Optimal strong Mal'cev conditions for congruence meet-semidistributivity in locally finite varieties

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Abstract. We show that the class of locally finite varieties omitting types **1** and **2** is definable by an idempotent, linear, strong Mal'cev condition, either in a language with two 3-ary function symbols or with one 4-ary function symbol. By an earlier result of the first author, these are the optimal strong Mal'cev conditions which define this class.

1. Introduction

The various conditions which are equivalent to congruence meet-semidistributivity in locally finite varieties of algebras have been explored in several previous papers and books [5], [7], [11], [15], [19], [13]. The reason for this activity is that congruence meet-semidistributive varieties are a very general, and yet well behaved class of varieties. For instance, this is equivalent condition to congruence neutrality [11] and [15]; in locally finite varieties it is characterized by omitting tame congruence theory types **1** and **2** [7]; it implies the truth of Park's conjecture [19], see also [12]; it is the weakest condition which allows the fixed-template constraint satisfaction problems to be accurately solved by using only the local consistency checking [14], [2], see also [1].

We are concerned in this paper with an optimal strong Mal'cev characterization for congruence meet-semidistributivity. Siggers proved in [18] that the weaker property, having a Taylor term (characterized in locally finite varieties by omitting type **1**) is a strong Mal'cev property. The Siggers' result was a big surprise at the time of publication and spurred an investigation of what other properties which were hitherto known to have a Mal'cev characterization would have a strong Mal'cev characterization in locally finite varieties. Congruence meet-semidistributivity was proved to be a strong Mal'cev property in the case of locally finite varieties in [13], while many other natural properties were proved not to have a strong Mal'cev characterization in the same paper. The paper [10] settled the question of optimal (syntactically simplest) strong Mal'cev characterization of having a Taylor term (= omitting type **1**) in locally finite varieties. The paper [8] managed to prove that there are no strong Mal'cev conditions which characterize congruence meet-semidistributivity in

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the language with at most one ternary operation and all other operations with arities less than 3. In the case of conditions with two ternary operations, the paper [8] isolates one candidate condition and proves that there are no other strong Mal'cev characterizations with two ternary operations of congruence meet-semidistributivity in locally finite varieties. In the present paper we prove that that condition isolated by [8] indeed does characterize congruence meet-semidistributivity in locally finite varieties. We also find a strong Mal'cev characterization in the language of one operation of arity 4. The two strong Mal'cev conditions we find are, thus, syntactically optimal.

Our paper is organized as follows: In Section 2, we give a list of definitions which will be used, beyond the classical universal algebra definitions and results which we assume the reader to be familiar with. Those readers who are not familiar with them are advised to check out the textbooks [4], [17] and/or [3]. In the same section we review and reprove a few results of commutator theory, as developed in [6], which we will use to prove the easier direction (that the Mal'cev condition implies congruence meet-semidistributivity) in our results. In fact, the same easy directions would be easier proved by invoking more commutator theory results, but we strove to make our paper more self-contained, and thus use only the basic definitions of commutator theory and their first consequences. We conclude Section 2 with definitions and a review of the constraint satisfaction problem terminology and results, particularly the main result of [1], which will be the main tool used to prove the harder direction of our two Mal'cev characterizations.

In Section 3 we prove the two main theorems of this paper. One of these gives a strong Mal'cev characterization of locally finite varieties in the language with two ternary operations, while the other one does the same in the language of one operation of arity 4. We conclude the paper with a list of topics for further research in Section 4.

2. Definitions and background

Definition 2.1. Let $\mathcal{V}$ be a variety. $\mathcal{V}$ is *congruence neutral* if for any algebra $\mathbf{A} \in \mathcal{V}$ and congruences $\alpha, \beta \in \operatorname{Con} \mathbf{A}$, $[\alpha, \beta] = \alpha \cap \beta$.

We recall the following characterization of congruence meet-semidistributivity, proved in [11], Corollary 4.7, see also [9], Theorem 8.1. To make our paper more self-contained, we provide the proof from [11] of the (easier) direction ($\Leftarrow$) which we intend to use. It is using the first consequences of the definition of centralizer $C(\alpha, \beta; \delta)$ from commutator theory, namely that $C(\beta, \alpha; \delta)$ and $C(\gamma, \alpha; \delta)$ imply $C(\beta \vee \gamma, \alpha; \delta)$.

Theorem 2.2. *Let $\mathcal{V}$ be a variety. $\mathcal{V}$ is congruence meet-semidistributive iff it is congruence neutral.*

Proof of the direction ($\Leftarrow$). We assume that $\mathcal{V}$ is congruence neutral and assume that in an algebra $\mathbf{A} \in \mathcal{V}$ there exist congruences $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$

such that $\delta := \alpha \cap \beta = \alpha \cap \gamma$. By congruence neutrality, it follows that $[\beta, \alpha] = [\gamma, \alpha]$. Therefore, $C(\beta, \alpha; \delta)$ and $C(\gamma, \alpha; \delta)$ hold in $\mathbf{A}$. However, then we get $C(\beta \vee \gamma, \alpha; \delta)$, so $\delta \geq [\beta \vee \gamma, \alpha] = (\beta \vee \gamma) \cap \alpha \geq \beta \cap \alpha = \delta$. So we get that $\delta = (\beta \vee \gamma) \cap \alpha$, i. e. that $\mathcal{V}$ is congruence meet-semidistributive. □

Corollary 2.3. *If $\mathcal{V}$ is a variety for all $\mathbf{A} \in \mathcal{V}$ and $\alpha \in \operatorname{Con} \mathbf{A}$, $[\alpha, \alpha] = \alpha$ is satisfied, then $\mathcal{V}$ is congruence meet-semidistributive.*

Proof. According to Theorem 2.2, we only need to show congruence neutrality. Let $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ for some $\mathbf{A} \in \mathcal{V}$. Since $C(\alpha, \beta; \alpha \cap \beta)$ always holds, $[\alpha, \beta] \subseteq \alpha \cap \beta$. Moreover, since any $(\alpha \cap \beta, \alpha \cap \beta)$-matrix is an (α, β)-matrix, then $C(\alpha \cap \beta, \alpha \cap \beta; [\alpha, \beta])$ holds, so $\alpha \cap \beta = [\alpha \cap \beta, \alpha \cap \beta] \subseteq [\alpha, \beta]$, so $\alpha \cap \beta = [\alpha, \beta]$ □

We turn to definitions of the constraint satisfaction problem and a $(2, 3)$-minimal instance of it. We follow [1] as we will use the main result of that paper a lot. The definition we give below is Barto's version of the variable-value instance of the constraint satisfaction problem (Definition 3.1 of [1]):

Definition 2.4. An instance of the constraint satisfaction problem (CSP) is a triple $(V; A; \mathcal{C})$ with

- V a nonempty, finite set of variables,
- A a nonempty, finite domain,
- $\mathcal{C}$ a finite nonempty set of constraints, where each constraint is a subset C of A^W. Here W is a subset of V called the scope of C and the cardinality $|W|$ of W is referred to as the arity of C.

