Posner-Robinson for Turing Degrees of Hyperjumps

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July 8, 2019

Previous Results

Theorem (Posner-Robinson 1981)

Suppose $0 <_{\mathrm{T}} Z \leq_{\mathrm{T}} A$ *and* $0' \leq_{\mathrm{T}} A$. *Then there exists* B *such that*

$$A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus 0'$$

The proof is relatively straight-forward.

Theorem (Kumabe & Slaman 1999)

Suppose Z *is not arithmetical and* $Z \leq_{\mathrm{T}} A$ *and* $0^{(\omega)} \leq_{\mathrm{T}} A$. *Then there exists* B *such that*

$$A \equiv_{\mathrm{T}} B^{(\omega)} \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus 0^{(\omega)}$$

The Main Result

Theorem

Suppose $0 <_{\mathrm{HYP}} Z \leq_{\mathrm{T}} A$ *and* $\mathcal{O} \leq_{\mathrm{T}} A$. *Then there exists* B *such that*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

The proof is more difficult than the classical version.

Theorem

Suppose $Z_0, Z_1, Z_2, \ldots$ *is a sequence of non-hyperarithmetical reals which is recursive in* A, *where* $\mathcal{O} \leq_{\mathrm{T}} A$. *Then there exists* B *such that*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus Z_k \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

for every $k \in \mathbb{N}$.

We present an unpublished proof due to Slaman using Kumabe-Slaman forcing.

1. Recursion Theory Background
2. Hyperarithmetical Theory Background
3. Turing Functionals and Kumabe-Slaman Forcing
4. Proof of the Main Results

Notation

Baire Space — $\mathbb{N}^{\mathbb{N}}$ with the product topology

Cantor Space — $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ with the product topology

Partial Function — $f :\subseteq A \to B$ is a function $f : C \to B$ with $\operatorname{dom} f = C \subseteq A$

$$f(a)\downarrow \iff f(a) \textit{ converge/is defined} \iff a \in \operatorname{dom} f$$
$$f(a)\uparrow \iff f(a) \textit{ diverges/is undefined} \iff a \notin \operatorname{dom} f$$

$f(x) \simeq g(x)$ means either both $f(x)$ and $g(x)$ converge and are equal, else they both diverge

Strings — S^* is the set of strings of elements in S.

- If $s_1, \ldots, s_n \in S$, then $\sigma = \langle s_1, \ldots, s_n \rangle \in S^*$ is defined by $\sigma(k) = s_{k+1}$. $\langle\rangle$ is the empty string.
- $|\sigma| = \operatorname{dom} \sigma$, i.e. $|\langle s_1, \ldots, s_n \rangle| = n$.
- $\langle s_1, \ldots, s_n \rangle^\frown \langle t_1, \ldots, t_m \rangle = \langle s_1, \ldots, s_n, t_1, \ldots, t_m \rangle$.
- $\sigma \subseteq \tau$ if $\sigma = \tau \upharpoonright |\sigma|$.

Partial Recursive Functions

Definition (Partial Recursive)

Suppose $f :\subseteq \mathbb{N}^k \to \mathbb{N}$ *is given.*

$$f \text{ is } \textbf{partial recursive} \iff f \text{ is algorithmically computable}$$

where 'algorithm' is interpreted in your favorite programming language.

If e is the *Gödel number* of such an algorithm, write

$$\varphi_e^{(k)}(m_1, \ldots, m_k) \simeq f(m_1, \ldots, m_k)$$

We call e an **index** of f.

Partial Recursive Functionals

Definition (Partial Recursive)

Suppose $\Psi :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k \to \mathbb{N}$ *is given.*

$$\Psi \textit{ is } \textbf{partial recursive} \iff \Psi \textit{ is algorithmically computable}$$

where 'algorithm' now includes oracle/black-box computations that make use of the function parameter.

If e is the *Gödel number* of such an algorithm, write

$$\varphi_e^{(k),f}(m_1, \ldots, m_k) \simeq \Psi(f, m_1, \ldots, m_k)$$

We call e an **index** of Ψ.

Turing Reducibility, Equivalence, and Degrees

Definition (Turing Reducibility)

Suppose $f, g \in \mathbb{N}^{\mathbb{N}}$. The **Turing reducibility** *preorder $\leq_{\mathrm{T}}$ is defined by*

$$\begin{aligned} f \leq_{\mathrm{T}} g &\iff f \text{ is algorithmically computable using oracle } g \\ &\iff f = \varphi_e^{(1),g} \text{ for some } e \end{aligned}$$

f and g are **Turing equivalent**, *$f \equiv_{\mathrm{T}} g$, if and only if $f \leq_{\mathrm{T}} g$ and $g \leq_{\mathrm{T}} f$. $\equiv_{\mathrm{T}}$ is an equivalence relation.*

Can similarly define *g-computability* ($g \in \mathbb{N}^{\mathbb{N}}$) for partial functions or predicates.

