The Limitations of Ruler-and-Compass Constructions

Hayden Jananthan

Vanderbilt University

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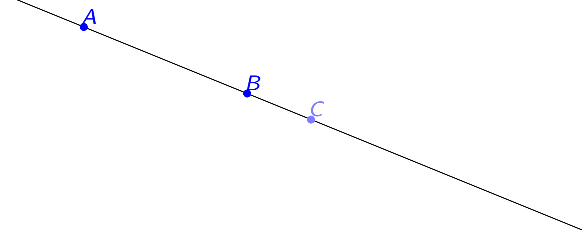
- If the line ℓ and circle C have been constructed already, then their intersection points have been constructed.

Construct Perpendicular Line $\ell_\perp$ to line $\ell = \overleftrightarrow{AB}$ through point C

Case 1: C incident to ℓ



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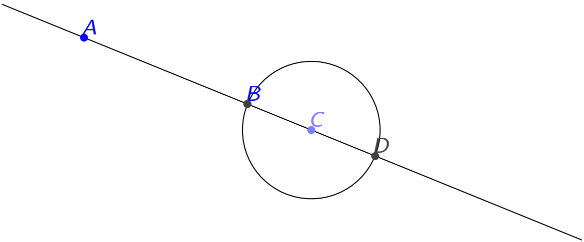

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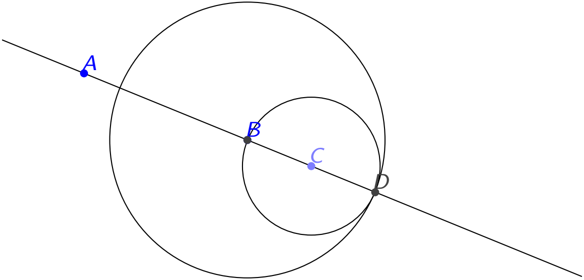

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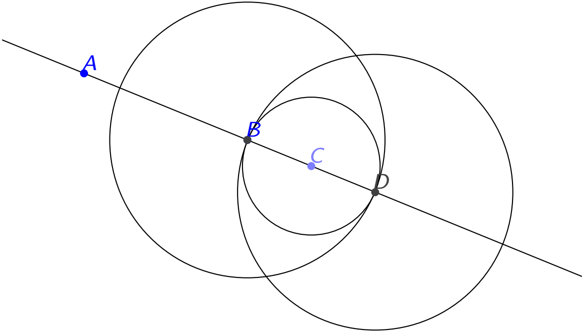

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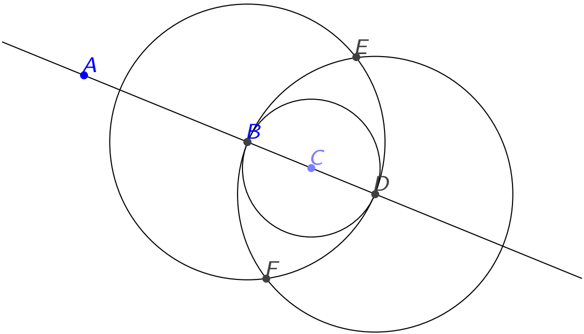

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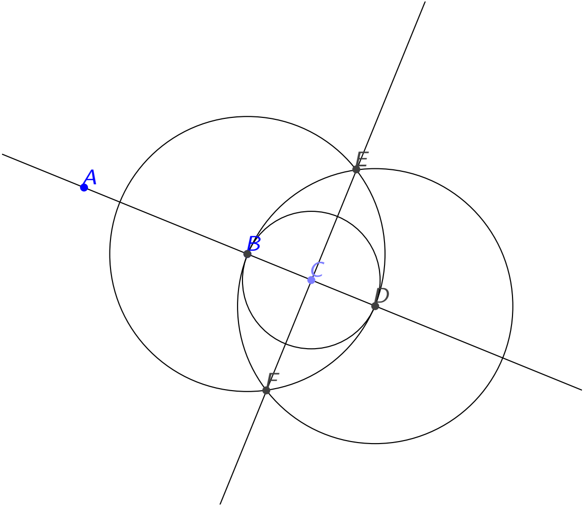