An instance is *trivial* if it contains the empty constraint. The instance $(V; A; \mathcal{C})$ has a solution, that is, a function $f : V \to A$ such that, for each constraint $C \in \mathcal{C}$, such that the scope of C is $W \subseteq V$, the restriction $f \restriction_W$ is in C. Next we define a 2-consistent and a $(2, 3)$-minimal instance:

Definition 2.5. An instance of CSP $(V; A; \mathcal{C})$ is 2-consistent, if for every $U \subseteq V$ such that $|U| \leq 2$ and every pair of constraints $C, D \in \mathcal{C}$ such that U is contained in the scopes of both C and D, the restrictions $C \restriction_U = D \restriction_U$. An instance of CSP $(V; A; \mathcal{C})$ is $(2, 3)$-minimal if it is 2-consistent and every at most 3-element set of variables is contained in the scope of some constraint in $\mathcal{C}$.

Definition 2.6. Let $\mathbb{A} = \langle A; \Gamma \rangle$ be a relational structure. An instance of the constraint satisfaction problem $CSP(\mathbb{A})$ is any instance of the CSP $(V; D; \mathcal{C})$ such that for each constraint $C \in \mathcal{C}$, the relation C is equal to a permutation of coordinates $C' \in \Gamma$.

We silently assume that all Γ contain the equality relation (to allow for identification of variables).

Let $\mathbf{A}$ be an algebra. When $\Gamma \subseteq \mathsf{SP}_{fin}(\mathbf{A})$, then we say that $CSP(\langle A; \Gamma \rangle)$ is compatible with $\mathbf{A}$. The following result (Corollary 6.5 of [1]) is the one that has as a consequence the main theorem of [1]:

Theorem 2.7. *Let* $\mathbf{A}$ *be an idempotent finite algebra which generates a congruence meet-semidistributive variety. Then for every* $CSP(\langle A;\Gamma\rangle)$ *which is compatible with* $\mathbf{A}$*, every* $(2,3)$*-minimal instance of* $CSP(\langle A;\Gamma\rangle)$ *which is not trivial has a solution.*

By a *strong Mal'cev condition* we mean a finite set of identities in some language. Informally, a strong Mal'cev condition is *realized* in an algebra $\mathbf{A}$ (or variety $\mathcal{V}$) if there is a way to interpret the function symbols appearing in the condition as term operations of $\mathbf{A}$ (or $\mathcal{V}$) so that the identities in the Mal'cev condition become true equations in $\mathbf{A}$ (or $\mathcal{V}$). A Mal'cev condition is a sequence $\{C_n : n \in \omega\}$ such that any variety which realizes C_n must also realize C_{n+1} for all $n \in \omega$. We say that the variety $\mathcal{V}$ realizes the Mal'cev condition $\{C_n : n \in \omega\}$ if there exists an $n \in \omega$ such that $\mathcal{V}$ realizes C_n.

We give two Mal'cev conditions which will be of further use to us:

- We say that a variety has Jónsson terms if there exists $n \geq 2$ such that $\mathcal{V}$ realizes the strong condition $CD(n)$. $CD(n)$ is in the language $\{d_0, d_1, \ldots, d_n\}$ consisting of ternary symbols, and consists of identities

$$
\begin{aligned}
d_0(x,y,z) &\approx x, && \\
d_i(x,y,x) &\approx x && \text{for all } 0 \leq i \leq n,\\
d_i(x,y,y) &\approx d_{i+1}(x,y,y) && \text{for all even } i \text{ such that } 0 \leq i \leq n,\\
d_i(x,x,y) &\approx d_{i+1}(x,x,y) && \text{for all odd } i \text{ such that } 0 \leq i \leq n,\\
d_n(x,y,z) &\approx z. &&
\end{aligned}
$$

- We say that a variety has a weak near-unanimity term if there exists $n \geq 3$ such that $\mathcal{V}$ realizes the strong condition $WNU(n)$. $WNU(n)$ is in the language $\{w\}$, the arity is $ar(w) = n$, and consists of identities

$$
\begin{gathered}
w(x,x,\ldots,x) \approx x,\\
w(y,x,x,\ldots,x) \approx w(x,y,x,\ldots,x) \approx \ldots \approx w(x,x,\ldots,x,y).
\end{gathered}
$$

It was proved in the 1960s by Jónsson that a variety is congruence distributive iff it realizes $CD(n)$ for some $n \geq 2$. This kind of equivalence is usually called a Mal'cev characterization of some property. In [16] it was proved that any locally finite variety $\mathcal{V}$ is congruence meet-semidistributive iff $\mathcal{V}$ realizes the strong Mal'cev conditions $WNU(n)$ for all but finitely many $n \in \omega \setminus \{0,1,2\}$. This was a Mal'cev characterization of congruence meet-semidistributivity *within the class of locally finite varieties.* The equivalence was not proved and probably does not hold in all varieties. We raise a few questions about this distinction in the last section of this paper.

3. Congruence meet-semidistributivity characterizations with two ternary terms and with one 4-ary term

Theorem 3.1. *Let $\mathcal{V}$ be a variety. If $\mathcal{V}$ is locally finite congruence meet-semidistributive, $\mathcal{V}$ realizes the strong Mal'cev condition* (1) *given by:*

$$\begin{aligned} &p(x,x,x) \approx x \approx q(x,x,x), \\ &p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,y,x) \text{ and} \\ &q(x,x,y) \approx q(x,y,y). \end{aligned} \tag{1}$$

On the other hand, if $\mathcal{V}$ realizes the strong Mal'cev condition (1)*, then $\mathcal{V}$ is congruence meet-semidistributive.*

The second statement was proved in [8] for locally finite varieties. After consulting the literature, one realizes that this direction holds in all varieties. We will prove Theorem 3.1 simultaneously with Theorem 3.2 stated below.