Definition (Turing Degree)

Suppose $f \in \mathbb{N}^{\mathbb{N}}$. The **Turing degree** *associated with f is*

$$\deg_{\mathrm{T}}(f) = \{g \in \mathbb{N}^{\mathbb{N}} \mid f \equiv_{\mathrm{T}} g\}$$

Basic Results

Suppose $f, g \in \mathbb{N}^{\mathbb{N}}$. The *join* $f \oplus g \in \mathbb{N}^{\mathbb{N}}$ is defined by

$$(f \oplus g)(2n) = f(n)$$
$$(f \oplus g)(2n+1) = g(n)$$

Proposition

$$\sup(\deg_{\mathrm{T}}(f), \deg_{\mathrm{T}}(g)) = \deg_{\mathrm{T}}(f) \vee \deg_{\mathrm{T}}(g) = \deg_{\mathrm{T}}(f \oplus g)$$

Proposition

There exists $X \in 2^{\mathbb{N}}$ such that $f \equiv_{\mathrm{T}} X$.

Relativized Arithmetical Hierarchy

Suppose $X \in 2^{\mathbb{N}}$.

Definition

$$\Sigma_0^{0,X} = \Pi_0^{0,X} = \Delta_0^{0,X} = \{R \mid R \subseteq (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^\ell \text{ an } X\text{-recursive predicate}\}$$

$$\Sigma_{n+1}^{0,X} = \{S \mid S(-) \equiv \exists n\, R(n,-) \text{ for } R \in \Pi_n^{0,X}\}$$

$$\Pi_{n+1}^{0,X} = \{S \mid S(-) \equiv \forall n\, R(n,-) \text{ for } R \in \Sigma_n^{0,X}\}$$

$$\Delta_n^{0,X} = \Sigma_n^{0,X} \cap \Pi_n^{0,X}$$

where quantifiers range over $\mathbb{N}$. If X is recursive, we drop mention of X, i.e. $\Sigma_n^{0,X} = \Sigma_n^0$ and $\Pi_n^{0,X} = \Pi_n^0$.
*S is X-**arithmetical** if S is $\Sigma_n^{0,X}$ for some n.*

Proposition

$S \subseteq (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^\ell$ is X-recursive if and only if S is $\Delta_1^{0,X}$.

Turing Jump Operator

Suppose $X, Y \in 2^{\mathbb{N}}$.

Definition

The **Turing jump** *of X is defined by $X' = \{e \in \mathbb{N} \mid \varphi_e^{(1),X}(0)\downarrow\}$.*

I.e. X' is the *Halting problem* for programs with oracle X.

Theorem

X' is a complete $\Sigma_1^{0,X}$ set, i.e. if $S \subseteq \mathbb{N}$ is another $\Sigma_1^{0,X}$ set then there is a recursive f such that $x \in S$ if and only if $f(x) \in X'$.

Corollary

1. $X <_{\mathrm{T}} X'$.
2. $X \leq_{\mathrm{T}} Y$ *implies* $X' \leq_{\mathrm{T}} Y'$.

Π^0_1 Classes

A Π^0_1 subset of $2^{\mathbb{N}}$ is a Π^0_1 **class**.

They have an effective enumeration

$$P_e = \{X \in 2^{\mathbb{N}} \mid \varphi_e^{(1),X}(0)\uparrow\}$$

Proposition

1. *$P \subseteq 2^{\mathbb{N}}$ is Π^0_1 if and only if P is the set of paths through some recursive subtree of $\{0,1\}^*$.*
2. *For $X \in 2^{\mathbb{N}}$, $X' = \{e \in \mathbb{N} \mid X \in P_e\}$.*

Basis Theorems for Π^0_1 Classes

Let $P \neq \emptyset$ be a Π^0_1 class.

Theorem (Kleene Basis Theorem)

There exists $B \in P$ such that $B \leq_{\mathrm{T}} 0'$.

This can be substantially strengthened:

Theorem (Jump Inversion)

Suppose P is special, i.e. contains no recursive elements. Then for every $A \geq_{\mathrm{T}} 0'$, there exists $B \in P$ such that

$$A \equiv_{\mathrm{T}} B' \equiv_{\mathrm{T}} B \oplus 0'$$

In a different direction:

Theorem (Jockusch-Soare Basis Theorem)

If $0 <_{\mathrm{T}} Z$, then there exists $B \in P$ such that $Z \not\leq_{\mathrm{T}} B$.

Relativized Analytical Hierarchy

Suppose $X \in 2^{\mathbb{N}}$.

Definition

$$\Sigma^{1,X}_0 = \Pi^{1,X}_0 = \Delta^{1,X}_0 = \{R \mid R \subseteq (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^\ell \textit{ an } X\textit{-arithmetical predicate}\}$$

$$\Sigma^{1,X}_{n+1} = \{S \mid S(-) \equiv \exists f\, R(f,-) \textit{ for } R \in \Pi^{1,X}_n\}$$

$$\Pi^{1,X}_{n+1} = \{S \mid S(-) \equiv \forall f\, R(f,-) \textit{ for } R \in \Sigma^{1,X}_n\}$$

$$\Delta^{1,X}_n = \Sigma^{1,X}_n \cap \Pi^{1,X}_n$$

where quantifiers range over $\mathbb{N}^{\mathbb{N}}$. If X is recursive, we drop mention of X, i.e. $\Sigma^{1,X}_n = \Sigma^1_n$ and $\Pi^{1,X}_n = \Pi^1_n$.

Hyperarithmetical Reducibility

Suppose $X, Y \in 2^{\mathbb{N}}$.

Theorem (Kleene, Kreisel)

The following are equivalent:

1. $X \leq_{\mathrm{T}} Y^{(\alpha)}$ *for some recursive ordinal* α.
2. $X \subseteq \mathbb{N}$ *is* $\Delta_1^{1,Y}$.
3. X *is an element of any* ω*-model of ZFC which contains* Y.