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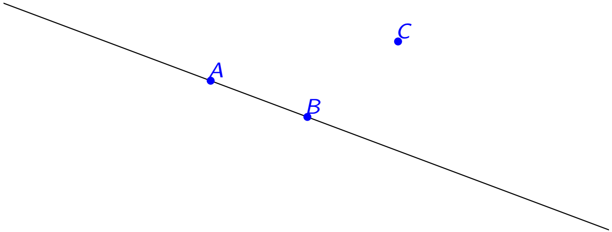

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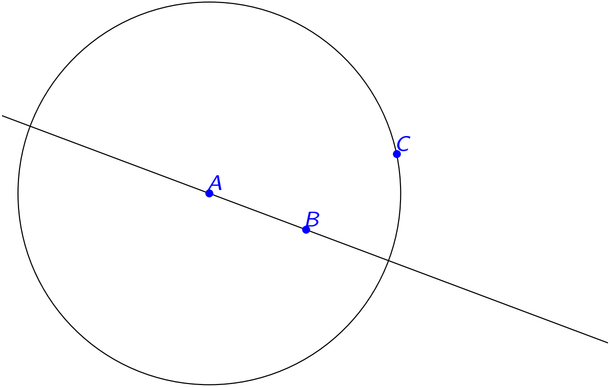

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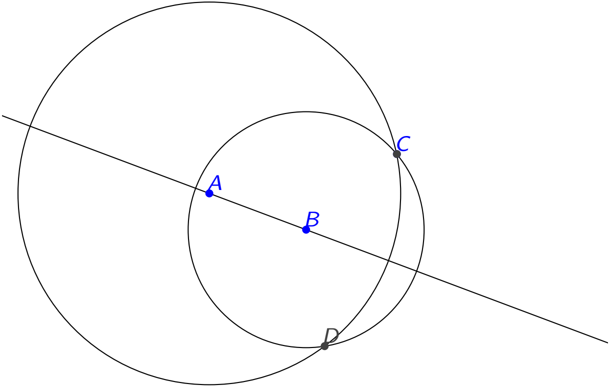

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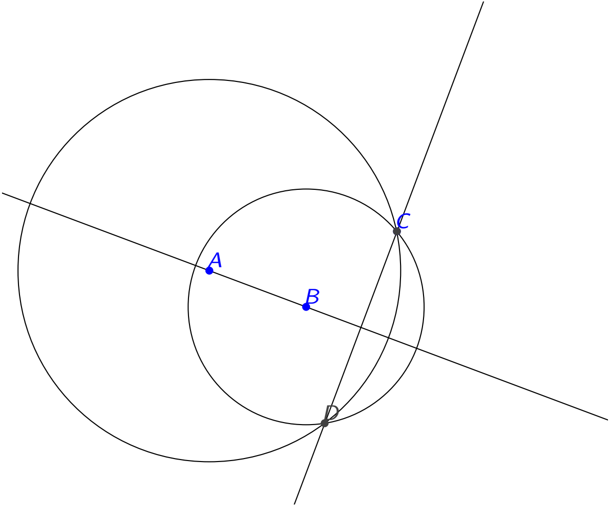

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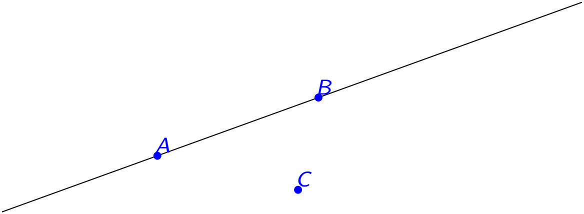

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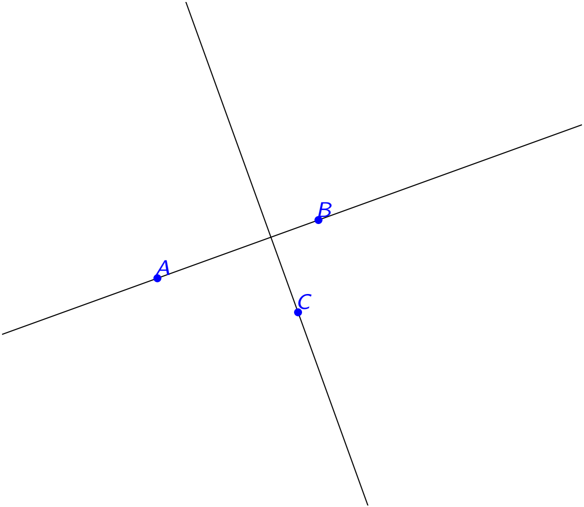

