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MALTSEV FAMILIES OF VARIETIES CLOSED UNDER JOIN OR MALTSEV PRODUCT

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ABSTRACT. Maltsev families of varieties which are closed under join or Maltsev product are investigated. New Maltsev conditions for congruence semi-distributivity are given.

1. INTRODUCTION

A variety of algebras is a class of algebras of the same type defined by a set of equations. Examples include groups, rings, modules over a fixed ring, lattices, and Boolean algebras. Varieties can also be characterized as classes closed under homomorphic images, subalgebras and direct products. We sometimes group varieties by certain properties they have and call these classes of varieties Maltsev families. For example all the members of the classical varieties such as groups, rings and modules have permutable congruence lattices; that is, the relational product of congruences is commutative. Congruence permutability implies that the congruence lattices are modular, as was essentially shown by Dedekind [5]. Maltsev [15] showed all the members of a variety having permutable congruences is equivalent to the existence of a term $p(x, y, z)$ satisfying

$$x \approx p(x, z, z) \text{ and } z \approx p(x, x, z).$$

Such a term is called a *Maltsev term* and conditions of this type (the existence of terms satisfying certain equations or implications) is now called a *Maltsev condition.* For groups $p(x, y, z) = xy^{-1}z$ is a Maltsev term. Collections of varieties satisfying a Maltsev condition are called *Maltsev families.*

See [1] and [17] for standard notions from universal algebra.

This paper is concerned with eleven Maltsev families of varieties that are certainly the families most frequently encountered in the research in universal algebra over the past half-century, and arguably, the most significant such families. These are, namely, the varieties having a cube-term, a Taylor term, a Maltsev term (permuting congruences), Jónsson terms (distributive congruence lattices), a near unanimity term, Day terms (modular congruence lattices), meet semi-distributive congruence lattices, a non-trivial congruence identity, semi-distributive congruence lattices, n-permuting congruences (for some n), and finally, semi-distributive congruence lattices of n-permuting congruences (for some n). The last five families, along with the family defined by Taylor terms, were identified, in the domain of locally finite varieties, in D. Hobby, R. McKenzie [11], as the families defined by

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omitting all types of congruence covers in a set J of types, where J ranges over the non-empty, non-full, order ideals in the ordered set of tame congruence types.

For each family we consider, a defining Maltsev condition can be expressed as the existence, for some n, of a finite set of terms $f_1, \dots, f_k$ satisfying over the variety the equations

$$f_i(x, \dots, x) \approx x \quad \text{(the idempotency law)}$$

and a further specified finite set Σ_n of equations, each of which takes the form $f_i(z_1, \dots, z_m) \approx f_j(w_1, \dots, w_m)$ where $z_1, \dots, w_m$ are variables, not necessarily distinct. Equations such as these, involving no superposition of operations, we call *linear.* In other words, all eleven Maltsev families considered in this paper are defined by linear, idempotent Maltsev conditions.

For a linear idempotent Maltsev condition expressed as finitely many linear equations in operations $f_1, \dots, f_k$ as above, there is an equivalent linear idempotent Maltsev condition expressed with one operation symbol f. Given two operations $s(x_1, \dots, x_m)$ and $t(x_1, \dots, x_n)$ we define $s \star t$ by

$$\begin{aligned} (1) \qquad s \star t &= [s \star t](x_1, x_2 \dots, x_{mn}) \\ &= s(t(x_1, \dots, x_n), t(x_{n+1}, \dots, x_{2n}), \dots, t(x_{(m-1)n+1}, \dots, x_{mn})) \,. \end{aligned}$$

Given such a linear idempotent Maltsev condition expressed with $f_1, \dots, f_k$, put

$$f = f_1 \star f_2 \star \cdots \star f_k \,.$$

Then the given condition is equivalent to the existence of an idempotent term operation f satisfying a finite set of equations of the form $f(\mathit{variables}) \approx f(\mathit{variables})$. These new equations are derived from the former ones via the observations that since all f_i are idempotent, all f_i are derived from f through replacing the variables in f by the variables of f_i in an appropriate way. To illustrate the process, suppose that our condition requires the existence of term operations f_1, f_2 so that $f_1(x, y, y) \approx f_2(y, x, x)$, $f_1(x, y, x) \approx x$ and $f_2(x, x, x) \approx x$. This is equivalent to the existence of a term operation $f(x_1, \dots, x_9)$ (and given f_1, f_2, we can take $f = f_1 \star f_2$) satisfying the equations

$$f(xxxyyyyyy) \approx f(yxxyxxyxx) \quad \text{and} \quad f(xxxyyyxxx) \approx x \,.$$

Such a linear idempotent Maltsev condition in one function symbol f of r variables, constituted by s equations plus the idempotency law, can also be expressed by two "equations", namely: $f(x, \dots x) \approx x$ and $f(Z) \approx f(W)$ where Z and W are $s \times r$ matrices of variables and $f(Z) \approx f(W)$ stands for the collection of s equations stating that f applied to the ith row of variables in Z gives the same result as f applied to the ith row of variables in W, for $1 \leq i \leq s$.

The Maltsev family most naturally defined in this way is the largest of our eleven, containing all of the other ten. A *Taylor term* for a variety $\mathcal{V}$ is a term $f(x_1, \dots, x_n)$ for some $n \geq 1$ satisfying $f(x, \dots, x) \approx x$ and $f(Z) \approx f(W)$ where for each $1 \leq i \leq n$, the ith columns in Z and W are distinct. For a given Taylor term $f(x_1, \dots, x_n)$ we can always arrange (by simple manipulations) to find another equation $f(Z') \approx f(W')$ holding in $\mathcal{V}$ where Z' and W' are $n \times n$ matrices with all entries from $\{x, y\}$, for two variables x and y, and the main diagonal of Z' consists entirely of x's while the main diagonal of W' consists entirely of y's. W. Taylor [20] proved that an idempotent variety $\mathcal{V}$ has a Taylor term if and only if $\mathcal{V}$ has no algebra $\mathbf{F}$ of more than one element in which every operation g is, for some m, one of the m-ary projections $g(x_1, \dots, x_m) = x_i$ $(1 \leq i \leq m)$.

The concept of a cube-term emerged very recently. A *cube-term* for a variety $\mathcal{V}$ is a term $f(x_1, \dots, x_n)$ for some $n \geq 1$ satisfying the idempotency law and some matrix equation $f(Z) \approx f(W)$ where for two variables x and y, Z and W are $n \times n$ matrices of x's and y's, the entries of Z are all x, and the diagonal entries of W are y. In other words, a cube-term is a term f satisfying a collection of linear equations $f(\bar{z}) \approx x$ (where $z_1, \dots, z_n, x$ are variables) such that for each $1 \leq i \leq n$ one of these equations has a variable $y \neq x$ replacing x_i. Unlike Taylor terms, there are standard forms for cube-terms. A k-dimensional cube-term c has $2^k - 1$ variables and satisfies the idempotency law and the equation $c(C) \approx c(X)$ where C and X are $k \times (2^k - 1)$ matrices of x's and y's, X has only x for entries, and the columns of C are precisely all k-tuples of x's and y's containing at least one y—with the $2^k - 1$ columns arranged in lexicographic order. A variety with a cube-term has a k-dimensional cube-term for some $k \geq 2$. To illustrate, a 2-dimensional cube term is just a Maltsev term with the variables in non-standard order; namely, the standard 2-dimensional cube-term equations are $f(x, x, x) \approx x$ and $f(x, y, y) \approx x$, $f(y, x, y) \approx x$. Cube-terms were introduced in J. Berman, P. Idziak, P. Markovic, R. McKenzie, M. Valeriote, R. Willard [2] where it was shown that a finite algebra $\mathbf{A}$ has a cube-term if and only if the relational clone consisting of all subalgebras of powers $\mathbf{A}^n$ is "sparse", in the sense that for some k, and for all $n \geq 2$, the number of subuniverses of $\mathbf{A}^n$ is no greater than $\exp(2, n^k)$. It is also known that every variety with a cube-term has Day terms; i.e., its congruence lattices are modular lattices.

This paper you are reading began with the observation in P. Markovic, M. Maroti and R. McKenzie [16] that, for any signature σ, the class $\mathcal{C}$ of finite idempotent algebras in the signature σ that have a cube-term is a pseudo-variety; that is, it is closed under homs, subs, and finite products. Moreover if $\mathbf{A}$ is a finite idempotent algebra of signature σ with a congruence θ so that both $\mathbf{A}/\theta$ and every θ-block have cube terms, then $\mathbf{A}$ has a cube term.