Definition

Suppose $X, Y \in 2^{\mathbb{N}}$. *The* **hyperarithmetical reducibility** *preorder* $\leq_{\mathrm{HYP}}$ *is defined by declaring* $X \leq_{\mathrm{HYP}} Y$ *if any of the equivalent conditions above hold.*

X *and* Y *are* **hyperarithmetically equivalent**, $X \equiv_{\mathrm{HYP}} Y$, *if and only if* $X \leq_{\mathrm{HYP}} Y$ *and* $Y \leq_{\mathrm{HYP}} X$. $\equiv_{\mathrm{HYP}}$ *is an equivalence relation.*

X is **hyperarithmetical** if $X \leq_{\mathrm{HYP}} 0$.

Hyperjumps

Is there is a hyperarithmetical analogue of the Turing jump operator? Perhaps a complete $\Sigma_1^{1,X}$ set?

It ends up the $\Pi_1^{1,X}$ sets are a better analogy for $\Sigma_1^{0,X}$ sets both structurally and in terms of the properties they satisfy.

Definition

The **hyperjump** *of X is a fixed complete $\Pi_1^{1,X}$ set $\mathcal{O}^X \subseteq \mathbb{N}$ (uniform in X), i.e. if $S \subseteq \mathbb{N}$ is another $\Pi_1^{1,X}$ set then there is a recursive f such that $x \in S$ if and only if $f(x) \in \mathcal{O}^X$.*

Proposition

1. $X <_{\mathrm{HYP}} \mathcal{O}^X$
2. $X \leq_{\mathrm{HYP}} Y$ *implies* $\mathcal{O}^X \leq_{\mathrm{T}} \mathcal{O}^Y$.

Σ^1_1 Classes

A Σ^1_1 subset of $2^{\mathbb{N}}$ is a Σ^1_1 **class**.

Proposition

Is $S \subseteq 2^{\mathbb{N}}$ is Σ^1_1, then there is a Π^0_1 subset P of $\mathbb{N}^{\mathbb{N}}$ such that

$$S(X) \equiv \exists f \, P(X \oplus f)$$

Letting $P_e = \{f \in \mathbb{N}^{\mathbb{N}} \mid \varphi_e^{(1),f}(0)\uparrow\}$, they have an effective enumeration

$$P_e^* = \{X \in 2^{\mathbb{N}} \mid \exists f \, (X \oplus f \in P_e)\}$$

Proposition

$\mathcal{O}^X \equiv_{\mathrm{T}} \{e \in \mathbb{N} \mid X \in P_e^*\}$.

Basis Theorems for Σ^1_1 Classes

Let K be a non-empty Σ^1_1 subset of $2^{\mathbb{N}}$.

Theorem (Gandy Basis Theorem)

There exists $B \in K$ such that $B <_{\mathrm{HYP}} \mathcal{O}$ and $B \leq_{\mathrm{T}} \mathcal{O}$.

Theorem (Hyperjump Inversion)

Suppose K is special, i.e. contains no hyperarithmetical elements. Then for every $A \geq_{\mathrm{T}} \mathcal{O}$, there exists $B \in K$ such that

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

Theorem (Kreisel Basis Theorem)

If Z is not hyperarithmetical, then there exists $B \in K$ such that $Z \not\leq_{\mathrm{HYP}} B$.

The Analogy between Recursion and Hyperarithmetical Theory

Recursion Theory	Hyperarithmetical Theory
Arithmetical Hierarchy	Analytical Hierarchy
Recursive $= \Delta^0_1$	Hyperarithmetical $= \Delta^1_1$
Turing reducibility	Hyperarithmetical reducibility
Π^0_1	Σ^1_1
Turing jump	Hyperjump
First-Order Logic	ω-logic

The Main Result

Theorem

Suppose $Z_0, Z_1, Z_2, \ldots$ is a sequence of non-hyperarithmetical reals which is recursive in A, where $\mathcal{O} \leq_{\mathrm{T}} A$. Then there exists B such that

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus Z_k \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

for every $k \in \mathbb{N}$.

The desired B will be a particular generic Turing functional.

Turing Functionals

Definition (Turing Functional)

A **Turing functional** *is a subset* $\Phi \subseteq \mathbb{N} \times \{0,1\} \times \{0,1\}^*$ *such that if* $(x, y_1, \sigma_1), (x, y_2, \sigma_2) \in \Phi$ *and* σ_1, σ_2 *are compatible, then* $y_1 = y_2$ *and* $\sigma_1 = \sigma_2$.
An element of Φ *is called a* **computation**.

Note that a Turing functional Φ is not necessarily recursive or even recursively enumerable.
We shall be interested in non-recursive Turing functionals and finite Turing functionals.

Computations along Reals

Let Φ be a Turing functional and $Z \in 2^{\mathbb{N}}$.

Definition (Computations along a Real)

$(x, y, \sigma) \in \Phi$ is a **computation along** *Z if $\sigma \subset Z$, written*

$$\Phi(Z)(x) = y$$

If for every $x \in \mathbb{N}$ there is $y \in \{0, 1\}$ such that $\Phi(Z)(x) = y$, then $\Phi(Z)$ defines an element of $2^{\mathbb{N}}$.