Theorem 3.2. *Let $\mathcal{V}$ be a variety. If $\mathcal{V}$ is locally finite and congruence meet-semidistributive, then $\mathcal{V}$ realizes the strong Mal'cev condition* (2) *given by*

$$\begin{aligned} &\mathcal{V} \models t(x,x,x,x) \approx x \\ &\mathcal{V} \models t(y,x,x,x) \approx t(x,y,x,x) \approx t(x,x,y,x) \approx t(x,x,x,y) \approx t(y,y,x,x) \end{aligned} \tag{2}$$

On the other hand, $\mathcal{V}$ realizes the strong Mal'cev condition (2)*, then $\mathcal{V}$ is congruence meet-semidistributive.*

Proofs of Theorems 3.1 and 3.2. First we prove the second statements of both theorems. Let us assume that terms $p(x,y,z)$ and $q(x,y,z)$ satisfy identities (1) in $\mathcal{V}$. According to Corollary 2.3, one must only show that for all algebras $\mathbf{A} \in \mathcal{V}$ and congruences $\alpha \in \operatorname{Con} \mathbf{A}$, $[\alpha,\alpha] = \alpha$. Let $\langle a,b\rangle \in \alpha$ and let $\delta = [\alpha,\alpha]$. Then from the term condition $C(\alpha,\alpha;\delta)$ and $q^{\mathbf{A}}(a,a,b) = q^{\mathbf{A}}(a,b,b)$ we conclude $a = q^{\mathbf{A}}(a,a,a)\ \delta\ q^{\mathbf{A}}(a,b,a)$. Now we get that $p^{\mathbf{A}}(a,a,a) = a\ \delta\ q^{\mathbf{A}}(a,b,a) = p^{\mathbf{A}}(b,a,a)$. Applying the term condition to that, we obtain $p^{\mathbf{A}}(a,b,a)\ \delta\ p^{\mathbf{A}}(b,b,a)$. Finally, from $p^{\mathbf{A}}(a,a,a) = a\ \delta\ q^{\mathbf{A}}(a,b,a) = p^{\mathbf{A}}(a,a,b)$ and the term condition we get $p^{\mathbf{A}}(b,b,a)\ \delta\ p^{\mathbf{A}}(b,b,b) = b$. By the transitivity of δ we have that $\langle a,b\rangle \in \delta$, i. e. that $\alpha \subseteq \delta = [\alpha,\alpha]$, so $[\alpha,\alpha] = \alpha$, as desired.

Similarly, assume that $t(x,y,z,u)$ is a term such that $\mathcal{V}$ satisfies the identities (2). Let $\mathbf{A}$, α, δ, a and b be as in the previous paragraph. From the term condition $C(\alpha,\alpha;\delta)$ and $t^{\mathbf{A}}(a,a,b,b) = t^{\mathbf{A}}(b,a,b,b)$ we conclude $a = t^{\mathbf{A}}(a,a,a,a)\ \delta\ t^{\mathbf{A}}(b,a,a,a)$. Then from (2) we obtain that $a = t^{\mathbf{A}}(a,a,a,a)\ \delta\ t^{\mathbf{A}}(a,b,a,a)$ and the term condition implies that $t^{\mathbf{A}}(b,a,a,a)\ \delta\ t^{\mathbf{A}}(b,b,a,a)$. Continuing like this and using $a\ \delta\ t^{\mathbf{A}}(a,a,b,a)$ and $a\ \delta\ t^{\mathbf{A}}(a,a,a,b)$ we get $t^{\mathbf{A}}(b,b,a,a)\ \delta\ t^{\mathbf{A}}(b,b,b,a)$ and $t^{\mathbf{A}}(b,b,b,a)\ \delta\ t^{\mathbf{A}}(b,b,b,b) = b$. Thus by transitivity we get that $\langle a,b\rangle \in \delta$, and therefore $\alpha \subseteq \delta = [\alpha,\alpha] \subseteq \alpha$, i. e. $\alpha = [\alpha,\alpha]$.

Now we prove the first statement of Theorem 3.1. Let $\mathcal{W}$ be the idempotent reduct of $\mathcal{V}$, which is the variety whose clone is the clone of idempotent term operations of $\mathcal{V}$ and whose fundamental operations are the distinct elements of this clone. Since congruence meet-semidistributivity can be characterized by an idempotent Mal'cev condition, $\mathcal{W}$ is a locally finite, idempotent, congruence meet-semidistributive variety. From idempotence follows that there exist term operations p and q in the language of $\mathcal{V}$ which satisfy identities (1) in $\mathcal{V}$ iff there exist such term operations which satisfy (1) in $\mathcal{W}$.

Let $\mathbf{F}$ be the two-generated free algebra in $\mathcal{W}$, freely generated by x and y. Let $|F| = n$. We define six subalgebras of powers of (i.e. compatible relations with) $\mathcal{F}$ (we write elements of F^k as vector columns):

$$E = \mathrm{Sg}^{\mathbf{F}^2}\left(\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}\right)$$

$$\leq = \mathrm{Sg}^{\mathbf{F}^2}\left(\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}\right)$$

$$R = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix}\right)$$

$$S = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \end{bmatrix}\right)$$

$$K = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \\ x \end{bmatrix}\right)$$

$$L = \mathrm{Sg}^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \\ y \end{bmatrix}\right)$$

Note that the projections of R to any two coordinates equal E, as does the projection of S to the first two coordinates, while the projections of S to first and third coordinate, also to the last two coordinates, are equal to $\leq$. The projection of K to the first and second coordinate is $\leq$, while the other projections of K to two coordinates are equal to E. The projections of K to any pair of coordinates are equal to $\leq$. Finally, the projection of any of these relations to any single coordinate is all of F, i. e. they are subdirect.

We define an instance of the constraint satisfaction problem on the template $\langle F; E, \leq, R, S, K, L\rangle$. The instance is described by $\langle V, F, \mathcal{C}\rangle$, where $V = x_0, \ldots, x_{n(2n+2)}$. We split the set of variables into disjoint sets $H = \{x_i \in V : (2n+2)|i\}$ (and call these variables "hot") and $W = V \setminus H$ (the variables in W are called "weak"). The constraint relations are all symmetrized (e.g. the constraint relation at tuple $\langle i, j\rangle$ is the relation $Q(x, y)$ iff the constraint at tuple $\langle j, i\rangle$ is the relation $Q(y, x)$). In other words, we need only describe the constraint relations where the tuples of variables are with strictly increasing indices. Moreover, there is at most one constraint per tuple of variables, actually they are defined on each pair and each triple of variables. We start describing the binary constraint relations in $\mathcal{C}$:

- For any pair $\langle x_i, x_j \rangle$, where $i < j$, we impose the relation $\rho_{i,j}$ so that $\rho_{i,j} = E$ if $x_j \in W$ and $\rho_{i,j} =\leq$ if $x_j \in H$.