The *Maltsev product* of two quasi-varieties $\mathcal{V}$ and $\mathcal{W}$ in the same signature, is the class of all algebras $\mathbf{C}$ in this signature having a congruence θ with $\mathbf{C}/\theta \in \mathcal{W}$ and such that every equivalence class c/θ is a subalgebra of $\mathbf{C}$ that belongs to $\mathcal{V}$. This class $\mathcal{V} \circ \mathcal{W}$ is a quasi-variety. It is idempotent if $\mathcal{V}$ is idempotent. $\mathcal{V} \circ \mathcal{W}$ is not necessarily a variety, even if both are idempotent. To see this let $\mathcal{V} = V(\mathbf{M}_3)$ and $\mathcal{W}$ be the variety of distributive lattices. ($\mathbf{M}_3$ is the five element modular, nondistributive lattice.) The first lattice of Figure 1 has an obvious homomorphism onto the two-element lattice and this witnesses that it is in $\mathcal{V} \circ \mathcal{W}$. It has the second lattice as an image. Since this lattice is simple, it is not in $\mathcal{V} \circ \mathcal{W}$.

We say that a Maltsev family $\mathcal{F}$ is *robust* if whenever $\mathcal{V}$ and $\mathcal{W}$ are idempotent varieties belonging to $\mathcal{F}$ then $H(\mathcal{V} \circ \mathcal{W})$ (the variety generated by the Maltsev product) also belongs to $\mathcal{F}$.

In this paper, we prove that robustness fails for the properties of having permuting congruences, having modular congruence lattices, or having distributive congruence lattices, but that all eight remaining Maltsev families we consider are robust.

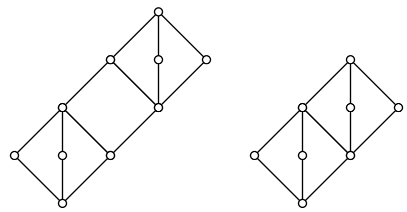


FIGURE 1. $\mathbf{M}_{33+}$

We stick with idempotent varieties, because it is easy to see that for each of our eleven Maltsev families $\mathcal{F}$, there are non-idempotent varieties of the same signature in $\mathcal{F}$ whose join does not belong to $\mathcal{F}$, in fact, has only trivial idempotent terms.

Here we record three easy robustness results. The equations that define a *near unanimity* term are $x \approx t(x, \ldots, x, y, x, \ldots, x)$, with y in the i^{th} place, $i = 1, \ldots, n$. If a variety has a near unanimity term, it is congruence distributive; see [18].

Theorem 1.1. *Each of the following Maltsev families is robust:*

(1) *varieties having a Taylor term,*
(2) *varieties having a cube term,*
(3) *varieties having a near unanimity term.*

Proof. Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of the same signature and let s and t be terms in the common language which are idempotent in both varieties. If s is a Taylor term for $\mathcal{V}$ and t is a Taylor term for $\mathcal{W}$, then the reader can show $s \star t$ is a Taylor term for $\mathcal{V} \circ \mathcal{W}$; and similarly for the other two properties. □

Since a Maltsev term is a cube term and a majority term is a near unanimity term, we get the following corollary.

Corollary 1.2. *The variety generated by the Maltsev product of two idemponent varieties each having a Maltsev term is congruence modular. The variety generated by the Maltsev product of two idemponent varieties each having a majority term is congruence distributive.*

To conclude the introduction, we remark that M. Valeriote [22], showed that if $\{\mathbf{A}_1, \ldots, \mathbf{A}_k\}$ are finitely many finite idempotent algebras of one signature and if tame congruence type $\mathbf{i}$ occurs in the variety $V(\mathbf{A}_1, \ldots, \mathbf{A}_k)$, then some type $\mathbf{j} \leq \mathbf{i}$ occurs in a homomorphic image of a subalgebra of one of the $\mathbf{A}_\ell$. This, combined with the omitting types characterizations mentioned above, enables an easy proof (which we omit) that among *finite* idempotent algebras, each of the six Maltsev classes characterized by omitting types is robust. In the remainder of this paper, we shall extend these robustness conclusions from locally finite idempotent varieties to arbitrary idempotent varieties.

2. Non-preservation examples

First we recall Polin's algebra [19]. Let $\mathbf{P}_i$, $i = 0, 1$, be algebras both with universe $\{0, 1\}$ and both having the meet operation and two unary operations: $x \mapsto x'$ and $x \mapsto x^+$. In $\mathbf{P}_0$, x' is the complement of x and $x^+ = 1$. In $\mathbf{P}_1$, x^+ is the complement and $x' = x$. Both algebras are term equivalent to the two element Boolean algebra but the variety they together generate is not congruence modular; in fact, $\mathbf{Con}\,(\mathbf{P}_0 \times \mathbf{P}_1) = \mathbf{N}_5$. Indeed, besides the top, bottom and projection congruences, the partition whose only nontrivial block is $\{(0,0),(1,0)\}$ is a congruence. The reader can check the details (or consult [4]).

A *Pixley* operation is a Maltsev operation t which also satisfies $t(x,y,x) \approx x$. If a variety has a Pixley term, it has a majority term; namely $t(x,t(x,y,z),z)$. Polin's example shows that the join of two varieties both having a Pixley term (and so both having a majority term) need not be congruence modular. As we saw in Corollary 1.2, if we restrict ourselves to idempotent varieties we cannot hope for something this strong.

The terms for some of the Maltsev conditions below were obtained with the help of the Universal Algebra Calculator [8]. UACalc algebra files for the algebras below can be obtained from `github.com/UACalc/AlgebraFiles` in the `Robust` directory.

Example 2.1. First, we exhibit two finite idempotent algebras $\mathbf{B}_0$ and $\mathbf{B}_1$, each with a Maltsev term, whose direct product does not have a Maltsev term. This shows also that neither the join nor the Maltsev product of two idempotent congruence permutable varieties need be congruence permutable. Our algebras will have similarity type $\{P,Q\}$, where P and Q are ternary operations symbols. $\{0,1\}$ is the universe of both algebras. $P^{\mathbf{B}_0} = Q^{\mathbf{B}_1}$ are Pixley operations and $Q^{\mathbf{B}_0}(x,y,z) = P^{\mathbf{B}_1}(x,y,z) = x \vee y \vee z$ (the maximum of the three inputs).

The set $S = \{0,1\}^2 \setminus \{(0,0)\}$ is a subuniverse of $\mathbf{B} = \mathbf{B}_0 \times \mathbf{B}_1$; let $\mathbf{S}$ be the corresponding subalgebra. The two projection congruences restricted to $\mathbf{S}$ do not permute. Thus $\mathbf{B}_0 \times \mathbf{B}_1$ has no Maltsev term.

However, the variety generated by $\mathbf{B}$ is 3-permutable with Hagemann-Mitschke terms:

$$x, \quad P(x,Q(x,y,x),Q(x,y,z)), \quad P(z,Q(y,y,z),Q(x,y,z)), \quad z\,.$$

(n-permutability and Hagemann-Mitschke terms are explained in Section 5; one can easily verify the identities of Theorem 5.1 hold in $\mathbf{B}_0$ and $\mathbf{B}_1$.)

Since both algebras have a Pixley and hence majority term, the variety generated by $\mathbf{B}$ is congruence distributive by Corollary 1.2. Since it is 3-permutable, it is 3-distributive. But actually it is 2-distributive; that is, it has a majority term:

$$\begin{aligned}&P(Q(x,P(x,x,y),y),\\ &\quad Q(x,P(x,x,y),Q(y,z,y)),\\ &\quad P(Q(x,P(x,x,z),z),Q(x,P(x,x,y),y),Q(y,P(y,x,z),z)))\end{aligned}$$

Next, we give more examples, with slightly different features, showing that congruence distributivity and modularity both fail to be robust.

Example 2.2. This example is an idempotent reduct of Polin's algebra constructed by Matthew Valeriote.[1] Let $\mathbf{A}_i$, $i = 0$, 1, be the reduct of $\mathbf{P}_i$ to the meet operation, which we denote with juxtaposition, and the ternary operations s and t given by

$$s(x,y,z) = (xyz')'(xy'z)'(x'yz)'(x'y'z')' \tag{2}$$

$$t(x,y,z) = (xz^+)^+y(x^+z)^+ \tag{3}$$

[1] He used it as an example of an idempotent algebra whose variety is not congruence modular and has no prime intervals in its congruence lattices of TCT type **1** or **5** and whose subalgebras have no tails; see [6]. There is a subalgebra of the square of the algebra with a tail.