Lemma

Suppose $\Phi(Z)$ is total (i.e. $\Phi(Z) \in 2^{\mathbb{N}}$). Then $\Phi(Z) \leq_{\mathrm{T}} \Phi \oplus Z$.

Outline of the Proof

Suppose $Z_0, Z_1, Z_2, \ldots$ is a sequence of distinct non-hyperarithmetical reals which is recursive in A. Without loss of generality, $\mathcal{O} \neq Z_k$ for every k.

1. Take a (code for an) countable ω-model M of ZFC which omits $\mathcal{O}, Z_0, Z_1, Z_2, \ldots$ and has $\mathcal{O}^M \equiv_{\mathrm{T}} A$.
2. Force over M using Kumabe-Slaman forcing to produce a generic Turing functional Φ with the following properties:
   - $\Phi(Z_k) \equiv_{\mathrm{T}} \mathcal{O}^\Phi$ for each k
   - $\Phi(\mathcal{O}) \equiv_{\mathrm{T}} A$

   The construction of Φ will be recursive in A.
3. Conclusion: for each k

$$A \equiv_{\mathrm{T}} \Phi(\mathcal{O}) \leq_{\mathrm{T}} \Phi \oplus \mathcal{O} \leq_{\mathrm{T}} \mathcal{O}^\Phi \equiv_{\mathrm{T}} \Phi(Z_k) \leq_{\mathrm{T}} \Phi \oplus Z_k \leq_{\mathrm{T}} A$$

so there is Turing equivalence throughout.

$B = \Phi$ is the desired real.

A Basis Theorem for Special Σ^1_1 Classes

To get our countable ω-model M, we will need the following basis theorem, which combines the Kreisel Basis Theorem and Hyperjump Inversion in a special Σ^1_1 class:

Theorem

Suppose K is a special Σ^1_1 class (so non-empty and $K \cap \mathrm{HYP} = \emptyset$) and $Z_0, Z_1, Z_2, \ldots$ is a sequence of non-hyperarithmetical reals which is recursive in $A \geq_{\mathrm{T}} \mathcal{O}$. Then there exists $B \in K$ such that

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

and such that $Z_k \not\leq_{\mathrm{HYP}} B$ for every $k \in \mathbb{N}$.

The proof uses some methods due to Gandy and Kreisel.

Corollary

Suppose $Z_0, Z_1, Z_2, \ldots$ is a sequence of non-hyperarithmetical reals which is recursive in $A \geq_{\mathrm{T}} \mathcal{O}$. Then there exists a (code for a) countable ω-model M of ZFC such that $\mathcal{O}^M \equiv_{\mathrm{T}} A$ and $Z_k \notin M$ for each k.

Proof.

The set of codes $\langle \omega, E \rangle$ for countable ω-models of ZFC is a Σ^1_1 class. □

(We assume ZFC is ω-consistent to ensure that there *are* countable ω-models of ZFC.)

Proof Sketch:

Gandy-Harrington Forcing. Define sequence of non-empty Σ^1_1 subsets of K

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$$

with $B \in \bigcap_{n=0}^{\infty} K_n$. Roughly, define K_n to achieve the following effects:

- Control $\mathcal{O}^B$ by deciding whether $B \in P^*_m$.
- Encode A into $\mathcal{O}^B$.
- Arrange for $Z_k \not\leq_{\mathrm{HYP}} B$.

Some subtleties:

- Need to ensure $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$. Must build B and 'witnesses' that $B \in K_n$ for each n along the way.
- Some difficulties in arranging for $Z_k \not\leq_{\mathrm{HYP}} B$, as enough of A may not have been encoded to fulfill. Keep trying.

A (Very) Brief Review of Forcing

The essential ingredients of forcing are the following:

- a model $(M, \in^M)$ of ZFC,
- a poset $(P, \leq)$ (in M), and
- an M-generic filter G of $(P, \leq)$.

The result is a new model $M[G]$ which 'extends' M.

The **forcing relation** $p \Vdash \theta(a_1, \ldots, a_n)$ between elements $p \in P$ and *sentences in the forcing language* $\theta(a_1, \ldots, a_n)$ that allow M to think about what its forcing extensions look like.

Extra subtlety: $(M, \in^M)$ is not necessarily well-founded!

Kumabe-Slaman Forcing

The following definition takes place in a countable ω-model M of ZFC:

$$\mathbb{P} = \left\{ (\Phi, \mathbf{X}) \;\middle|\; \begin{array}{c} \Phi \text{ finite, use-monotone Turing functional,} \\ \mathbf{X} \text{ finite set of subsets of } \mathbb{N} \end{array} \right\}$$

Suppose $p = (\Phi_p, \mathbf{X}_p)$ and $q = (\Phi_q, \mathbf{X}_q)$. $p \leq q$ if and only if

- $\Phi_p \subseteq \Phi_q$,
- $\mathbf{X}_p \subseteq \mathbf{X}_q$,
- if $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and $(x_p, y_p, \sigma_p) \in \Phi_p$, then $|\sigma_p| < |\sigma_q|$, and
- for every x, y and $X \in \mathbf{X}_p$, if $\Phi_q(X)(x) = y$, then $\Phi_p(X)(x) = y$.