Now we describe the ternary constraint relations in $\mathcal{C}$:

- For any triple $\langle x_i, x_j, x_k \rangle$, where $i < j < k$ and both $x_j, x_k \in W$, we impose the constraint relation $\rho_{i,j,k} = R$.
- For any triple $\langle x_i, x_j, x_k \rangle$, where $i < j < k$, $x_j \in W$ and $x_k \in H$, we impose the constraint relation $\rho_{i,j,k} = S$.
- For any triple $\langle x_i, x_j, x_k \rangle$, where $i < j < k$, $x_j \in H$ and $x_k \in W$, we impose the constraint relation $\rho_{i,j,k} = K$.
- For any triple $\langle x_i, x_j, x_k \rangle$, where $i < j < k$, and both $x_j, x_k \in H$, we impose the constraint relation $\rho_{i,j,k} = L$.

From the definitions of these constraints it should be clear that the restrictions $\pi_{0,1}(\rho_{i,j,k}) = \rho_{i,j}$, $\pi_{0,2}(\rho_{i,j,k}) = \rho_{i,k}$ and $\pi_{1,2}(\rho_{i,j,k}) = \rho_{j,k}$. The 2-consistency follows from our analysis of the projections of ternary relations after we defined them. Since every triple of coordinates is the scope of some constraint, then it is $(2, 3)$-minimal, according to Definition 2.5. Since it is also compatible with $\mathbf{F}$ (therefore, with the idempotent term operations of $\mathcal{V}$), it must have a solution, according to Theorem 2.7.

Now we apply this solution $f : V \to F$. We get that there are $n + 1$ hot variables, so since $|F| = n$, then there must exist i, j such that $0 \leq i < j \leq n$ and that $f(x_{i(2n+2)}) = f(x_{j(2n+2)})$. Moreover, there exist at least $2n+1$ weak variables x_k such that $i(2n + 2) < k < j(2n + 2)$. This means that there are three weak variables $x_k, x_l, x_m \in W$ such that $i(2n + 2) < k < l < m < j(2n + 2)$ and that $f(x_k) = f(x_l) = f(x_m)$. From $[f(x_k), f(x_l), f(x_m)]^T \in R$ we get that there exists a term $p(x, y, z)$ such that

$$p^{\mathbf{F}^3}\left(\begin{bmatrix} y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x_k) \\ f(x_l) \\ f(x_m) \end{bmatrix},$$

and from $f(x_k) = f(x_l) = f(x_m)$ follows $p^{\mathbf{F}}(y, x, x) = p^{\mathbf{F}}(x, y, x) = p^{\mathbf{F}}(x, x, y)$. Since $\mathbf{F}$ is the free algebra, then the corresponding identity holds in $\mathcal{W}$, and this implies that p is an idempotent term of $\mathcal{V}$ such that $\mathcal{V} \models p(y, x, x) \approx p(x, y, x) \approx p(x, x, y)$.

Finally, from $[f(x_{i(2n+2)}), f(x_k), f(x_{j(2n+2)})]^T \in S$ we get that there exists a term $q_1(x, y, z)$ such that

$$q_1^{\mathbf{F}^3}\left(\begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x_{i(2n+2)}) \\ f(x_k) \\ f(x_{j(2n+2)}) \end{bmatrix},$$

Similarly as above, from $f(x_{i(2n+2)}) = f(x_{j(2n+2)})$ follows $\mathcal{V} \models q_1(x, y, x) \approx q_1(x, y, y)$ and also from $q_1(x, x, y) = f(x_k) = p(x, x, y)$ follows that $\mathcal{V} \models q_1(x, x, y) \approx p(x, x, y) \approx p(x, y, x) \approx p(y, x, x)$. Finally, by taking $q(x, y, z) = q_1(x, z, y)$ we are done.

Now we turn to proving the first statement of Theorem 3.2. We keep the notation $\mathcal{W}$ for the idempotent reduct of $\mathcal{V}$, $\mathbf{F}$ as the $\mathcal{W}$-free algebra $\mathbf{F}_{\mathcal{W}}(x, y)$ such that $|F| = n$, and the subpowers of $\mathbf{F}$ we denoted by E, $\leq$, R, S, K and L are still defined the same way. We define one more subpower of $\mathbf{F}$, which we call Q, by

$$Q = \mathrm{Sg}^{\mathbf{F}^5}\left(\begin{bmatrix} y \\ x \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ x \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ x \\ x \\ y \end{bmatrix}\right)$$

Note that the projection of Q to the first and fifth coordinate, and also to second and fifth coordinate, equals $\leq$, while its projection to any other pair of coordinates equals E. Thus, the projection $\pi_{1,2,3}(Q) = S$, $\pi_{1,3,4}(Q) = K$, $\pi_{1,3,5}(Q) = K$, $\pi_{2,3,4}(Q) = K$, $\pi_{2,3,5}(Q) = K$, while any other projection of Q to three coordinates is equal to R.

We describe an instance $(V, F, \mathcal{C})$ of the constraint satisfaction problem with the template $\mathbb{A} = \langle F; E, \leq, R, S, K, L, Q\rangle$. The set of variables V has $(2n+1)^{n+2} - 2$ elements. In order to ease our description of the constraint relations, we will arrange them all into the nodes of an downward-branching full $(2k+1)$-ary tree of height $n+1$. This is a ranked partially ordered set $\langle T; \leq\rangle$ where each node at level l between 0 and n has $2n+1$ lower covers which are at the level $l+1$ below it, while all $(2n+1)^{n+1}$ elements at the level $n+1$ are minimal. No element has more than one upper cover and the only maximal element (and therefore the greatest element) is at level 0 and is called the root. We arrange the variables so that each element $t \in T$ is assigned to a unique variable x_t, except for the root which is not assigned to any variable.

Firstly, we describe the binary constraints:

- If $s < t$ in T then $[x_s, x_t]^T \in \leq$ (the ambiguity in notation of both T and $\leq$ should be clear from the context),
- If $s, t \in T$ are incomparable, then $[x_s, x_t]^T \in E$.

Next we describe the ternary constraints:

- If $p, r, s \in T$ are such that $p < r < s$, then $[x_p, x_r, x_s]^T \in L$.
- If $p, r, s \in T$ are such that $p < s$ and $r < s$, while p and r are incomparable, then $[x_p, x_r, x_s]^T \in S$.
- If $p, r, s \in T$ are such that $p < r$, while s is incomparable to both p and to r, then $[x_p, x_r, x_s]^T \in K$.
- If $p, r, s \in T$ are pairwise incomparable, then $[x_p, x_r, x_s]^T \in R$.