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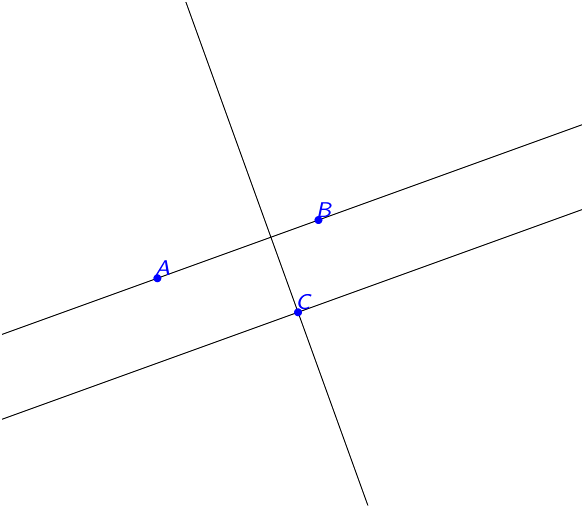
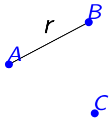


Translate Line Segment

Case 1: Non-collinear



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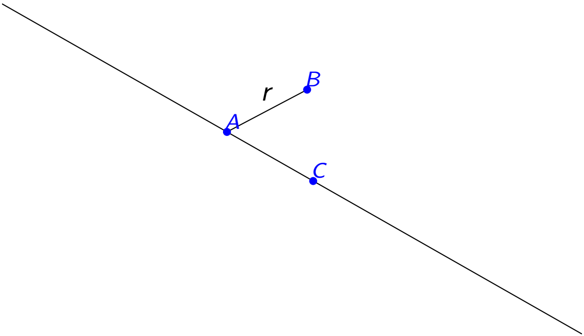

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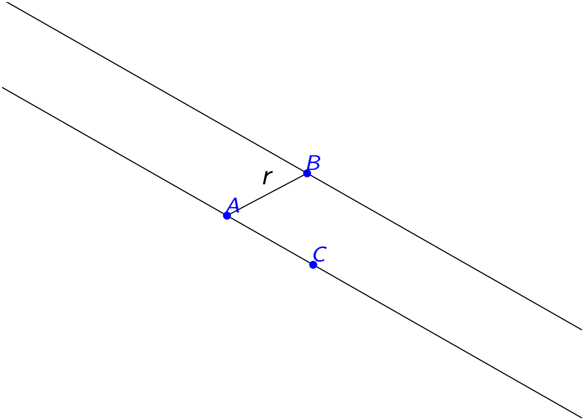

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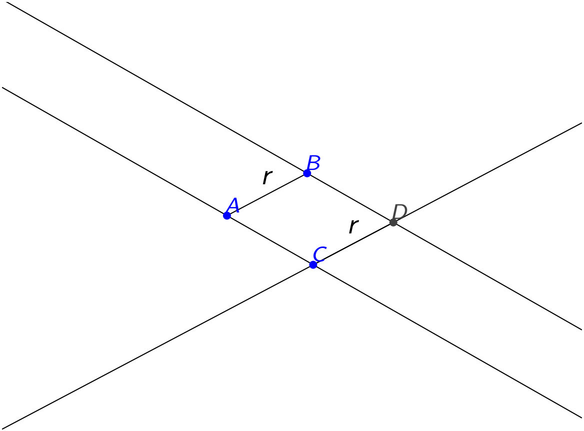

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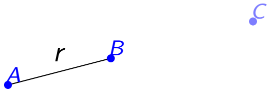
Case 2: Collinear



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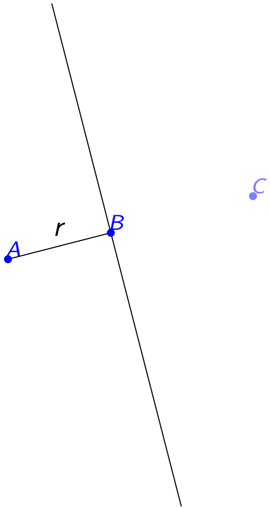