In other words:

$$p \leq q \iff \begin{array}{c} \Phi_q \supseteq \Phi_p,\ \mathbf{X}_q \supseteq \mathbf{X}_p, \\ \text{and all new computations are longer} \\ \text{and do not apply to elements of } \mathbf{X}_p \end{array}$$

Generic Turing Functionals

Suppose G is an M-generic filter (in the sense of Kumabe-Slaman Forcing). Then for every $X \in M$,

$$M \vDash (X \subseteq \mathbb{N}) \iff \text{there is } p \in G \text{ with } X \in \mathbf{X}_p$$

Thus, the essential parts of a generic filter G are the Φ_p.

Definition

Φ *is* M-**generic for** $\mathbb{P}$ *if and only if there exists a* M*-generic filter* G

$$(x, y, \sigma) \in \Phi \iff \textit{there exists } p \in G \textit{ such that } M \vDash ((x^M, y^M, \sigma^M) \in \Phi_p)$$

(here we are making use of the fact that M *is an* ω*-model)*

Φ may be identified with an element $(\dot{\Phi})_G$ in $M[G]$, where

$$M \vDash (\dot{\Phi} = \{(p, \dot{c}) \mid p \in \mathbb{P} \wedge c \in \Phi_p\})$$

and $\dot{c}$ is a canonical 'name' for $c \in M$.

An Extension Lemma

Lemma

Suppose M is an ω-model of ZFC, $D \in M$ is dense in $\mathbb{P} \in M$, and $X_1, \ldots, X_n \in 2^{\mathbb{N}}$. Then for any $p \in \mathbb{P}$, there is $q \geq p$ such that $q \in D$ and Φ_q does not add any new computations along any X_k.

To prove the lemma, we will need the following notion:
$\vec{\tau} \in (\{0,1\}^*)^n$ is **essential for** (p, D) if any extension of p in D adds a computation along a string compatible with a component of $\vec{\tau}$.

Define

$$T_n(p, D) = \{\vec{\tau} \in (\{0,1\}^*)^n \mid \vec{\tau} \text{ essential for } (p, D) \text{ and } |\tau_1| = \cdots = |\tau_n|\}$$

$T_n(p, D)$ is a finitely-branching tree in M.

Our proof will show that if the claim is false, then $T_n(p, D)$ is infinite. This will provide a contradiction of the density of D.

Proof.

Suppose otherwise, so every $q > p$ has $q \notin D$ or q adds a new computation along some X_k.

Claim 1: $(X_1 \upharpoonright m, \ldots, X_n \upharpoonright m)$ is essential for (p, D) for each m.

Proof. If $q > p$ and $q \in D$, by hypothesis there is q adds a computation (x, y, σ) along some X_k. Then σ is compatible with $X_k \upharpoonright m$. □

Claim 2: M thus has a path through $T_n(p, D)$ of the form $(Y_1 \upharpoonright m, \ldots, Y_n \upharpoonright m)$ for $Y_1, \ldots, Y_n \in M$.

Proof. $T_n(p, D)$ is infinite. M is a model of ZFC, so the Weak Konig's Lemma implies the existence of a path $(\vec{\tau}_n)_{n\in\mathbb{N}}$. Let $Y_k = \bigcup_{n\in\mathbb{N}} \vec{\tau}_n(k)$. □

Let $p_1 = (\Phi_p, \mathbf{X}_p \cup \{Y_1, \ldots, Y_n\})$ and suppose $q \geq p_1$ with $q \in D$. By construction, for each m, there is $(x_m, y_m, \sigma_m) \in \Phi_q \setminus \Phi_p$ such that σ_m is compatible with $Y_k \upharpoonright m$ for some k. Letting m be sufficiently large yields $(x, y, \sigma) \in \Phi_q \setminus \Phi_p$ such that $\sigma \subseteq Y_k$ for some k. Contradiction.

□

Proof of the Main Result: Outline Redux

Suppose $Z_0, Z_1, Z_2, \ldots$ is a sequence of distinct non-hyperarithmetical reals which is recursive in A. Without loss of generality, $\mathcal{O} \neq Z_k$ for every k.

1. Take a (code for an) ω-model M of ZFC which omits $\mathcal{O}, Z_0, Z_1, Z_2, \ldots$ and has $\mathcal{O}^M \equiv_{\mathrm{T}} A$.
2. Let $D_0, D_1, D_2, \ldots$ be an enumeration of the dense open subsets of $\mathbb{P}$ in M recursive in A.
3. Define a sequence

$$p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_n \leq \cdots$$

so that $\Phi = \bigcup_{n=0}^{\infty} \Phi_{p_n}$. Roughly, we do the following at Stage n:
Stage $n = 2^m$: Meet the open dense set D_m.
Stage $n = 2^m \cdot 3$: Arrange for $\Phi(\mathcal{O}) \equiv_{\mathrm{T}} A$.
Stage $n = 2^m \cdot 5^{k+1}$: Arrange for $\Phi(Z_k) \equiv_{\mathrm{T}} \mathcal{O}^{\Phi}$.
All other stages n: Do nothing.

4. Conclusion: for each k

$$A \equiv_{\mathrm{T}} \Phi(\mathcal{O}) \leq_{\mathrm{T}} \Phi \oplus \mathcal{O} \leq_{\mathrm{T}} \mathcal{O}^{\Phi} \equiv_{\mathrm{T}} \Phi(Z_k) \leq_{\mathrm{T}} \Phi \oplus Z_k \equiv_{\mathrm{T}} A$$

Proof of the Main Result: Stages $n = 2^m$ and $n = 2^m \cdot 3$

Let $p_0 = (\emptyset, \emptyset)$.
Suppose p_{n-1} has been constructed.