Finally, we describe the 5-ary constraints:

- For any $p, r, s, u, v \in T$ such that $p < s$ and $r < s$, but all other pairs of elements are incomparable, we impose the constraint $[x_p, x_r, x_s, x_u, x_v]^T \in Q$.

We note that any triple of coordinates is assigned a unique ternary constraint since in an downward-branching tree the only possible restrictions of the order to a three-element subset are:

- the three-element chain,
- the poset with two minimal elements above the same greatest element,
- a two-element chain with an additional element incomparable to both of those and
- the three-element antichain.

Thus any three-element set of variables is within the scope of some constraint. Our analysis of the projections of Q to any two- and three-element subset of coordinates, taken together with our previous analysis of the restrictions of R, S, K and L to two-element sets of coordinates yields that the instance $\langle V, F, \mathcal{C}\rangle$ is 2-consistent, so it is $(2,3)$-minimal. Since it is also compatible with $\mathbf{F}$, and therefore with all idempotent term operations of $\mathcal{V}$, it must have a solution, according to Theorem 2.7.

Now we apply this solution $f : V \to F$. We select $3(n+1)$ vertices $\bigcup_{i=1}^{n+1} \{x_{p_i}, x_{r_i}, x_{s_i}\}$ inductively in the following way:

- Let p_1, r_1, s_1 be vertices on the level 1 of the tree (lower covers of the root) such that $f(x_{p_1}) = f(x_{r_1}) = f(x_{s_1})$ and
- for all $1 \leq i \leq n$ let $p_{i+1}, r_{i+1}, s_{i+1}$ be lower covers of the vertex p_i such that $f(x_{p_{i+1}}) = f(x_{r_{i+1}}) = f(x_{s_{i+1}})$.

By the pigeonhole principle, there must exist i, j such that $1 \leq i < j \leq n+1$ and that $f(x_{p_i}) = f(x_{p_j})$. Therefore we have $f(x_{p_i}) = f(x_{r_i}) = f(x_{s_i}) = f(x_{p_j}) = f(x_{r_j}) = c \in F$. Note that $p_j < p_i$, $r_j < p_i$ and no other comparabilities exist among p_i, r_i, s_i, p_j, s_j, so $[f(x_{p_j}), f(x_{r_j}), f(x_{p_i}), f(x_{r_i}), f(x_{s_i})]^T = [c, c, c, c, c]^T \in Q^{\mathbb{A}}$. Therefore, there must exist a term $t(x, y, z, u, v)$ such that

$$t^{\mathbf{F}}\left(\begin{bmatrix} y \\ x \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ x \\ y \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \\ x \\ x \\ y \end{bmatrix}\right) = \begin{bmatrix} c \\ c \\ c \\ c \\ c \end{bmatrix}.$$

This implies that

$$\mathcal{W} \models t(y, x, x, x) \approx t(x, y, x, x) \approx t(y, y, x, x) \approx t(x, x, y, x) \approx t(x, x, x, y),$$

and thus those same identities, together with idempotence of t, hold in $\mathcal{V}$. □

4. Further improvements

We have proved that congruence meet-semidistributivity can be characterized by an idempotent strong Mal'cev condition in the language consisting of two ternary operations and in the language consisting of one operation of arity 4. According to [8], it is impossible to do it in the language which consists of one ternary operation and any number of operations of arity less than 3. So, our Theorems 3.1 and 3.2 are optimal in the sense of [10].

Still, one can further improve these Mal'cev conditions. The characterization with two ternary operations which we proved in Theorem 3.1 is the only strong Mal'cev condition in the language with two ternary operations which characterizes congruence meet-semidistributivity, according to [8]. Therefore the characterization we proved in Theorem 3.1 is the simplest for checking by computer since ternary operations are much easier for it than 4-ary. The area in which there are possible further improvements is in syntactic strength and reduction of number of equations. These two are opposite to each other, so we must decide on preference. Say we want the fewest number of equations while having just one operation in the language, then we prove a lower bound:

Theorem 4.1. *Any strong Mal'cev condition in the language with one operation f with arity n which consists of idempotence plus one other equation and which is realized in a nontrivial semilattice, can also be realized in a nontrivial module. The module may even be assumed to be finite.*

Proof. Let the strong Mal'cev condition in question be

$$\begin{array}{rcl} f(x, x, \dots, x) & \approx & x \\ f(y_1, y_2, \dots, y_n) & \approx & f(z_1, z_2, \dots, z_n), \end{array} \tag{4.1}$$

where all y_i and all z_j are in the set $\{x_1, x_2, \dots, x_m\}$. First of all, we prove that the statement will hold iff it holds under the additional assumption that the identities are *balanced*, i. e. that $\{y_1, y_2, \dots, y_n\} = \{z_1, z_2, \dots, z_n\} = \{x_1, x_2, \dots, x_m\}$.

Assume that $y_t \notin \{z_1, \dots, z_n\}$. If $J \subseteq \{1, \dots, n\}$, denote by $\pi_J(y_1, \dots, y_n)$ the tuple of length $|J|$ consisting of all y_i such that $i \in J$ listed in the increasing order of indices. The Mal'cev condition (4.1) is realized by a nontrivial semilattice iff the condition

$$\begin{array}{rcl} g(x, x, \dots, x) & \approx & x \\ g(\pi_J(y_1, y_2, \dots, y_n)) & \approx & g(\pi_J(z_1, z_2, \dots, z_n)) \end{array} \tag{4.2}$$

is realized by a nontrivial semilattice, where $J = \{i : 1 \leq i \leq n \wedge y_i \neq y_t\}$ and the arity of the symbol g is $|J|$. That is because any interpretation of f in a nontrivial semilattice $\mathbf{S}$ must be a meet of variables which are all in J (otherwise the evaluation of y_j as a, and all other variables as b such that $a < b$ would falsify the Mal'cev condition (4.1)). On the other hand, any algebra in which the Mal'cev condition (4.2) is realized must realize the original condition (4.1), by just adding the new dummy variables.

To summarize, if we assume the following implication:
if the condition (4.2) *is realized in a nontrivial semilattice,*
then (4.2) is realized in a nontrivial module,
then we get the implication
if the condition (4.1) *is realized in a nontrivial semilattice,*
then (4.1) is realized in a nontrivial module.

Proof: **S** realizes (4.1) $\Rightarrow$ **S** realizes (4.2) $\Rightarrow$ a module realizes (4.2) $\Rightarrow$ a module realizes (4.1). Thus we prune off one after another the variables which occur only on one side. Inductively, we may assume without loss of generality that the equations in condition (4.1) are balanced.