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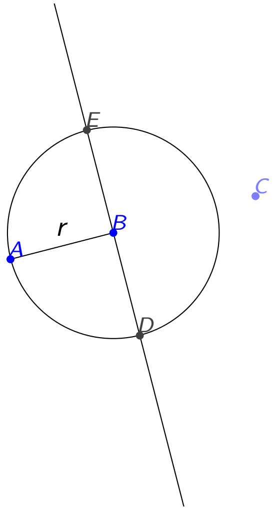

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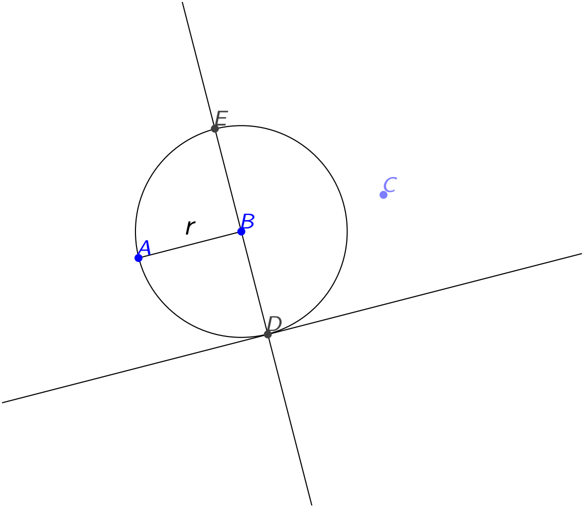

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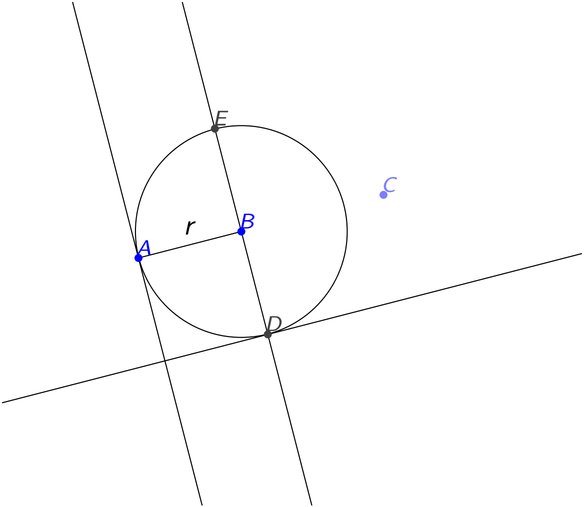

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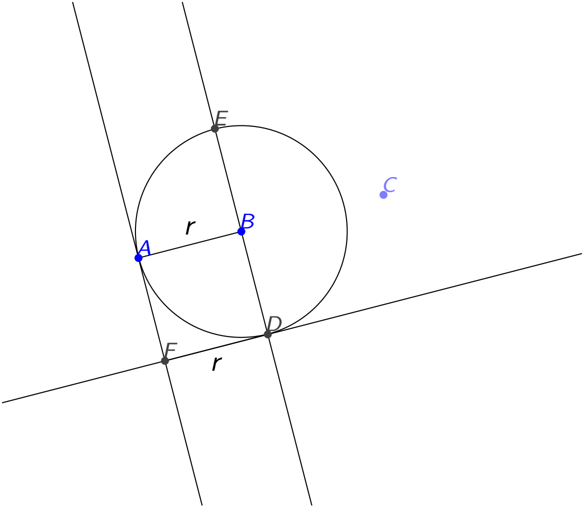

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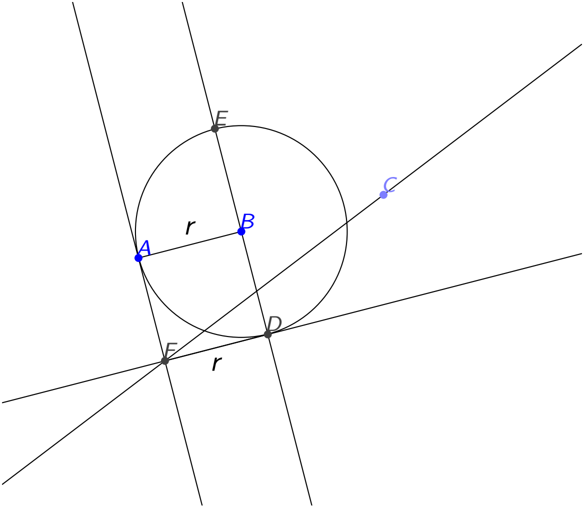