Stage $n = 2^m$: By the Extension Lemma, there is $p_n \in D_m$ extending p_{n-1} which does not add any new computations along $\mathcal{O}, Z_0, \ldots, Z_n$.

Stage $n = 2^m \cdot 3$: Extend p_{n-1} to p_n by adding $(m, A(m), \sigma)$, where $\sigma \subset \mathcal{O}$ is sufficiently long, i.e. longer than any existing strings in elements of $\Phi_{p_{n-1}}$.

Stage $n = 2^m \cdot 5^{k+1}$ involves similar ideas, but requires a case-analysis based on what is forced about $\mathcal{O}^\Phi$.

Proof of the Main Result: Stage $n = 2^m \cdot 5^{k+1}$

If there is y and $\sigma \subseteq Z_k$ such that $(m, y, \sigma) \in \Phi_{p_{n-1}}$, let $p_n = p_{n-1}$ and proceed to the next stage. Otherwise, proceed as follows:
Recall one definition of $\mathcal{O}^B$:

$$\mathcal{O}^B \equiv_{\mathrm{T}} \{m \in \mathbb{N} \mid m \text{ encodes a } B\text{-recursive well-ordering of } \omega\}$$

Define $D = D_1 \sqcup D_2 \sqcup D_3$ where

$$D_1 = \left\{ q \in \mathbb{P} \;\middle|\; q \Vdash \begin{pmatrix} (m \text{ encodes } \dot{\Phi}\text{-recursive linear order on } \omega) \wedge (m \in \mathcal{O}^{\dot{\Phi}}) \\ \wedge \exists \alpha (\alpha \in \mathrm{Ord}^M \wedge |m| = \alpha) \end{pmatrix} \right\}$$

$$D_2 = \{q \in \mathbb{P} \mid q \Vdash ((m \text{ encodes } \dot{\Phi}\text{-recursive linear order on } \omega) \wedge (m \notin \mathcal{O}^{\dot{\Phi}}))\}$$

$$D_3 = \{q \in \mathbb{P} \mid q \Vdash \neg(m \text{ encodes } \dot{\Phi}\text{-recursive linear order on } \omega)\}$$

D is dense.
By the Extension Lemma, take $q \in D$ extending p_{n-1} which does not add any new computations along $\mathcal{O}, Z_0, \ldots, Z_n$.

Proof of Main Result: Stage $n = 2^m \cdot 5^{k+1}$, continued

Extend q to p_n by adding (m, y, σ), where $\sigma \subset Z_k$ is sufficiently long and y depends on the following cases:

Case 1: $q \in D_1$. Then
$q \Vdash ((m \text{ encodes } \dot{\Phi}\text{-recursive linear order on } \omega) \wedge (m \in \mathcal{O}^{\dot{\Phi}}) \wedge \exists\alpha(\alpha \in \mathrm{Ord}^M \wedge |m| = \alpha))$. We break into two subcases:

Case 1a: If α is *actually* an ordinal, set $y = 1$.
Case 1b: If α is *not actually* an ordinal, set $y = 0$.

Case 2: $q \in D_2$. Then
$q \Vdash (m \text{ encodes } \dot{\Phi}\text{-recursive linear order on } \omega \wedge m \notin \mathcal{O}^{\dot{\Phi}})$.
Set $y = 0$.

Case 3: $q \in D_3$. Then $q \Vdash \neg(m \text{ encodes } \dot{\Phi}\text{-recursive linear order on } \omega)$.
Set $y = 0$.

Proof of Main Result: Final Comments

The construction of Φ is recursive in A: assuming p_{n-1} is given

1. Stage $n = 2^m$ is recursive in $\mathcal{O}^M \equiv_{\mathrm{T}} A$,
2. Stage $n = 2^m \cdot 3$ is recursive in $\mathcal{O} \leq_{\mathrm{T}} A$,
3. Stage $n = 2^m \cdot 5^{k+1}$ is recursive in $\mathcal{O}^M \oplus \bigoplus_{i=0}^{n} Z_i \leq_{\mathrm{T}} A$, and
4. all other stages are recursive.

This shows that $\Phi \leq_{\mathrm{T}} A$.

Finally, $\Phi(Z_k) \equiv_{\mathrm{T}} \mathcal{O}^{\Phi}$ as they only differ at finitely many places. Likewise, $\Phi(\mathcal{O}) \equiv_{\mathrm{T}} A$.

This justifies the string of Turing reductions and equivalences

$$A \equiv_{\mathrm{T}} \Phi(\mathcal{O}) \leq_{\mathrm{T}} \Phi \oplus \mathcal{O} \leq_{\mathrm{T}} \mathcal{O}^{\Phi} \equiv_{\mathrm{T}} \Phi(Z_k) \leq_{\mathrm{T}} \Phi \oplus Z_k \equiv_{\mathrm{T}} A$$

Open Questions

- What Σ^1_1 classes K have the following property (and the stronger version with a sequence replacing Z)?