Next we prove that any balanced condition of the form (4.1) is realized in the vector space of rational numbers **Q** viewed as a space over themselves. If there is any i such that $y_i = z_i$, then just make the interpretation as the ith projection, and this will satisfy the condition (4.1) in any algebra. Any interpretation $f^{\mathbf{Q}}$ is of the form $f(u_1, u_2, \ldots, u_n) = \sum_{i=1}^{n} \alpha_i u_i$ for some $\alpha_i \in Q$. For any i such that $1 \leq i \leq n$, denote by $I_i = \{j : 1 \leq j \leq n \wedge y_j = x_i\}$ and $J_i = \{j : 1 \leq j \leq n \wedge z_j = x_i\}$. We claim that condition (4.1) holds in **Q** iff the system of equations

$$\begin{array}{rcll} \sum_{i=1}^{n} \alpha_i & = & 1 & \text{together with equations} \\ \sum_{j\in I_i} \alpha_j & = & \sum_{j\in J_i} \alpha_j & \text{for each } 1 \leq i \leq m \end{array} \tag{4.3}$$

has a solution in **Q** (here α_i are viewed as variables). In one direction by evaluating all x_i as 1 in (4.1) from idempotence we get the first equations of (4.3), while the evaluation of x_i as 1 and of all other variables x_k as 0 implies the equation $\sum_{j\in I_i} \alpha_j = \sum_{j\in J_i} \alpha_j$. On the other hand, assume that the system (4.3) has a solution $\langle a_1, \ldots, a_n\rangle$. Interpret $f^{\mathbf{Q}}(u_1, \ldots, u_n) = \sum_{i=1}^{n} a_i u_i$. Then $f^{\mathbf{Q}}(x, \ldots, x) = \sum_{i=1}^{n} a_i x = 1x = x$. Moreover,

$$f^{\mathbf{Q}}(y_1, \ldots, y_n) = \sum_{i=1}^{m} \left(\sum_{j\in I_i} \alpha_j \right) x_i = \sum_{i=1}^{m} \left(\sum_{j\in J_i} \alpha_j \right) x_i = f^{\mathbf{Q}}(z_1, \ldots, z_n).$$

It remains to show that the system (4.3) has a solution in **Q** no matter which partitions $\{I_i : 1 \leq i \leq m\}$ and $\{J_i : 1 \leq i \leq m\}$ the Mal'cev condition (4.1) imposes. About those partitions, the assumption that $y_i \neq z_i$ for all i reflects as the property that $I_i \cap J_i = \emptyset$ for all i, and this is the only property which we will assume.

We convert the system (4.3) into the system

$$\begin{array}{rcll} \sum_{i=1}^{n} \alpha_i & = & 1 & \text{and} \\ \sum_{j\in I_i} \alpha_j - \sum_{j\in J_i} \alpha_j & = & 0 & \text{for each } 1 \leq i \leq m \end{array} \tag{4.4}$$

Let the matrix M of this system (of dimensions $m + 1 \times n$) have rank r. It consists of entries which are 0, 1 or -1, and each column has one 1 in the top row, exactly one more 1 and one -1 and the other entries are zeros. Each row has at least one 1 and, unless it is the first row which consists of all 1s it must also have at least one -1 as a consequence of (4.1) having balanced identities.

The system (4.4) will have a solution unless the augmented matrix of the system has rank $r+1$. This will occur iff the column of free coeficients and the first row are used in constructing the minor of order $r+1$ which is regular (since all entries in the column of free coefficients are zeros except for the first one). Computing the determinant of this minor by the last column yields that the augmented matrix of the system has rank $k+1$ iff there is a minor of order k of the matrix obtained from M by deleting the first row which is regular. Therefore, the system (4.4) has no solutions in some vector space iff the first row vector of M (consisting of all entries 1) is in the linear span of all other row vectors.

Let the row vectors be $\mathbf{b}_1, \ldots, \mathbf{b}_m$, let $\mathbf{1}$ be the row consisting only of 1s, and let $\mathbf{1} = \sum_{i=1}^{m} q_i \mathbf{b}_i$. If we restrict our attention initially to $\mathbf{Q}$, we may select the value q_k such that $|q_k|$ is maximal. If $q_k > 0$, we know that for some $1 \leq j \leq m$, $\mathbf{b}_k(j) = -1$. From the way matrix M looks like (and since $\mathbf{b}_i$ are its rows which are not the top one) we know that there is precisely one l such that $\mathbf{b}_l(j) = 1$ and all other rows $\mathbf{b}_i(j) = 0$ whenever $i \neq k$ and $i \neq l$. So we get that

$$1 = \mathbf{1}(j) = \sum_{i=1}^{m} q_i \mathbf{b}_i(j) = q_l - q_k.$$

This implies that $q_l > q_k > 0$ which contradicts the maximality of $|q_k|$. The case when $q_k < 0$ is dealt with analogously, we just need to select a j such that $b_k(j) = 1$ and we will get that $q_l < q_k < 0$, a contradiction again.

So we have proved that the Mal'cev condition (3.2) is realized in $\mathbf{Q}$ whenever (3.2) is realized in a nontrivial semilattice. We proceed to prove that it is also realized in $\mathbf{Z}_p$ viewed as a vector space over itself for a suitably selected p. The fact that (3.2) is realized in $\mathbf{Q}$ implies that the system (4.4) has a solution $q_1, \ldots, q_n$ in $\mathbf{Q}$. Let k be the positive integer such that all numbers $c_i = kq_i$ are integers. Then the system

$$\begin{array}{rcll} \sum_{i=1}^{n} \alpha_i & = & k & \text{and} \\ \sum_{j \in I_i} \alpha_j - \sum_{j \in J_i} \alpha_j & = & 0 & \text{for each } 1 \leq i \leq m \end{array} \tag{4.5}$$

has the solution $(c_1, c_2, \ldots, c_n)$ in the ring of integers. Select a prime number p which is relatively prime to k, and for all i, let d_i be the element of Z_p which is congruent to c_i modulo p. Let $l \in Z_p$ be such that lk is congruent to 1 modulo p. Then by multiplying all equations of the system (4.5) by l in the field $\mathbb{Z}_p$ we get that the system (4.4) has the solution $(ld_1, ld_2, \ldots, ld_n)$ in $\mathbf{Z}_p$, as desired. □

Corollary 4.2. *There exists no idempotent linear strong Mal'cev characterization of locally finite congruence meet-semidistributive varieties in the language with a only one operation and one equation other than idempotence.*