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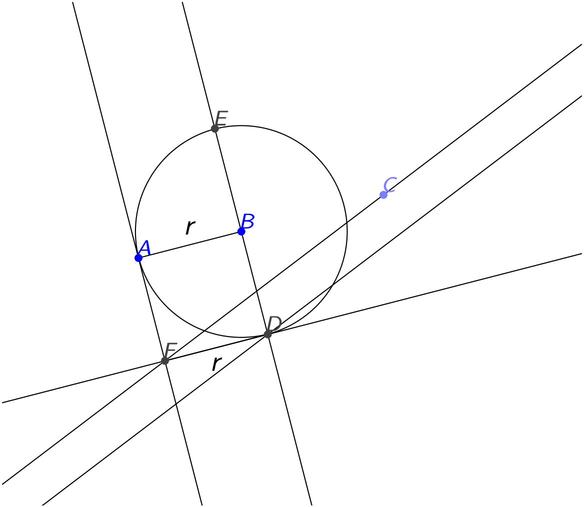

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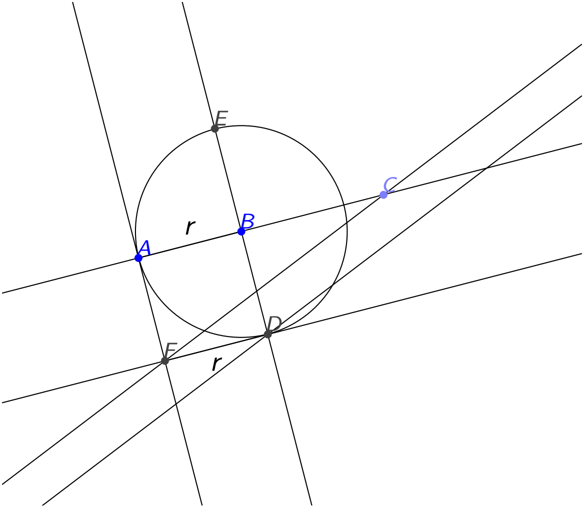

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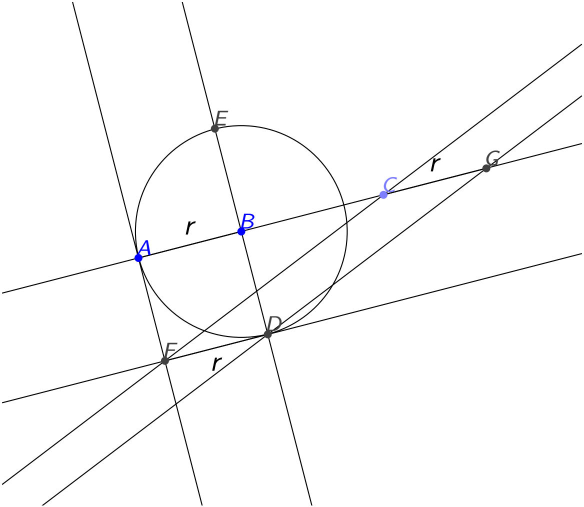

Limitations of Ruler and Compass Constructions

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It wasn't until the 1800s that these constructions were proven to *not* be possible.

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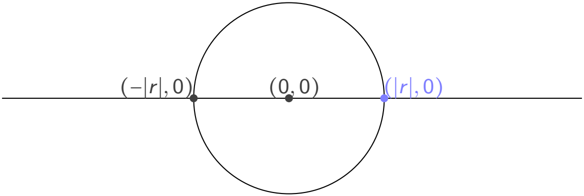
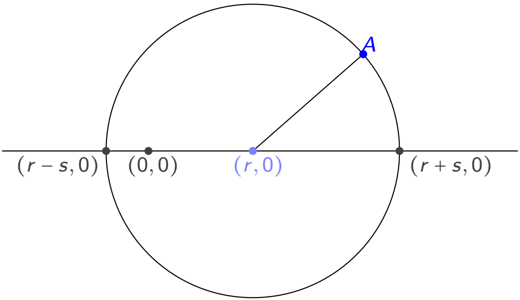
2 $\Leftarrow$ 1 : Suppose $\overline{AB}$ is a line segment of length $|r|$. Translate $\overline{AB}$ to $\overline{OC}$, where O = origin. Draw circle centered at O going through C. This intersects the x-axis at the point $(|r|, 0)$.