Since $\mathbf{A}_0 \times \mathbf{A}_1$ is a reduct of $\mathbf{P}_0 \times \mathbf{P}_1$, it also has a nonmodular congruence lattice. (In fact the congruence lattice is also $\mathbf{N}_5$.)

One can easily check that $s^{\mathbf{A}_0}(x, y, z) = x + y + z \pmod 2$ and $t^{\mathbf{A}_0}(x, y, z) = y$, and so $s(x, s(xy, xz, yz), z)$ is a Pixley term for $\mathbf{A}_0$.

For $\mathbf{A}_1$, $s^{\mathbf{A}_1}(x, y, z) = xyz$ and

$$t^{\mathbf{A}_1}(x, y, z) = \begin{cases} 1 & \text{if } y = 1 \text{ and } x = z \\ 0 & \text{otherwise} \end{cases}$$

By Corollary 1.2 $\mathbf{A}_1$ cannot have a Maltsev term. Its variety is, however, 3-permutable with Hagemann-Mitschke terms x, $t(y, x, z)$, $t(x, z, y)$, z. Also by Corollary 1.2 $\mathbf{A}_1$ cannot be 2-distributive (have a majority term). But it is 3-distributive with Jónsson terms x, $t(s(x, y, z), x, t(y, x, z))$, $t(y, z, s(x, x, y))$, z. The variety generated by $\mathbf{A}_0 \times \mathbf{A}_1$ cannot be 3-permutable since 3-permutable varieties are congruence modular. Nevertheless it is 4-permutable with Hagemann-Mitschke terms x, $t(y, x, z)$, $s(x, y, z)$, $t(x, z, y)$, z.

If we let $\mathbf{A}_2$ be a third algebra on $\{0, 1\}$ with the meet operation, $t^{\mathbf{A}_2}(x, y, z) = y$ and $s^{\mathbf{A}_2}(x, y, z) = t^{\mathbf{A}_1}(x, y, z)$. Then the variety of $\mathbf{A}_2$, like that of $\mathbf{A}_1$, is 3-permutable but not permutable. The variety generated by $\mathbf{A}_1 \times \mathbf{A}_2$ is 5-permutable (but not 4-permutable) with Hagemann-Mitschke terms x, $t(y, x, z)$, $s(y, x, z)$, $s(x, z, y)$, $t(x, z, y)$, z.

3. SD($\wedge$) is robust

A variety is congruence meet semi-distributive, or satisfies SD($\wedge$), if in the congruence lattice of any algebra in this variety, for congruences α, β, γ we have

$$\alpha \wedge \beta = \alpha \wedge \gamma \ \text{ implies } \ \alpha \wedge (\beta \vee \gamma) = \alpha \wedge \beta\,.$$

A *system of Willard terms* for a class $\mathcal{K}$ of similar algebras is a finite set

$$\{(f_i(x, y, z), g_i(x, y, z)) : 0 \le i \le n\}$$

of pairs of three-variable terms such that $\mathcal{K}$ satisfies the equations $f_i(x, y, x) \approx g_i(x, y, x)$, $f_i(x, x, x) \approx x$ for $0 \le i \le n$ and for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$, we have that $a = b$ if and only if for all $0 \le i \le n$,

$$f_i(a, b, b) = g_i(a, b, b) \leftrightarrow f_i(a, a, b) = g_i(a, a, b)\,.$$

R. Willard [23] proved that a variety $\mathcal{V}$ satisfies SD($\wedge$) if and only if it has a system of Willard terms.

Certain concepts of Abelian algebras, and of Abelian congruences, which have played many important roles in universal algebra were introduced, notably in D. Hobby, R. McKenzie [11] and R. Freese, R. McKenzie [9], in the following way. Given congruences α, β, γ in an algebra $\mathbf{A}$, we say that α *centralizes* β $\pmod\gamma$, and write this as $C(\alpha, \beta; \gamma)$, if for every term operation $t(x_0, \ldots, x_n)$ of $\mathbf{A}$ and for all pairs of elements $(a_0, b_0) \in \alpha$, $(c_i, d_i) \in \beta$, $i = 1, \ldots n$ we have that $t(a_0, \bar{c}) \equiv t(a_0, \bar{d}) \pmod\gamma$ if and only if $t(b_0, \bar{c}) \equiv t(b_0, \bar{d}) \pmod\gamma$. We say that α is an *Abelian congruence* if and only if $C(\alpha, \alpha; 0_A)$ holds, where 0_A is the equality relation on A (the smallest congruence), and we say that $\mathbf{A}$ is an *Abelian algebra* if and only if 1_A is an Abelian congruence where $1_A = A \times A$ (the largest congruence). Note subalgebras of Abelian algebras are Abelian.

Combining the above with results from [12], we get the following theorem, which is due K. Kearnes and E. Kiss [12], K. Kearnes and Á. Szendrei [13], and R. Willard [23].

Theorem 3.1. *The following are equivalent for a variety $\mathcal{V}$ of algebras.*

(1) *$\mathcal{V}$ satisfies SD($\wedge$).*
(2) *$\mathcal{V}$ has Willard terms.*
(3) *$\mathcal{V}$ has no non-trivial Abelian algebras.*

Theorem 3.2. *If $\mathcal{V}$ and $\mathcal{W}$ are idempotent varieties of one signature satisfying SD($\wedge$) then the variety $H(\mathcal{V} \circ \mathcal{W})$ satisfies SD($\wedge$).*

Proof. Suppose that $\mathbf{A}$ is a non-trivial Abelian algebra in $H(\mathcal{V} \circ \mathcal{W})$. Then we have an algebra $\mathbf{B} \in \mathcal{V} \circ \mathcal{W}$ and a surjective homomorphism $f : \mathbf{B} \to \mathbf{A}$. Write λ for the kernel of f, a congruence of $\mathbf{B}$. Since $\mathbf{B} \in \mathcal{V} \circ \mathcal{W}$, we have a second congruence θ of $\mathbf{B}$ such that $\mathbf{B}/\theta \in \mathcal{W}$ and every θ-class is an algebra in $\mathcal{V}$.

We now establish that $\theta \subseteq \lambda$. Suppose this fails. Then there is some element $a \in B$ so that f is non-constant on the θ-class a/θ, which is a subalgebra of $\mathbf{B}$. Thus $f(a/\theta)$ is a subalgebra of $\mathbf{A}$ with at least two elements. This algebra is Abelian and belongs to $\mathcal{V}$. But this cannot happen since $\mathcal{V}$ contains no non-trivial Abelian algebras by Theorem 3.1.

Thus, indeed, $\theta \subseteq \lambda$. It follows that $\mathbf{A} \cong \mathbf{B}/\lambda \in H(\mathbf{B}/\theta) \subseteq \mathcal{W}$. But $\mathcal{W}$ contains no non-trivial Abelian algebras by Theorem 3.1. Thus $H(\mathcal{V} \circ \mathcal{W})$ contains no non-trivial Abelian algebras and so satisfies SD($\wedge$). $\square$

4. SD is robust

The next theorem combines some known characterization of join semi-distributivity (or SD($\vee$)) with some new ones. The Maltsev condition given in part (4) is similar to the one given in Hobby and McKenzie [11] and Kearnes and Kiss [12] but is apparently more useful, as we shall see. For another application of (4) see [7].

Condition (5) arose in [14] and the term t is called a SD($\vee$)-*term.* The authors there showed having such a term implies congruence semi-distributivity and that every congruence distributive variety has such a term. Here we strengthen this by showing that every congruence semi-distributive variety has such a term and so the condition is equivalent to SD($\vee$).

The equivalence of (1) and (2) is in Hobby and McKenzie [11, Exercise 7.14 (10)]. That (1) and (2) are equivalent to (6) is proved by Kearnes and Kiss in Theorem 8.14 of [12]. We will give a new proof that (2) implies (1) by deriving Willard's Maltsev condition for SD($\wedge$) [23] from the Maltsev condition given in (4).

Theorem 4.1. *The following are equivalent for a idempotent variety $\mathcal{V}$:*

(1) *$\mathcal{V}$ is congruence semi-distributive.*
(2) *$\mathcal{V}$ is congruence join semi-distributive.*
(3) *$\mathcal{V}$ satisfies*

$$\gamma \cap (\alpha \circ \beta) \subseteq (\alpha \wedge \beta) \vee (\beta \wedge \gamma) \vee (\alpha \wedge \gamma)$$

for congruences.