Proof. It follows from [7], Theorem 9.10, (2) ⇔ (5) and Theorem 4.1. □

We are not able to provide any strong Malcev characterization of congruence meet-semidistributive locally finite varieties in the language of a single operation, with idempotence and two more equations. However, we are able to provide a candidate strong Mal'cev condition:

$$\begin{array}{rcl} t(x,x,x,x) \approx & x & \\ t(x,x,y,z) \approx & t(y,z,y,x) \approx & t(x,z,z,y) \end{array} \tag{4.6}$$

This condition (4.6) via its substitution instances implies

$$t(x,x,x,y) \approx t(x,x,y,x) \approx t(x,y,x,x) \approx$$
$$t(y,x,x,x) \approx t(y,y,x,x) \approx t(y,x,y,x) \approx t(x,y,y,x)$$

which, together with idempotence, implies congruence meet-semidistributivity by Theorem 3.2. If one were to use $p(x,y,z) := t(x,x,y,z)$ and $q(x,y,z) := t(x,y,z,z)$, the equations (1) of Theorem 3.1 would be satisfied for these terms whenever the above strong Mal'cev condition was satisfied. So, we have a syntactically very strong candidate condition, and yet with minimal number of equations and in one of the optimal languages (optimal in the sense of [10]). This motivates the following:

Problem 4.3. Does the condition (4.6) characterize congruence meet-semidistributivity in locally finite varieties?

On the topic of strength of conditions, it may be worthwhile to try to find the place of some of these strong Mal'cev conditions within the lattice of interpretability types of strong Mal'cev conditions. For instance, there is a proof in [16], Theorem 1.3 which says that any congruence distributive variety realizes the Mal'cev condition (not strong!) of having weak near-unanimity terms of all arities greater than 2, and the weak near-unanimity terms are syntactically found from Jónsson terms. Having weak near-unanimity operations of all arities greater than 2 is a Mal'cev characterization of congruece meet-semidistributivity in locally finite varieties. None of the strong Mal'cev we present here are known to be implied by congruence distributivity, but some of the syntactically weaker ones, like the condition in Theorem 3.1 have a chance. The two strong Mal'cev conditions which seem to be the weakest, syntactically, while being equivalent to congruence meet semidistributivity in locally finite varieties, are realized in all varieties which realize $CD(4)$, as we demonstrate below:

Proposition 4.4. *Let $\mathcal{V}$ be a variety which realizes $CD(4)$, i. e. such that there exists a short sequence of term operations, $d_0 = x$, $d_1(x,y,z)$, $d_2(x,y,z)$, $d_3(x,y,z)$ and $d_4 = z$, which satisfy the Jónsson identities in $\mathcal{V}$. Then there exist terms $p(x,y,z)$, $q(x,y,z)$ and $w(x,y,z,u)$ such that p and q are the realization of strong Mal'cev condition* (1) *of Theorem 3.1 in $\mathcal{V}$, while p and w are weak near-unanimity terms with the same derived binary operation (the strong Mal'cev condition featured in Theorem 2.8 of* [13]*).*

Proof. Following [16], we introduce the representation of tuples with words, so for instance $a^ib^jca^k$ represents the $(i+j+k+1)$-tuple that has the value a in the first i coordinates, value b in the next j coordinates, value c in the $i+j+1$st coordinate and again a in the final k coordinates.

Let $\mathcal{W}$ be the idempotent reduct of $\mathcal{V}$, as defined at the beginning of the proof of Theorem 3.1. We denote by $\mathbf{F} = \mathbf{F}_{\mathcal{W}}(x, y)$, the $\mathcal{W}$-free algebra freely generated by x and y. We define the following subalgebras of powers of $\mathbf{F}$:

- $G = \mathrm{Sg}^{\mathbf{F}^3}(x^2y, xyx, yx^2)$,
- $H = \mathrm{Sg}^{\mathbf{F}^3}(x^3, yxy, xy^2)$ and
- $K = \mathrm{Sg}^{\mathbf{F}^4}(x^3y, x^2yx, xyx^2, yx^3)$.

We want to prove that there was some $c \in \mathbf{F}$ such that $c^3 \in G \cap H$ and that $c^4 \in K$. This would suffice, since there would exist terms $p(x, y, z), q(x, y, z)$ and $t(x, y, z, u)$ such that $p^{\mathbf{F}^3}(x^2y, xyx, yx^2) = c^3 = q^{\mathbf{F}^3}(x^3, yxy, xy^2)$ and that $w^{\mathbf{F}^4}(x^3y, x^2yx, xyx^2, yx^3) = c^4$. This implies that the desired equations, except for idempotence, hold in $\mathbf{F}$ when we compute the operations $p^{\mathbf{F}^3}$, $q^{\mathbf{F}^3}$ and $w^{\mathbf{F}^4}$ coordinatewise. As $\mathbf{F}$ is the free algebra, and the equalities hold when the terms are applied to the tuples of free generators, this implies that all desired identities hold in $\mathcal{W}$. Idempotence of terms p, q and w in $\mathcal{V}$ (and in $\mathcal{W}$, as well) follows from the definition of $\mathcal{W}$ as the idempotent reduct of $\mathcal{V}$.

Note that G and K are subalgebras of powers of $\mathbf{F}$ which are invariant under all permutations of coordinates (totally symetric subpowers), as explained in Definition 4.2 of [16], so if we prove that, say, $abc \in G$, this will imply that any permutation of the word abc is also in G, and similarly in the case of K.

Now we define three new elements of F: $x_1 := d_1^{\mathbf{F}}(x, x, y) = d_2^{\mathbf{F}}(x, x, y)$, $y_1 := d_2^{\mathbf{F}}(y, x_1, x_1) = d_3^{\mathbf{F}}(y, x_1, x_1)$ and $y_2 := d_2^{\mathbf{F}}(y_1, x_1, x_1) = d_3^{\mathbf{F}}(y_1, x_1, x_1)$. We will prove that $c = y_2$ satisfies the requirements of the first paragraph. First we prove that $y_2^3 \in G$. Note that $\mathbf{G}$ is totally symmetric which we will repeatedly use without mentioning it. Also recall that terms apply coordinatewise like this: $t(abc, def, ghi) = (t(a, d, g), t(b, e, h), t(c, f, i))$.