□

Consider below diagram:



$-r$ is one of $|r|$ or $-|r|$.

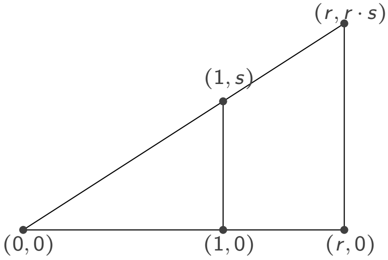


For $r, s \geq 0$, the above construction shows $r + s$ and $r - s$ are constructible.
If r is negative, then

$$r + s = -\big((-r) + (-s)\big)$$

Properties of Constructible Numbers

Closure under Multiplication



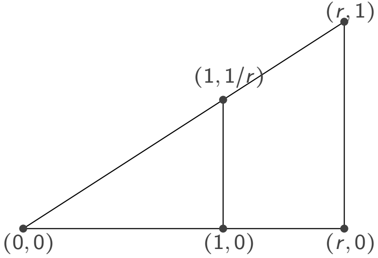
For $r, s \geq 0$, the above construction shows there is a line segment of length $r \cdot s$.

If either r or s are negative, then

$$r \cdot s = -\big((-r) \cdot s\big) = -\big(r \cdot (-s)\big) = (-r) \cdot (-s)$$

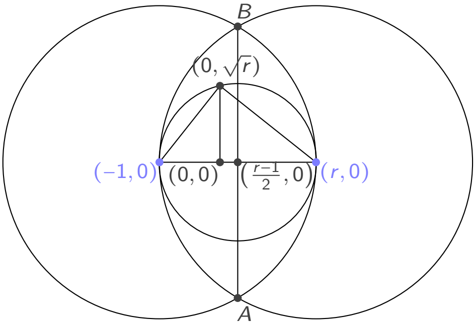
Properties of Constructible Numbers

Closure under Reciprocation



For $r > 0$, the above construction shows there is a line segment of length $1/r$.
If $r < 0$, then

$$\frac{1}{-r} = -\frac{1}{r}$$



We construct the circle of radius $\frac{1+r}{2}$ centered at $\left(\frac{r-1}{2}, 0\right)$.
This circle intersects the y-axis at $(0, \pm\sqrt{r})$.

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- 1 ⇒ 2 Must analyze three kinds of intersections:
  - two lines
  - a circle and a line
  - two circle

How Constructed Points Arise

Intersection of two Lines

Suppose ℓ_1, ℓ_2 are the lines

$$\ell_1 a_1 x + b_1 y = c_1$$
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$$a_1 x + b_1 y = c_1 \Leftrightarrow x = \frac{c_1}{a_1} - \frac{b_1}{a_1} y$$

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This is linear in y.
Thus, x and y are obtained from $a_1, b_1, c_1, a_2, b_2, c_2$ using the operations of addition, subtraction, multiplication, and division.

Suppose C is the circle

$$C: \quad (x-h)^2 + (y-k)^2 = r^2$$

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Intersection of a Circle and Line

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and ℓ is the line

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where h, k, r, a, b, c are numbers which have already been constructed.

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where h, k, r, a, b, c are numbers which have already been constructed. Then either $a \neq 0$ or $b \neq 0$. Say $a \neq 0$, so

$$ax + by = c \Leftrightarrow x = \frac{c}{a} - \frac{b}{a}y$$

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Intersection of a Circle and Line

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$$C: \quad (x-h)^2 + (y-k)^2 = r^2$$

and ℓ is the line

$$\ell: \quad ax + by = c$$

where h, k, r, a, b, c are numbers which have already been constructed. Then either $a \neq 0$ or $b \neq 0$. Say $a \neq 0$, so

$$ax + by = c \Leftrightarrow x = \frac{c}{a} - \frac{b}{a}y$$

Substituting into the equation for C gives a quadratic equation in y.

How Constructed Points Arise

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Substituting into the equation for C gives a quadratic equation in y. Thus, x and y are obtainable from h, k, r, a, b, c using the operations of addition, subtraction, multiplication, division, and taking square roots.