(4) *For some k $\mathcal{V}$ has terms $d_0(x,y,z),\dots,d_k(x,y,z)$ satisfying*

$$\begin{aligned} &d_0(x,y,z) \approx x; \\ &d_i(x,y,y) \approx d_{i+1}(x,y,y) \quad \textit{if } i \equiv 0 \textit{ or } 1 \pmod 3; \\ &d_i(x,y,x) \approx d_{i+1}(x,y,x) \quad \textit{if } i \equiv 0 \textit{ or } 2 \pmod 3; \\ &d_i(x,x,y) \approx d_{i+1}(x,x,y) \quad \textit{if } i \equiv 1 \textit{ or } 2 \pmod 3; \\ &d_k(x,y,z) \approx z; \end{aligned}$$

(5) *$\mathcal{V}$ has an idempotent term $t(x_1,\dots,x_n)$ satisfying for each $1 \le i \le n$ an equation $t(\bar u) \approx t(\bar v)$ where $\bar u$, $\bar v$ are tuples of x's and y's and $u_j = v_j = x$ for $1 \le j < i$, $u_i = x$ and $v_i = y$.*

(6) *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in the variety of semilattices and in every non-trivial variety of modules.*

Proof. Clearly, (1) implies (2). That (2) implies (3) is proved by considering the congruences $\gamma = \mathrm{Cg}(x,z)$ $\alpha = \mathrm{Cg}(x,y)$ and $\beta = \mathrm{Cg}(y,z)$ on $\mathbf{F}_{\mathcal{V}}(x,y,z)$. We assume that $\mathcal{V}$ satisfies SD($\vee$). Since, $\alpha \vee \beta = \alpha \vee \gamma = \beta \vee \gamma$ it follows that $\alpha \vee (\beta \wedge \gamma) = \beta \vee (\alpha \wedge \gamma) = \alpha \vee \beta$. Then

$$\alpha \vee (\beta \wedge \gamma) \vee (\alpha \wedge \gamma) = \alpha \vee \beta\,;$$

$$\beta \vee (\beta \wedge \gamma) \vee (\alpha \wedge \gamma) = \alpha \vee \beta\,;$$

and finally applying SD($\vee$) again we get

$$(\alpha \wedge \beta) \vee (\beta \wedge \gamma) \vee (\alpha \wedge \gamma) = \alpha \vee \beta\,,$$

and it is easy to see (3) follows from this.

Next, (4) is an easy consequence of (3)—or rather of the corollary that (x,z) belongs to the join of the three binary meets—using the usual characterization of the relations $f(x,y,z) \equiv g(x,y,z)$ mod α, mod β, and mod γ.

Let ($5'$) be the following condition:

($5'$) $\mathcal{V}$ has an idempotent term $t(x_0,\dots,x_{n-1})$ such that for all nonempty subsets S of $\{x_0,\dots,x_{n-1}\}$

> $\mathcal{V}$ satisfies an equation in the variables $\{x,y\}$ of the form
>
> $$t(a_0,\dots,a_{n-1}) \approx t(b_0,\dots,b_{n-1}),$$
>
> where $a_i = x$ for all i such that $x_i \in S$, and $b_i = x$ for all i such that $x_i \in S$ except for exactly one such i (for which $b_i = y$, of course).

Let $(*)_S$ denote the displayed condition. We will show that (4) implies ($5'$) and that ($5'$) implies (5) (so ($5'$) is also equivalent to $\mathcal{V}$ being congruence join semidistributive).

To see that (4) implies ($5'$) suppose $\mathcal{V}$ has terms as in condition (4). We may assume k is 1 or 2 mod 3 (but adding the third projection to the end if necessary) so that $d_{k-1}(x,y,y) \approx y$ holds in $\mathcal{V}$. Define t_i, $i = 1,\dots,k-1$ by

$$t_i = d_i \star d_{i-1} \star \cdots \star d_1$$

and let $t = t_{k-1}$.

Claim. For $1 \le i < k$ the term $t_i(x_0,\dots,x_{3^i-1})$ satisfies $(*)_I$ for all nonempty subsets I of its variables except possibly $I = \{x_0\}$.

Since $t_1(x,x,x) \approx t_1(x,y,y)$ and $t_1(x,x,x) \approx t_1(x,y,x)$, the claim holds for $i = 1$. Assume that the claim holds for t_i. Suppose $I \not\subseteq \{x_0, x_{3^i}, x_{2\cdot 3^i}\}$. Then there is a j such that $x_j \in I$ but not in $\{x_0, x_{3^i}, x_{2\cdot 3^i}\}$. So either $0 < j < 3^i$ or $3^i < j < 2\cdot 3^i$ or $2\cdot 3^i < j < 3^{i+1}$. The arguments for the three cases are similar, so we shall assume $3^i < j < 2\cdot 3^i$ and let $I_1 = I \cap \{x_{3^i}, \ldots, x_{2\cdot 3^i - 1}\}$. By induction the claim holds for $t_i(x_0, \ldots, x_{3^i-1})$ which implies that $(*)_S$ holds for each nonempty subset S of $\{x_{3^i}, \ldots, x_{2\cdot 3^i - 1}\}$ except possibly $\{x_{3^i}\}$. Thus $(*)_{I_1}$ holds for $t_i(x_{3^i}, \ldots, x_{2\cdot 3^i-1})$, so there is an equation witnessing this. We can extend this to a witness that t_{i+1} satisfies $(*)_I$ by setting the first third and last third of the variables to x.

So we may assume $I \subseteq \{x_0, x_{3^i}, x_{2\cdot 3^i}\}$. Let $\bar{x}$ be a string of 3^{i-1} x's and $\bar{\bar{x}}$ be a string of 3^i x's and similarly for y. Then

$$\begin{aligned}
t_{i+1}(\bar{x},\bar{y},\bar{y},\bar{x},\bar{y},\bar{y},\bar{x},\bar{y},\bar{y}) &\approx t_{i+1}(\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{y}}) && i = 0 \text{ or } 1 \pmod 3\\
t_{i+1}(\bar{x},\bar{y},\bar{x},\bar{x},\bar{y},\bar{x},\bar{x},\bar{y},\bar{x}) &\approx t_{i+1}(\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{x}}) && i = 0 \text{ or } 2 \pmod 3\\
t_{i+1}(\bar{x},\bar{x},\bar{y},\bar{x},\bar{x},\bar{y},\bar{x},\bar{x},\bar{y}) &\approx t_{i+1}(\bar{\bar{x}},\bar{\bar{x}},\bar{\bar{y}}) && i = 1 \text{ or } 2 \pmod 3
\end{aligned}$$

For example both sides of the first equation can be simplified to $d_i(x,y,y)$.

To complete the proof of the claim we need to verify $(*)_I$ for all nonempty subsets I of $\{x_0, x_{3^i}, x_{2\cdot 3^i}\}$ except $\{x_0\}$. Using the above equations, this is straightforward.

Using $d_{k-1}(x,y,y) \approx y$ one can show

$$t_{k-1}(y,x,\ldots,x) \approx t_{k-1}(x,x,\ldots,x) \approx x$$

which witnesses $(*)_{\{x_0\}}$. Thus t_{k-1} is an SD($\vee$)-term.

To see that (5′) implies (5) let $t(x_0,\ldots,x_{n-1})$ be the term given by (5′). We shall show that after permuting the variables, this term will satisfy (5). Let S be all the variables. The condition $(*)_S$ says there is a j such that $t(x,\ldots,x) = t(x,\ldots,x,y,x,\ldots,x)$, with the y in the j^{th} position. We interchange j and the last position so the modified t satisfies $t(x,\ldots,x) = t(x,\ldots,x,y)$. Now take S to be all but the last variable. $(*)_S$ and another permutation of the variables give an equation of the form

$$t(x,\ldots,x,x,a_{n-1}) = t(x,\ldots,x,y,b_{n-1})$$

where a_{n-1} and $b_{n-1} \in \{x,y\}$. Continuing we see that (5) holds.