$$\begin{aligned}
yx_1x &= d_1^{\mathbf{G}}(yxx,\ xxy,\ xyx),\\
yx_1x_1 &= d_1^{\mathbf{G}}(yxx_1,\ xxy,\ xyx_1),\\
y_1x_1x &= d_2^{\mathbf{G}}(yxx,\ x_1xy,\ x_1yx),\\
y_1x_1x_1 &= d_2^{\mathbf{G}}(yxx_1,\ x_1xy,\ x_1yx_1),\\
y_2x_1x_1 &= d_2^{\mathbf{G}}(y_1xx_1,\ x_1xy,\ x_1yx_1),\\
y_1y_2x_1 &= d_3^{\mathbf{G}}(yxx_1,\ x_1xy,\ x_1y_2x_1),\\
y_2y_2x_1 &= d_3^{\mathbf{G}}(y_1xx_1,\ x_1xy,\ x_1y_2x_1),\\
y_2y_2y_2 &= d_3^{\mathbf{G}}(y_1x_1y_2,\ x_1x_1y,\ x_1y_2y_2).
\end{aligned}$$

Next, we prove that $y_2^3 \in H$:

$$\begin{aligned}
x_1xx &= d_1^{\mathbf{H}}(xyy,\, xxx,\, yxy),\\
x_1yy &= d_1^{\mathbf{H}}(xyy,\, xxx,\, yxy),\\
y_1xx &= d_3^{\mathbf{H}}(yxy,\, x_1yy,\, x_1xx),\\
y_1yy &= d_3^{\mathbf{H}}(yxy,\, x_1xx,\, x_1yy).
\end{aligned}$$

Now we let t be a binary term such that $y_2 = t^{\mathbf{F}}(x, y)$. Then

$$\begin{aligned}
x_1y_2y_2 &= t^{\mathbf{H}}(x_1xx,\, x_1yy),\\
y_1y_2y_2 &= t^{\mathbf{H}}(y_1xx,\, y_1yy),\\
y_2y_2y_2 &= d_2^{\mathbf{G}}(y_1y_2y_2,\, x_1y_2y_2,\, x_1y_2y_2).
\end{aligned}$$

Finally, we prove that $y_2^4 \in K$:

$$\begin{aligned}
yx_1x^2 &= d_1^{\mathbf{K}}(yx^3,\, x^3y,\, xyx^2),\\
yx_1^2x &= d_1^{\mathbf{K}}(yxx_1x,\, x^3y,\, xyx_1x),\\
yx_1^3 &= d_1^{\mathbf{K}}(yxx_1^2,\, x^3y,\, xyx_1^2),\\
y_1x_1^2x &= d_2^{\mathbf{K}}(yxx_1x,\, x_1x^2y,\, x_1yx_1x),\\
y_1x_1^3 &= d_2^{\mathbf{K}}(yxx_1^2,\, x_1x^2y,\, x_1yx_1^2),\\
y_2x_1^3 &= d_2^{\mathbf{K}}(y_1xx_1^2,\, x_1xx_1y,\, x_1yx_1^2),\\
y_2^2x_1^2 &= d_3^{\mathbf{K}}(y_1x_1^3,\, x_1^3y,\, x_1y_2x_1^2),\\
y_2^3x_1 &= d_3^{\mathbf{K}}(y_1x_1^3,\, x_1^3y,\, x_1y_2^2x_1),\\
y_2^4 &= d_3^{\mathbf{K}}(y_1x_1^2y_2,\, x_1^3y,\, x_1y_2^3).
\end{aligned}$$

□

The following problem is inspired by Proposition 4.4. It seems to be difficult.

Problem 4.5. (1) Does every congruence distributive variety $\mathcal{V}$ satisfy the strong Mal'cev condition

$$\begin{aligned}
&\mathcal{V} \models p(x,x,x) \approx x \approx q(x,x,x),\\
&\mathcal{V} \models p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,y,x) \text{ and} \qquad (1)\\
&\mathcal{V} \models q(x,x,y) \approx q(x,y,y)?
\end{aligned}$$

(2) Does every congruence distributive variety $\mathcal{V}$ satisfy the strong Mal'cev condition

$$\begin{aligned}
&\mathcal{V} \models p(x,x,x) \approx x \approx q(x,x,x,x),\\
&\mathcal{V} \models p(x,x,y) \approx p(x,y,x) \approx p(y,x,x) \approx q(x,x,x,y) \qquad (3)\\
&\approx q(x,x,y,x) \approx q(x,y,x,x) \approx q(y,x,x,x)?
\end{aligned}$$

Finally, we wish to address a different improvement of the Mal'cev condition for congruence meet-semidistributivity. Namely, one may modify just a couple of words in the proof of Theorem 2.8 of [13] to obtain the following Mal'cev characterization of congruence meet-semidistributive locally finite varieties.

This is a usual Mal'cev condition, not a strong one, but syntactically stronger than the strong Mal'cev condition that was proved there.

Proposition 4.6. *Let $\mathcal{V}$ be a locally finite variety. $\mathcal{V}$ is congruence meet-semidistributive iff there exists a binary term $t(x,y)$ and for all arities $n \geq 3$ terms $w_n(x_1,\dots,x_n)$ such that*

(1) *All w_n are weak near-unanimity terms in $\mathcal{V}$ and*
(2) *For all n, $\mathcal{V} \models w_n(x,x,\dots,x,y) \approx t(x,y)$.*

Proof. We will prove that for any n_0 there exist $t(x,y)$ such that the item Proposition holds for all $3 \leq n \leq n_0$. This will suffice, as the following argument shows: Let every element of $\mathbf{F}_{\mathcal{V}}(x,y)$, represented by the term $t(x,y)$, be assigned a number k which is the least such that for every term operation p of arity k, if p is weak near-unanimity in $\mathcal{V}$, then $\mathcal{V} \not\models p(x,x,\dots,x,y) \approx t(x,y)$. Our proof will show that for every n_0 there exists t such that t is not assigned any number in the interval $[3,n_0]$. Since $\mathbf{F}_{\mathcal{V}}(x,y)$ is finite, there must exist an element of $\mathbf{F}_{\mathcal{V}}(x,y)$ to which no number is assigned. For every arity $k \geq 3$, therefore, this element is $\mathcal{V}$-equal to the nearly unanimous evaluation of some weak near-unanimity term of arity k.

Now we imitate the proof of Theorem 2.8 with the following two modifications: the set of variables is $\{x_1,\dots,x_n\}$ where $n > (n_0-1)|F_{\mathcal{V}}(x,y)|$ and we impose the appropriate constraints on all subsets of variables with cardinality between 3 and n_0. The proof is now identical to the one in [13]. □

So our third and final problem is related to the above proposition.

Problem 4.7. Can Proposition 4.6 be further strengthened to say that $\mathcal{V} \models t(x,t(x,y)) \approx t(x,y)$, (i. e. so that the weak near-unanimity terms w_n are special in the terminology of [16], Definition 4.6)?

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