How Constructed Points Arise

Intersection of Circles

Suppose C_1, C_2 are circles

$$C_1 : (x - h_1)^2 + (y - k_1)^2 = r_1^2$$
$$C_2 : (x - h_2)^2 + (y - k_2)^2 = r_2^2$$

where $h_1, k_1, h_2, k_2, r_1, r_2$ are numbers which have already been constructed.

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Then

$$\begin{aligned}(x - h_1)^2 + (y - k_1)^2 - r_1^2 &= (x - h_2)^2 + (y - k_2)^2 - r_2^2 \\ x^2 - 2h_1x + h_1^2 + y^2 - 2k_1y + k_1^2 - r_1^2 &= x^2 - 2h_2x + h_2^2 + y^2 - 2k_2y + k_2^2 - r_2^2 \\ 2(h_2 - h_1)x + (h_1^2 + k_1^2 - r_1^2) &= 2(k_1 - k_2)y + (h_2^2 + k_2^2 - r_2^2) \\ ax + by &= c\end{aligned}$$

where $a = 2(h_2 - h_1)$, $b = 2(k_2 - k_1)$, and $c = h_1^2 - h_2^2 + k_1^2 - k_2^2 + r_2^2 - r_1^2$.
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$$2(h_2 - h_1)x + (h_1^2 + k_1^2 - r_1^2) = 2(k_1 - k_2)y + (h_2^2 + k_2^2 - r_2^2)$$
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Some Algebra Background

Proposition

Suppose r is a root of the irreducible polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where each a_k is a root of an irreducible polynomial p_k with rational coefficients. Then r is the root of an irreducible polynomial with rational coefficients.

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where each a_k is a root of an irreducible polynomial p_k with rational coefficients. Then r is the root of an irreducible polynomial with rational coefficients.
Moreover, the degree of that irreducible polynomial divides $n \cdot \deg(p_0) \cdots \deg(p_n)$.

Application to Constructibility

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Suppose r is constructible. Then r is the root of an irreducible polynomial with rational coefficients of degree 2^n *for some n.*

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Suppose r uses $k + 1$ square roots, so $r = a + b\sqrt{c}$ where a, b, c use at most k square roots. Then r is a root of the quadratic

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Proposition implies r is root of an irreducible polynomial with rational coefficients with degree dividing

$$2 \cdot 2^k \cdot 2^k \cdot 2^k = 2^{3k+1}$$

□

Impossibility of Squaring the Circle

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π is *transcendental*, not the root of any polynomial, so neither is $\sqrt{\pi}$. $\square$

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If $\sqrt[3]{2}$ is constructible, it is the root of a polynomial $p(x)$ with rational coefficients of degree 2^n for some n.

But $\sqrt[3]{2}$ is also the root of the irreducible polynomial $x^3 - 2$, so $x^3 - 2$ must divide $p(x)$. This is impossible since 3 does not divide 2^n. □

Impossibility of Trisecting Arbitrary Angles

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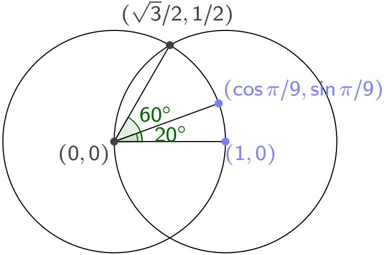
Proof.

If trisecting arbitrary angles is possible, then a 20° angle in primary position can be constructed since a 60° angle in primary position can be constructed. This angle intersects the unit circle at $\left(\cos\frac{\pi}{9}, \sin\frac{\pi}{9}\right)$. Suffices to show $\cos\frac{\pi}{9}$ is not constructible.



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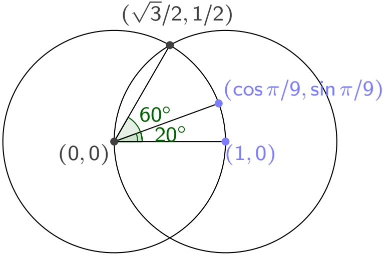

The triple angle formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ implies that $\cos\frac{\pi}{9}$ is a root of the cubic $4x^3 - 3x - \frac{1}{2}$. But 3 does not divide 2^n for any n, so $\cos\frac{\pi}{9}$ cannot be constructible.

□