It is easily seen that (5) does not hold in any non-trivial module or semilattice. Thus (5) implies (6). As mentioned above the equivalence of (1), (2) and (6) is proved in [12]. □

We shall now directly show that SD($\vee$) implies SD($\wedge$) by showing any variety that has terms satisfying (4) of the last theorem has Willard terms. Let Γ be the set of all terms $f(x,y,z) = d_i(u,v,w)$ with $0 \le i \le k$ and $\{u,v,w\} = \{x,y,z\}$. Take for Σ the set of all pairs $(f(x,y,z), g(x,y,z))$ such that $f,\ g \in \Gamma$ and $\mathcal{V}$ satisfies $f(x,y,x) \approx g(x,y,x)$. We claim Σ constitutes a system of Willard terms for congruence meet semidistributivity. To do this we need to prove $\mathcal{V}$ satisfies

$$x = y \leftrightarrow \bigwedge_{(f,g)\in\Sigma} [f(x,x,y) = g(x,x,y) \leftrightarrow f(x,y,y) = g(x,y,y)].$$

So let $a,\ b \in A$ where $\mathbf{A} \in \mathcal{V}$ and suppose that whenever $(f,g) \in \Sigma$ we have $f(a,b,b) = g(a,b,b) \leftrightarrow f(a,a,b) = g(a,a,b)$. We proceed to show that for all

$0 \le i \le k$, $d_i(a,b,b) = d_i(a,b,a) = d_i(a,a,b) = a$, by induction on i. For $i = 0$ it is trivial.

Suppose $i < k$ and it is true for i. Suppose that $d_{i+1}(a,b,b) \neq a$. Then $i \equiv 2 \pmod 3$. It follows that $(d_i, d_{i+1}) \in \Sigma$. But then $d_{i+1}(a,a,b) = d_i(a,a,b)$ from equations (4), and hence $d_{i+1}(a,b,b) = d_i(a,b,b) = a$ after all.

Next, suppose that $d_{i+1}(a,b,a) \neq a$. Then $i \equiv 1 \pmod 3$. Then $(f,g) \in \Sigma$ where $f(x,y,z) = d_i(x,z,y)$ and $g(x,y,z) = d_{i+1}(x,z,y)$. Since $f(a,b,b) = g(a,b,b)$, from equations (4), it follows that $f(a,a,b) = g(a,a,b)$—i.e., $d_{i+1}(a,b,a) = d_i(a,b,a) = a$ after all.

The proof that $d_{i+1}(a,a,b) = a$ follows the pattern of the demonstration that $d_{i+1}(a,b,b) = a$.

Finally, it follows that $b = d_k(a,b,b) = a$, as required.

Theorem 4.2. *If $\mathcal{V}$ and $\mathcal{W}$ are idempotent varieties of one signature satisfying SD then the variety $H(\mathcal{V} \circ \mathcal{W})$ satisfies SD.*

Proof. We can assume that the term $s(x_1, \ldots, x_n)$ satisfies condition (5) of Theorem 4.1 over $\mathcal{V}$ and $t(x_1, \ldots, x_m)$ satisfies Theorem 4.1 (5) over $\mathcal{W}$.

Consider the term

$$r(x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, x_{22}, \ldots, x_{m1}, \ldots, x_{mn}) = s(t(\bar{x}^1), \ldots, t(\bar{x}^n))$$

where $\bar{x}^j = (x_{1j}, \ldots, x_{mj})$. The $m \times n$ variables are ordered lexicographically as displayed, so that for $1 \le i, i' \le m$ and $i \le j, j' \le n$, $ij < i'j'$ if and only if $i < i'$ or $i = i'$ and $j < j'$. With this ordering, the term r satisfies a set of equations over $\mathcal{V} \circ \mathcal{W}$ that fulfills the condition of Theorem 4.1 (5) for this class, and thus for $H(\mathcal{V} \circ \mathcal{W})$.

Indeed, for any variable x_{ij}, choose an equation $t(\bar{u}) = t(\bar{v})$ valid in $\mathcal{W}$ where the tuples of x's and y's $\bar{u}$, $\bar{v}$ satisfy $u_k = v_k = x$ for $1 \le k < i$ and $u_i = x$ and $v_i = y$. Choose an equation $s(\bar{z}) = s(\bar{w})$ valid in $\mathcal{V}$ where the tuples $\bar{z}, \bar{w}$ of x's, y's satisfy $z_k = w_k = x$ for $1 \le k < j$ and $z_j = x$ and $w_j = y$. In the second equation, substitute $t(\bar{u})$ for x everywhere, and $t(\bar{v})$ for y everywhere. This gives a linear equation $r(\mathit{variables}) \approx r(\mathit{variables})$ which is valid in $\mathcal{V} \circ \mathcal{W}$, has $x_{i'j'} = x$ on both sides when $i'j' < ij$, and has $x_{ij} = x$ on the left side, and $x_{ij} = y$ on the right side. □

The next theorem combines some curious observations.

Theorem 4.3. *The following are equivalent for an idempotent variety $\mathcal{V}$:*

(1) *$\mathcal{V}$ is congruence semi-distributive.*
(2) *$\mathcal{V}$ satisfies*
$$\gamma \cap (\alpha \circ \beta) \subseteq \alpha \vee (\gamma \wedge \beta)$$
for congruences.
(3) *Whenever $\mathbf{S}, \mathbf{T}, \mathbf{A}, \mathbf{B} \in \mathcal{V}$ and $\mathbf{T} \le \mathbf{A} \times \mathbf{B}$ and $\mathbf{S}$ is a block of a congruence θ on $\mathbf{T}$, then $(a,b), (a',b') \in S$ and $(a',b) \in T$ imply $(a',b) \in S$. In particular, the blocks of a congruence on $\mathbf{A} \times \mathbf{B}$ are rectangles.*

Proof. Statement (2) is a consequence of Theorem 4.1 (3), hence (1) implies (2) in this theorem. Also, (2) easily is seen to fail in semilattices and in any non-trivial variety of modules. Hence again by Theorem 4.1, (1) and (2) are equivalent.

In the situation of statement (3), take γ to be the first projection congruence restricted to T, put $\alpha = \theta$, and take β to be the second projection congruence restricted to T. Then

$$((a',b'),(a',b)) \in \gamma \cap (\alpha \circ \beta)\,.$$

Since $\gamma \cap \beta = 0_T$, then (2) implies

$$((a',b'),(a',b)) \in \alpha$$

which entails $(a',b) \in S = (a',b')/\alpha$. Thus (2) implies (3).

To see that (3) implies (2), let $\mathbf{G} \in \mathcal{V}$, and α, β, γ be congruences of $\mathbf{G}$, and u, v, w elements of $\mathbf{G}$ with $(u,v) \in \alpha$, $(v,w) \in \beta$, $(u,w) \in \gamma$. We need to see that $(u,w) \in \alpha \vee (\beta \wedge \gamma) = \alpha'$. Take $\mathbf{A} = \mathbf{G}/\beta$, $\mathbf{B} = \mathbf{G}/\gamma$, and let $\pi : \mathbf{G} \to \mathbf{A} \times \mathbf{B}$ be the map

$$\pi(x) = (x/\beta, x/\gamma)\,,$$

and $\mathbf{T} = \pi(\mathbf{G}) \leq \mathbf{A} \times \mathbf{B}$. The kernel of π is $\beta \wedge \gamma$. Put

$$\theta = \pi(\alpha') = \{(\pi(x), \pi(y)) : (x,y) \in \alpha'\}\,.$$

Since α' contains the kernel of π, θ is a congruence of $\mathbf{T}$. Finally, take $S = \pi(u)/\theta$. Now where $a = u/\beta$, $b = u/\gamma$, $a' = v/\beta$, $b' = v/\gamma$, we have

$$\pi(u) = (a,b) \;\; \pi(v) = (a',b'),\;\; \pi(w) = (a',b)$$

and these elements belong to T. Note that $\{\pi(u), \pi(v)\} \in S$. So it follows by (3) that $\pi(w) \in S$; equivalently, $(u,w) \in \alpha'$. □

5. Congruence n-permutability for some n is robust

For $n \geq 2$, a variety is congruence n-permutable if and only if for all $\mathbf{A} \in \mathcal{V}$ and congruences θ, ψ of $\mathbf{A}$, we have that $\theta \vee \psi = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n$ (relation product) where $\alpha_i = \theta$ for odd i and $\alpha_i = \psi$ for even i.

Theorem 5.1 (J. Hagemann, A. Mitschke [10])**.** *For any $n \geq 2$ an idempotent variety $\mathcal{V}$ is congruence n-permutable if and only if $\mathcal{V}$ has terms $f_0, \ldots, f_n$ satisfying*

(1) $f_0(x,y,z) = x$.
(2) $f_i(x,x,y) = f_{i+1}(x,y,y)$ *for* $0 \leq i < n$.
(3) $f_n(x,y,z) = z$.