Property

Suppose $0 <_{\mathrm{HYP}} Z \leq_{\mathrm{T}} A$ *and* $\mathcal{O} \leq_{\mathrm{T}} A$. *Then there exists* $B \in K$ *such that*

$$A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

- The e-**th pseudo-hyperjump of** X is $\mathsf{HJ}_e(X) = X \oplus W_e^X$ where W_e^X is the e-th $\Pi_1^{1,X}$ set. Does the following result hold?

Conjecture

Suppose $0 <_{\mathrm{HYP}} Z \leq_{\mathrm{T}} A$ *and* $\mathcal{O} \leq_{\mathrm{T}} A$. *For every* e, *there exists* B *such that*

$$A \equiv_{\mathrm{T}} \mathsf{HJ}_e(B) \equiv_{\mathrm{T}} B \oplus Z \equiv_{\mathrm{T}} B \oplus \mathcal{O}$$

Thank you!

A Brief Review of Forcing – Basic Definitions

Suppose $(M, \in^M) \vDash \mathrm{ZFC}$ (not necessarily well-founded!) and $M \vDash$ "$(P, \leq)$ is a poset".

Definition

Suppose $D \in M$ and $M \vDash (D \subseteq P)$, and $G \subseteq \{p \in M \mid M \vDash (p \in P)\}$.

- *D is* **dense** *if $M \vDash (\forall p \in P)(\exists q \in D)(p \leq q)$.*
- *D is* **open** *if $M \vDash (\forall p, q \in P)((p \in D \wedge p \leq q) \to q \in D)$.*
- *G is a* **filter** *if it is upwards-closed and downwards-directed (as evaluated in V).*
- *G is an M-***generic filter** *if it is a filter and for every open dense set $D \subseteq^M P$ there is $p \in G$ such that $M \vDash (p \in D)$.*

Suppose G is M-generic. Associate to each $a \in M$ a symbol a_G. Define

$$M[G] = \{a_G \mid a \in M\}$$

$$a_G \in_G b_G \iff \text{there is } p \in G \text{ such that } M \vDash ((p, b) \in a)$$

Some Differences from Forcing with Transitive Standard Models

If M is transitive and standard, then

$$M[G] = \text{smallest model of ZFC containing } M \cup \{G\}$$

In general, we only get an *embedding* of $M \cup \{G\}$ into $M[G]$.
For $a \in M$, define by transfinite recursion (in M) $\dot{a}$ to be the unique element in M for which

$$M \vDash (\dot{a} = P \times \{\dot{b} \mid b \in a\})$$

$\dot{a}$ gives a canonical 'name' for a. $a \mapsto (\dot{a})_G$ embeds M into $M[G]$.
Likewise, define $\dot{G}$ to be the unique element in M for which

$$M \vDash (\dot{G} = \{(p, \dot{p}) \mid p \in P\})$$

$(\dot{G})_G$ embeds G into $M[G]$, which is to say that

$$p \in G \iff (\dot{p})_G \in_G (\dot{G})_G$$

A Brief Review of Forcing – Forcing Relation

Definition (Forcing Language)

The forcing language consists of $\mathcal{L}_{\in}$ along with constant symbols a for each $a \in M$.
$M[G] \vDash \theta(a_1, \dots, a_n)$ if and only if $\theta(a_1, \dots, a_n)$ is true in $M[G]$, where quantifiers range over $M[G]$ and $a_1, \dots, a_n$ are interpreted as $(a_1)_G, \dots, (a_n)_G$.

$\dot{G}$ allows us to talk about G without necessarily knowing what G is.

Definition (Forcing Relation)

Let $p \in P$ and θ be a sentence of the forcing language.

$$p \Vdash \theta \iff M[G] \vDash \theta \text{ for all } M\text{-generic filters } G \ni p$$

A Brief Review of Forcing – Basic Results

Suppose $\theta(x_1, \ldots, x_n)$ is a formula of $\mathcal{L}_\in$, M a countable model of ZFC, and G is an M-generic filter of $\mathbb{P}$.

Theorem (Definability of Forcing)

There is $\theta^*(p, x_1, \ldots, x_n)$ such that, for every $p \in P$ and $a_1, \ldots, a_m \in M$,

$$p \Vdash \theta(a_1, \ldots, a_n) \iff M \vDash \theta^*(p, a_1, \ldots, a_n)$$

Theorem (Forcing Equals Truth)

For every $a_1, \ldots, a_n \in M$, $M[G] \vDash \theta(a_1, \ldots, a_n)$ if and only if there is $p \in G$ such that $p \Vdash \theta(a_1, \ldots, a_n)$.

Theorem

$(M[G], \in_G)$ is a model of ZFC with the same ordinals as M (under the embedding $a \mapsto (\dot{a})_G$).

Tools needed for Proof of Σ^1_1 Basis Theorem

Proposition

Suppose K is special Σ^1_1 class and $T = \{\rho \in \mathbb{N}^* \mid (\exists X \in K)(\rho \subset X)\}$. For every $\sigma \in T$ there are infinitely-many pairwise-incompatible $\sigma' \supset \sigma$ in T. Moreover, the partial functions

$$\sigma(n, e) \simeq \text{least } \sigma \in T_e \text{ with } |\sigma| = n$$
$$\rho_n(\sigma, e) \simeq n\text{-th extension } \sigma' \supset \sigma \text{ such that } \sigma' \in T_e$$

are recursive in $\mathcal{O}$, where $T_0, T_1, \ldots$ is an effective enumeration of the recursive subtrees of $\mathbb{N}^*$.

