \in B$ then

$$\mathbb{A} \models \varphi(b_1, \ldots, b_n) \quad \text{if and only if} \quad \mathbb{B} \models \varphi(b_1, \ldots, b_n)$$

Theorem (Lowenheim-Skolem Theorem)

Suppose κ is an infinite cardinal with $\kappa \geq |\mathscr{F} \cup \mathscr{R}|$. Suppose $\mathbb{A}$ is an infinite model of Φ. Then there exists a model $\mathbb{B}$ of Φ with $|B| = \kappa$ and

- *$\mathbb{B}$ is an elementary submodel of $\mathbb{A}$ if $\kappa \leq |A|$*
- *$\mathbb{B}$ is an elementary extension of $\mathbb{A}$ if $\kappa \geq |A|$*

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Contradiction. □

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Remark: In fact, countable models of ZFC are some of the most wildly studied since they can be used with forcing.

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A model of Φ' gives a model of PA but which contains "infinite" elements.
Lowenheim-Skolem implies that we have such models that are countable.

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Let Φ be the set of sentences in this language satisfied by $\mathbb{R}$ and

$$\Phi' = \Phi \cup \{P_<(f_r, c) \mid r \in \mathbb{R}\}$$

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$$\mathscr{F} = \{f_F \mid F : \mathbb{R}^k \to \mathbb{R}, k \in \mathbb{N}\} \cup \{c\}$$
$$\mathscr{R} = \{P_R \mid R \subset \mathbb{R}^k, k \in \mathbb{N}\}$$

Let Φ be the set of sentences in this language satisfied by $\mathbb{R}$ and

$$\Phi' = \Phi \cup \{P_<(f_r, c) \mid r \in \mathbb{R}\}$$

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Compactness and Lowenheim-Skolem imply that Φ' has a model of cardinality $|\mathbb{R}|$; call it ${}^*\mathbb{R}$ - the *hyperreals*!.

Thank you!

Questions?