The next theorem, due to M. Kozik, A. Krokhin, M. Valeriote and R. Willard, gives a matrix like condition for n-permutability in the same vein as condition (5) of Theorem 4.1. However we have not been able to use this theorem to proof robustness of "n-permutable for some n". Instead, we use Hagemann-Mitschke terms from Theorem 5.1 to prove that if $\mathcal{V}$, $\mathcal{W}$ are idempotent n-permutable varieties then $H(\mathcal{V} \circ \mathcal{W})$ is m-permutable for some $m > n$. Note that we already showed in Example 2.1 that 2-permutability is not robust. However the arguments below show that the Maltsev product of two idempotent 2-permutable varieties is 4-permutable. In the following section we shall give an arguement of a different nature showing that such a Maltsev product is actually 3-permutable.

Theorem 5.2 (Kozik, Krokhin, Valeriote, Willard [14])**.** *A variety $\mathcal{V}$ is congruence n-permutable for some $n \geq 2$ if and only if it has a term $f(x_1, \ldots, x_m)$ for some $m \geq 3$ satisfying an equation $f(Z) = f(W)$ where Z and W are $m \times m$ matrices of x's and y' and Z has x on the main diagonal and everywhere below the main diagonal, while W has y on the main diagonal and everywhere above the main diagonal.*

Theorem 5.3. *The condition "congruence n-permutable for some n" is robust.*

Proof. Suppose that $\mathcal{V}$ is k-permutable and $\mathcal{W}$ is ℓ-permutable and both varieties are idempotent (and in the same signature). Then both varieties are n-permutable where n is the larger of k and ℓ.

By Theorem 5.1 we have Hagemann-Mitschke terms $f_0(x,y,z),\dots,f_n(x,y,z)$ for $\mathcal{V}$ and Hagemann-Mitschke terms $g_0(x,y,z),\dots,g_n(x,y,z)$ for $\mathcal{W}$. There is no harm in assuming $f_0 = g_0 = x$ and $f_n = g_n = z$. We write $\mathcal{U}$ for the quasi-variety $\mathcal{V}\circ\mathcal{W}$.

For a class $\mathcal{K}$ and terms $s(x,y,z)$ and $t(x,y,z)$ we write $\mathcal{K} \models s \Rightarrow t$ to denote $\mathcal{K} \models s(x,x,z) \approx t(x,z,z)$. The transitive closure of this relation is denoted $\mathcal{K} \models s \to t$. We have that $\mathcal{V}$, $\mathcal{W} \models x \to z$ and want to show $\mathcal{U} \models x \to z$. In what follows, $\Rightarrow$ and $\to$ and $\approx$ denote the relations over $\mathcal{U}$. We will show that

$$x \to g_1(x,y,z) \to g_2(x,y,z) \to \cdots \to g_n(x,y,z) = z. \tag{4}$$

First

$$\begin{aligned} x &\Rightarrow f_1(x, g_1(x,y,y), g_1(x,z,z)) \\ &\Rightarrow f_2(x, g_1(x,y,y), g_1(x,z,z)) \\ &\Rightarrow f_3(x, g_1(x,y,y), g_1(x,z,z)) \\ &\vdots \\ &\Rightarrow f_{n-1}(x, g_1(x,y,y), g_1(x,z,z)) \\ &\Rightarrow g_1(x,y,z)\,. \end{aligned}$$

To see $f_2(x, g_1(x,y,y), g_1(x,z,z)) \Rightarrow f_3(x,y,y), g_1(x,z,z))$, for example, we need to show $\mathcal{U} \models f_2(x, g_1(x,x,x), g_1(x,z,z)) \approx f_3(x, g_1(x,z,z), g_1(x,z,z))$. Of course $g_1(x,x,x) \approx x$ and if θ is a congruence of an idempotent algebra $\mathbf{A}$ with $\mathbf{A}/\theta \in \mathcal{W}$ and whose blocks are in $\mathcal{V}$, then $g_1(x,z,z)\ \theta\ g_0(x,x,z) = x$. So x and $g_1(x,z,z)$ are in the same block. And so $f_2(x, g_1(x,x,x), g_1(x,z,z)) \approx f_3(x, g_1(x,z,z), g_1(x,z,z))$ holds.

Now

$$\begin{aligned} g_1(x,y,z) &\Rightarrow f_1(g_1(x,x,z), g_2(x,y,y), g_2(x,z,z)) \\ &\to f_2(g_1(x,x,z), g_2(x,y,y), g_2(x,z,z)) \\ &\to f_3(g_1(x,x,z), g_2(x,y,y), g_2(x,z,z)) \\ &\vdots \\ &\to f_{n-1}(g_1(x,x,z), g_2(x,y,y), g_2(x,z,z)) \\ &\to f_n(g_1(x,x,z), g_2(x,y,y), g_2(x,z,z)) \\ &\Rightarrow g_2(x,y,z)\,. \end{aligned}$$

To see $f_2(g_1(x,x,z), g_2(x,y,y), g_2(x,z,z)) \to f_3(g_1(x,x,z), g_2(x,y,y), g_2(x,z,z))$, for example, we first calculate

$$\begin{aligned} f_2(g_1(x,x,z), g_2(x,y,y), g_2(x,z,z)) &\Rightarrow f_2(g_1(x,x,z), g_2(x,x,x), g_2(x,z,z)) \\ &\approx f_2(g_1(x,x,z), x, g_2(x,z,z)) \end{aligned}$$

Now, since $x \to g_1(x,y,z) \Rightarrow g_1(x,x,z)$, we have

$$\begin{aligned} f_2(g_1(x,x,z),x,g_2(x,z,z)) &\to f_2(g_1(x,x,z),g_1(x,x,z),g_2(x,z,z)) \\ &\approx f_3(g_1(x,x,z),g_2(x,z,z),g_2(x,z,z)) \\ &\Rightarrow f_3(g_1(x,x,z),g_2(x,y,y),g_2(x,z,z)) \end{aligned}$$

Thus $g_1(x,y,z) \to g_2(x,y,z)$. Continuing in this way we see that (4) holds. □

The proof of this theorem obviously produces a very long sequence of Hagemann-Mitschke terms. Each $\to$ of (4) is longer (has more $\Rightarrow$'s) than the one before it (about n times as long). Also the depth of the terms increases by one at each step of (4). One way to decrease the length is to work from both ends towards the middle. The next theorems do this explicitly for $n = 2$ and 3.

Theorem 5.4. *If $\mathcal{V}$ and $\mathcal{W}$ are idempotent, congruence permutable varieties of the same type, then the variety generated by $\mathcal{V} \circ \mathcal{W}$ is 4-permutable. If f and g are Maltsev terms for $\mathcal{V}$ and $\mathcal{W}$, respectively, then $\mathcal{V} \circ \mathcal{W}$ has Hagemann-Mitschke terms $h_0(x,y,z) = x$, $h_4(x,y,z) = z$ and*

$$\begin{aligned} h_1(x,y,z) &= f(x,g(x,y,y),g(x,z,z)) \\ h_2(x,y,z) &= g(x,y,z) \\ h_3(x,y,z) &= f(g(x,x,z),g(y,y,z),z) \end{aligned}$$

Theorem 5.5. *If $\mathcal{V}$ and $\mathcal{W}$ are idempotent, congruence 3-permutable varieties of the same type, then the variety generated by $\mathcal{V} \circ \mathcal{W}$ is 15-permutable. If f_i and g_i, $i = 0,1,2,3$ are Hagemann-Mitshchke terms for $\mathcal{V}$ and $\mathcal{W}$, respectively, then $\mathcal{V} \circ \mathcal{W}$ has Hagemann-Mitschke terms $h_0(x,y,z) = x$, $h_{15}(x,y,z) = z$ and*

$$\begin{aligned} h_1(x,y,z) &= f_1(x,g_1(x,y,y),g_1(x,z,z)) \\ h_2(x,y,z) &= f_2(x,g_1(x,y,y),g_1(x,z,z)) \\ h_3(x,y,z) &= g_1(x,y,z) \\ h_4(x,y,z) &= f_1(g_1(x,x,z),g_2(x,y,y),g_2(x,z,z)) \\ h_5(x,y,z) &= f_1(g_1(x,x,z),f_1(x,g_1(x,y,y),g_1(x,z,z)),g_2(x,z,z)) \\ h_6(x,y,z) &= f_1(g_1(x,x,z),f_2(x,g_1(x,y,y),g_1(x,z,z)),g_2(x,z,z)) \\ h_7(x,y,z) &= f_1(g_1(x,x,z),g_1(x,y,z),g_2(x,z,z)) \\ h_8(x,y,z) &= f_2(g_1(x,x,z),g_2(x,y,z),g_2(x,z,z)) \\ h_9(x,y,z) &= f_2(g_1(x,x,z),f_1(g_2(x,x,z),g_2(y,y,z),z),g_2(x,z,z)) \\ h_{10}(x,y,z) &= f_2(g_1(x,x,z),f_2(g_2(x,x,z),g_2(y,y,z),z),g_2(x,z,z)) \\ h_{11}(x,y,z) &= f_2(g_1(x,x,z),g_1(y,y,z),g_2(x,z,z)) \\ h_{12}(x,y,z) &= g_2(x,y,z) \\ h_{13}(x,y,z) &= f_1(g_2(x,x,z),g_2(y,y,z),z) \\ h_{14}(x,y,z) &= f_2(g_2(x,x,z),g_2(y,y,z),z) \end{aligned}$$

Remark 5.6. Example 2.1 gives two idempotent congruence permutable varieties whose join is 3-permutable but not permutable. Example 2.2 gives two idempotent varieties, one permutable and the other 3-permutable whose join is 4 but not 3-permutable. It also gives two idempotent, 3-permutable varieties whose join is 5, but not 4, permutable. Of course these numbers differ from those of Theorems 5.4 and 5.5.