Proposition

Suppose K is a special Σ^1_1 class, $e \in \mathbb{N}$, $Z \notin \mathrm{HYP}$, and $b \in \mathcal{O}$. There is a special Σ^1_1 class $\tilde{K} \subseteq K$ such that $Z \neq \varphi_e^{(1), H_b^Y}$ for every $Y \in \tilde{K}$.

Proof.

We consider the following cases:

Case 1: Suppose $(\exists Y \in K)\,(\varphi_e^{H_b^Y}$ not total). Let

$$\tilde{K} = K \cap \{\, Y \in \mathbb{N}^{\mathbb{N}} \mid \exists n\,(\varphi_e^{(1),H_b^Y}(n)\uparrow)\}$$

Case 2: Suppose Case 1 fails and $(\exists Y_1, Y_2 \in K)\,(\exists m \in \mathbb{N})\,(\varphi_e^{(1),H_b^{Y_1}}(m) \neq \varphi_e^{(1),H_b^{Y_2}}(m))$. Let

$$\tilde{K} = K \cap \{\, Y \in \mathbb{N}^{\mathbb{N}} \mid \varphi_e^{(1),H_b^Y}(m) \neq Z(m)\}$$

Case 3: Otherwise, the common function $h = \varphi_e^{H_b^Y}$ is a Σ_1^1 singleton and hence Δ_1^1. Let

$$\tilde{K} = K$$

□

Proof shows that an index of $\tilde{K}$ is recursive in $\mathcal{O} \oplus Z$ as a function of an index of K.

Futher Details about Σ^1_1 Basis Theorem Proof

We define sequences of special Σ^1_1 subsets

$$K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n \supseteq \cdots$$

strings

$$\begin{array}{ccccccccc} \sigma_0 & \subseteq & \sigma_1 & \subseteq & \cdots & \subseteq & \sigma_n & \subseteq & \cdots \\ \tau_{0,0} & \subseteq & \tau_{1,0} & \subseteq & \cdots & \subseteq & \tau_{n,0} & \subseteq & \cdots \\ \tau_{0,1} & \subseteq & \tau_{1,1} & \subseteq & \cdots & \subseteq & \tau_{n,0} & \subseteq & \cdots \\ \vdots & & \vdots & & \ddots & & \vdots & & \ddots \end{array}$$

finite subsets of $\mathbb{N}$

$$\emptyset = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

and $j : \mathbb{N} \to \mathbb{N}$ such that

$$K_n = P^*_{j(n)} = \bigcap_{k \in I_n} \{X \mid X \supset \sigma_n \wedge (\exists f \supset \tau_{n,k})(X \oplus f \in P_k)\}$$

Assume that j encodes all of the information from previous steps (i.e. a course-of-values computation).

Futher Details about Σ^1_1 Basis Theorem Proof, Continued

WLOG, $\omega_1^X = \omega_1^{\mathrm{CK}}$ for all $X \in K$.
Let $\tilde{e}$ be such that $K = P^*_{\tilde{e}}$.

Stage $n = 0$: Define

$$K_0 = K \qquad \sigma_0 = \langle\rangle \qquad \tau_{0,k} = \langle\rangle \qquad j(0) = \tilde{e} \qquad I_0 = \{\tilde{e}\}$$

Stage $n = 3e + 1$: Take

$$K_n = K_{n-1} \cap P^*_e$$

if that intersection is non-empty, and $K_n = K_{n-1}$ otherwise, with book-keeping.

Stage $n = 3e + 2$: Encode $A(e)$ into B by extending σ_{n-1} to one of infinitely-many pairwise-incompatible extensions which extend to elements of K_{n-1}, with book-keeping.

Futher Details about Σ^1_1 Basis Theorem Proof, Continued

Stage $n = 3^{b+1} \cdot 5^e \cdot 7^k \cdot 11^s$: Take

$$K_n = \tilde{K_n}$$

if enough of A has been encoded to carry out that computation. Book-keeping analogous to Stage $n = 3e + 1$. Virtually identical way to Stage $3e + 1$, with replacements

$$P_e^* \to \widetilde{K_{n-1}} \qquad e \to \text{index of } \widetilde{K_{n-1}}$$

Otherwise, do nothing.

All Other Stages n: Do nothing.

This completes the construction.

Futher Details about Σ^1_1 Basis Theorem Proof, Continued

Let

$$B = \bigcup_{n \in \omega} \sigma_n \quad \text{and} \quad g_k = \bigcup_{n \in \omega} \tau_{n,k}$$

Observations:

- Stage $n = 3e + 1$ determines whether $e \in \mathcal{O}^B$.
- Stage $n = 3e + 2$ encodes $A(e)$ into B, recoverable from $\mathcal{O}^B$ (or j or $B \oplus \mathcal{O}$).
- Stage $n = 3^{b+1} \cdot 5^e \cdot 7^k \cdot 11^s$ ensures $Z_k \neq \varphi_e^{(1),H_b^B}$ for sufficiently large s.
- Book-keeping ensures $B \oplus g_n \in K_n$ for each n.
- Entire construction is recursive in any one of j, A, $\mathcal{O}^B$, and $B \oplus \mathcal{O}$, so

$$j \equiv_{\mathrm{T}} A \equiv_{\mathrm{T}} \mathcal{O}^B \equiv_{\mathrm{T}} \mathcal{O} \oplus B$$