6. Maltsev products of permutable varieties

In the last section we saw that the Maltsev product of two idempotent, congruece permutable varieties is 4-permutable. While this seemed like it would be the best possible result, it isn't. Matt Valeriote is able to show that the join of two idempotent, congruece permutable varieties is 3-permutable:

Theorem 6.1 (M. Valeriote, [21])**.** *If $\mathcal{V}$ and $\mathcal{W}$ are idempotent, congruence permutable varieties of the same signature, then $\mathcal{V}\vee\mathcal{W}$ is 3-permutable with Hagemann-Mitschke terms*

$$
\begin{aligned}
&x\\
&p(q(x,p(x,y,z),p(x,y,z)),q(x,p(y,z,z),z),q(x,y,z))\\
&p(q(x,y,z),q(x,p(x,y,y),z),q(p(x,y,z),p(x,y,z),z))\\
&z
\end{aligned}
$$

where p and q are Maltsev terms for $\mathcal{V}$ and $\mathcal{W}$, respectively.

Valeriote also found algebras that show the above terms are not adequate to show $\mathcal{V}\circ\mathcal{W}$ is 3-permutable.

Here we show that in fact the Maltsev product of two idempotent, congruence permutable varieties is 3-permutable. However the proof, which is classic universal algebra, does not produce Hagemann-Mitschke terms.

Theorem 6.2. *The Maltsev product of two idempotent, congruence permutable varieties is congruence* 3*-permutable.*

The idea of the proof is to show that if $\mathbf{A}$ is an algebra and $\theta\in\mathbf{Con}(\mathbf{A})$ is such that $\mathbf{A}/\theta$ has a Maltsev term $q(x,y,z)$, and there is a term $p(x,y,z)$ which is Maltsev on each block of θ, then the congruences of $\mathbf{A}$ 3-permute. The theorem then follows from this by stardard arguments. First we show that θ itself 3-permutes with all congruences of $\mathbf{Con}(\mathbf{A})$.

Theorem 6.3. *Let $\mathbf{A}$ be an algebra, $\theta\in\mathbf{Con}(\mathbf{A})$.*

(1) *If $\mathbf{A}/\theta$ has a Maltsev term, then for all $\psi\in\mathbf{Con}(\mathbf{A})$*
$$\psi\circ\theta\circ\psi\subseteq\theta\circ\psi\circ\theta$$
(2) *If $p(x,y,z)$ is a term which is Maltsev on each θ-block, then for $\psi\in\mathbf{Con}(\mathbf{A})$*
$$\theta\circ\psi\circ\theta\subseteq\psi\circ\theta\circ\psi$$

Thus if the hypotheses of both (1) and (2) hold, then θ 3-permutes with all $\psi\in\mathbf{Con}(\mathbf{A})$.

Proof. First assume the hypothesis of (1) holds and that the Maltsev term is $q(x,y,z)$. Suppose $(a,d)\in\psi\circ\theta\circ\psi$ so there exists b and c such that
$$a\ \psi\ b\ \theta\ c\ \psi\ d$$
Then
$$a\ \theta\ q(a,b,b)\ \theta\ q(a,b,c)\ \psi\ q(b,b,d)\ \theta\ d$$
so $(a,d)\in\theta\circ\psi\circ\theta$.

Now assume the hypothesis of (2) holds and suppose $(a, d) \in \theta \circ \psi \circ \theta$ so there exists b and c such that

$$a \;\theta\; b \;\psi\; c \;\theta\; d$$

We calculate

$$a = p(a, b, b) \;\psi\; p(a, b, c) \;\theta\; p(b, b, d) \;\psi\; p(c, c, d) = d$$

showing $(a, d) \in \psi \circ \theta \circ \psi$. □

Lemma 6.4. *Assume* $\mathbf{A}$ *and* θ *satisfy the hypotheses of (1) and (2) of Theorem 6.3.*

(1) *If* α, $\beta \geq \theta$, *then they permute.*
(2) *If* α, $\beta \leq \theta$, *then they permute.*

Proof. Since $\mathbf{A}/\theta$ lies in a CP variety, (1) holds by an easy argument. Suppose α, $\beta \leq \theta$ and that $a \;\alpha\; b \;\beta\; c$. Then a, b and c all lie in the same θ block and so $d = p(a, b, c)$ satisfies $a \;\beta\; d \;\alpha\; c$. □

Assume $\mathbf{A}$ is an algebra and $\theta \in \mathbf{Con}(\mathbf{A})$ satisfies the hypotheses of (1) and (2) of Theorem 6.3, but $\mathbf{A}$ does not have 3-permutable congruences. Hence there are α and $\beta \in \mathbf{Con}(\mathbf{A})$ such that

$$\beta \circ \alpha \circ \beta \neq \alpha \vee \beta. \tag{5}$$

The elements of $\mathbf{A}$ that witness (5) lie in a single block of $\alpha \vee \beta$, of course. If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, then $\mathbf{B}$ and the restriction of θ to B also satisfies the hypotheses (1) and (2) of Theorem 6.3. Thus we can replace $\mathbf{A}$ with $\mathbf{B}$, where $\mathbf{B}$ is the subalgebra on the above block. This shows we may also assume

$$\alpha \vee \beta = 1.$$

Since "having a cube term" is preserved under Maltsev products (and Maltsev terms are cube terms), $\mathbf{A}$ lies in a congruence modular variety. Hence the sublattice, which we denote $\mathbf{L}$, of $\mathbf{Con}(\mathbf{A})$ generated by α, β and θ is an image of the well known free modular lattice on three generators. Let $\mathbf{L}_0$ be the modular lattice freely generated by α, β and θ subject to $\alpha \vee \beta = 1$. $\mathbf{L}_0$ is diagrammed in Figure 2. Then $\mathbf{L}$ is a homomorphic image of $\mathbf{L}_0$.

As indicated in Figure 2, let β_* be the lower cover of β and let β_{**} be the lower cover of β_*; define α^* and α^{**} dually.

Using the commutator theory we see that $[\beta_*, \beta_*] \leq \alpha^*$; see [9]. Hence β_* and α^* permute; so $\alpha^{**} = \alpha^* \circ \beta_*$. Suppose that

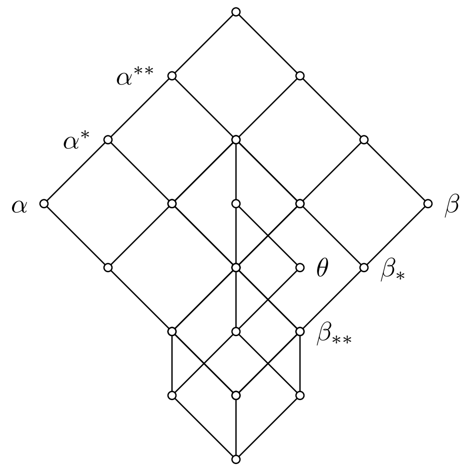
$$1 = \beta \circ \alpha^{**} \circ \beta \tag{6}$$

$$\alpha^* = \beta_{**} \circ \alpha \circ \beta_{**} \tag{7}$$

Then we calculate

$$\begin{aligned} 1 &= \beta \circ \alpha^{**} \circ \beta \\ &= \beta \circ \alpha^* \circ \beta_* \circ \beta \\ &= \beta \circ \alpha^* \circ \beta \\ &= \beta \circ \beta_{**} \circ \alpha \circ \beta_{**} \circ \beta \\ &= \beta \circ \alpha \circ \beta \end{aligned}$$

a contradiction.



$\mathbf{L}_0$

FIGURE 2

Thus either (6) or (7) must fail. If (6) fails we can replace α by α^{**}. In this case $\mathbf{L}$ is an image of $\mathbf{L}_2$ given in Figure 3. If (7) fails we can replace β by β_{**} and $\mathbf{L}$ is an image of $\mathbf{L}_1$ also given in Figure 3.

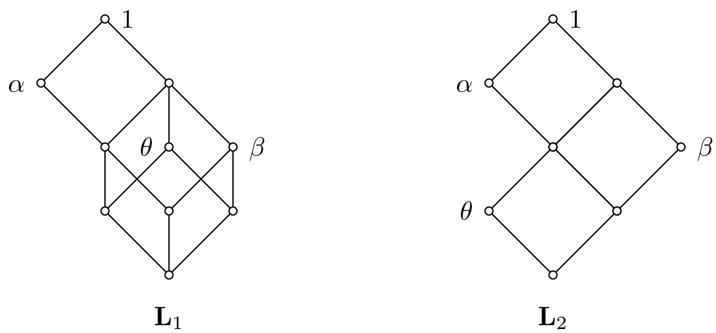


FIGURE 3

Suppose that $\mathbf{L}$ is a (not necessarily proper) homomorphic image of $\mathbf{L}_1$. In $\mathbf{L}_1$ let $\alpha_1 = \alpha \wedge \theta$ and $\beta_1 = \beta \wedge \theta$. By the lemma $\theta = \alpha_1 \circ \beta_1 = \beta_1 \circ \alpha_1$. Also θ 3-permutes with α so

$$\begin{aligned} 1 &= \theta \circ \alpha \circ \theta \\ &= \beta_1 \circ \alpha_1 \circ \alpha \circ \alpha_1 \circ \beta_1 \\ &= \beta_1 \circ \alpha \circ \beta_1 \\ &\subseteq \beta \circ \alpha \circ \beta \subseteq 1. \end{aligned}$$

Thus $\beta \circ \alpha \circ \beta = \alpha \vee \beta$, contradicting our assumption (5).

Now suppose $\mathbf{L}$ is a (not necessarily proper) homomorphic image of $\mathbf{L}_2$. We clain that

$$1 = \alpha \vee \beta = \alpha \circ \beta \circ \alpha = \alpha \circ \beta \circ \theta = \theta \circ \beta \circ \alpha. \tag{8}$$

To see this let $\gamma = \theta \vee \beta$ in $\mathbf{L}_2$. Then, since θ and β 3-permute and, by the corollary, α and γ permute,

$$1 = \alpha \circ \gamma = \alpha \circ (\theta \circ \beta \circ \theta) = \alpha \circ \beta \circ \theta.$$

Similar arguments prove the rest of the claim.

Let a and d be in A. By (8) there are elements b, c, e and $f \in A$ such that

$$a \; \alpha \; b \; \beta \; c \; \theta \; d \; \alpha \; e \; \beta \; f \; \theta \; a$$

see Figure 4.

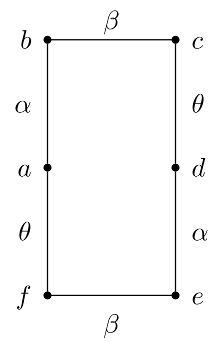


FIGURE 4

Since p is Maltsev on the blocks of θ, we have

$$p(d, c, b) \; \beta \; p(d, c, c) = d \quad \text{and} \quad p(e, e, a) \; \beta \; p(f, f, a) = a$$

Since $\theta \leq \alpha$, $p(d, c, b) \; \alpha \; p(e, e, a)$. Hence $a \; \beta \; p(e, e, a) \; \alpha \; p(d, c, b) \; \beta \; d$. Since a and d were arbitrary, $1 = \alpha \vee \beta = \beta \circ \alpha \circ \beta$, showing this case cannot occur and proving Theorem 6.2.

7. CONGRUENCE EQUATIONS

A Hobby-McKenzie term is an idempotent term $f(x_1, \ldots, x_n)$ satisfying an equation $f(Z) = f(W)$ where Z, W are $n \times n$ matrices of x's and y's so that Z has nothing but x on and below the main diagonal while W has nothing but y on the main diagonal.

Theorem 7.1. *The following are equivalent for an idempotent variety $\mathcal{V}$.*

(1) *$\mathcal{V}$ contains no non-trivial algebra that is a reduct of a semilattice.*
(2) *$\mathcal{V}$ has a Hobby-McKenzie term.*
(3) *There is a non-trivial lattice equation obeyed by the congruence lattices of all algebras in $\mathcal{V}$.*

Proof. D. Hobby and R. McKenzie [11] proved the equivalence of (1) and (2). K. Kearnes and E. Kiss [12] proved the equivalence of (1) and (3). Note that (1), for an idempotent variety $\mathcal{V}$, is equivalent to $\mathcal{V}$ satisfying an idempotent Maltsev condition that fails in the two-element semilattice. $\square$

Theorem 7.2. *The class of idempotent varieties possessing a congruence equation is robust.*

Proof. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are idempotent varieties and each has a non-trivial congruence equation. By Theorem 7.1 it suffices to show that $H(\mathcal{V} \circ \mathcal{W})$ does not contain a reduct of the two-element semilattice. Suppose, to the contrary that $\mathbf{C} \in \mathcal{V} \circ \mathcal{W}$ and h is a surjective homomorphism of $\mathbf{C}$ onto a reduct of the two-element semilattice $\mathbf{Q}$. Let λ be the kernel of h and θ be a congruence of $\mathbf{C}$ with $\mathbf{C}/\theta \in \mathcal{W}$ and every $c/\theta \in \mathcal{V}$.

We argue that $\theta \subseteq \lambda$. If h is not constant on some theta class c/θ then $h(c/\theta) = h(C)$ and this algebra belongs to $\mathcal{V}$, since the algebra c/θ does. This contradicts Theorem 7.1. Thus $\theta \subseteq \lambda$. Now it follows that the reduct of $\mathbf{Q}$, namely $h(\mathbf{C})$ is a quotient of $\mathbf{C}/\theta$. Then $h(\mathbf{C}) \in \mathcal{W}$, yielding again a contradiction. □

Remark 7.3. It follows that if $\mathcal{V}$ and $\mathcal{W}$ have Hobby-McKenzie terms s, t respectively, and both terms are idempotent in both varieties, then these varieties have a common Hobby-McKenzie term. We have been unable to discover any direct proof of that fact.

Remark 7.4. One of the hallmark results of K. Kearnes, E. Kiss and Á. Szendrei [12, 13] is that a variety $\mathcal{V}$ is congruence SD($\vee$) if and only if it is congruence SD($\wedge$) and possesses a congruence equation. Using this it follows from Theorem 3.2 and Theorem 7.2 that SD is robust. However, unlike Theorem 4.2, this does not give the SD($\vee$)-term that witnesses the result.

Remark 7.5. Of course, the eleventh Maltsev condition, namely SD plus congruence n-permutability for some n, is robust because it is the conjunction of two robust conditions.

8. Concluding remarks and further work

First we point out that knowing a property is robust leads to faster algorithms for testing if an idempotent algebra $\mathbf{A}$ has the property. Namely, we can reduce the problem to testing the property on simple sections of $\mathbf{A}$. Specifically, find a coatom of $\mathbf{A}$, test the property on $\mathbf{A}/\theta$ and then recursively continue on the blocks of θ. This was first noted for cube terms in [16].

We note that for certain properties we were able to prove they are robust but were not able to produce actual terms. These properties include congruence meet-semidistributivity, having a nontrivial congruence identity (Hobby-McKenzie term), and the 3-permutability of the Maltsev product of two permutable varieties. It would be nice to have actual terms witnessing each of these.

Many of the numerical bounds we have given are not optimal and it would be interesting to find better bounds. For example Theorem 5.5 shows that the Maltsev product of two idempotent 3-permutable varieties is 15-permutable and the remark following the theorem shows that 5 is a lower bound. But we do not know the best bound.

Theorem 1.1 shows that the Maltsev product of two idempotent varieties with near unamimity terms of arities n and m has a near unamimity term of arity nm. But in [3] M. Campanella, S. Conley and M. Valeriote show that there is a near unamimity term of arity $n + m - 1$, and that this is the best bound.

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