Absorption and directed Jónsson terms

A. Kazda, M. Kozik*, R. McKenzie, M. Moore

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Abstract

We prove that every congruence distributive variety has directed Jónsson terms, and every congruence modular variety has directed Gumm terms. The directed terms we construct witness every case of absorption witnessed by the original Jónsson or Gumm terms. This result is equivalent to a pair of claims about absorption for admissible preorders in congruence distributive and congruence modular varieties, respectively. For finite algebras, these absorption theorems have already seen significant applications, but until now, it was not clear if the theorems hold for general algebras as well. Our method also yields a novel proof of a result by P. Lipparini about the existence of a chain of terms (which we call Pixley terms) in varieties that are at the same time congruence distributive and k-permutable for some k.

1 Introduction

In 1967, Bjarni Jónsson [6] proved that a variety $\mathcal{V}$ is congruence distributive (CD) if and only if it has, for some n, a sequence of terms $J_0(x,y,z)$, $\dots$, $J_n(x,y,z)$ satisfying a certain system of equations, namely, $J_0(x,y,z) = x$, $J_n(x,y,z) = z$, $J_i(x,y,x) = x$ for each $0 \le i \le n$, and for each $0 \le i < n$, either the equation $J_i(x,x,y) = J_{i+1}(x,x,y)$ or the equation $J_i(x,y,y) = J_{i+1}(x,y,y)$. This Maltsev condition can be formulated more specifically in several equivalent ways. The following formulation is convenient for our purposes: for some $n \ge 0$ and terms $J_0(x,y,z), \dots, J_{2n+1}(x,y,z)$, consider the system of equations

$$\begin{aligned}
&J_1(x,x,y) = x, \qquad J_{2n+1}(x,y,y) = y, && \\
&J_i(x,y,x) = x, && \text{for } 0 \le i \le 2n+1, \\
&J_{2i+1}(x,y,y) = J_{2i+2}(x,y,y) && \text{for } 0 \le i \le n-1, \\
&J_{2i}(x,x,y) = J_{2i+1}(x,x,y) && \text{for } 1 \le i \le n,
\end{aligned} \tag{J(n)}$$

and call this package of equations J(n). By a chain of *Jónsson terms* for a variety $\mathcal{V}$, we mean a sequence of terms satisfying over $\mathcal{V}$ the equations J(n)

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for some n. Jónsson proved that an algebra $\mathbf{A}$ has terms obeying the equations J(n), for some n, if and only if the congruence lattice of every algebra in the variety generated by $\mathbf{A}$ is distributive. A system of *directed Jónsson terms* for $\mathcal{V}$ consists, for some $n \geq 1$, of terms $D_1(x,y,z), \ldots, D_n(x,y,z)$ satisfying over $\mathcal{V}$ the equations DJ(n):

$$\begin{aligned} &D_1(x,x,y) = x, \qquad D_n(x,y,y) = y, && \\ &D_i(x,y,x) = x && \text{for } 1 \leq i \leq n, \\ &D_i(x,y,y) = D_{i+1}(x,x,y) && \text{for } 1 \leq i < n. \end{aligned} \tag{DJ(n)}$$

Our chief purpose is to show that a variety has Jónsson terms if and only if it has directed Jónsson terms. Moreover, in such a case, one can find a sequence of terms which satisfy J(n) and DJ($2n+1$) for some n at the same time. These two results are contained in Corollary 4.1 and Observation 1.2.

H.P. Gumm [4] proved that a variety $\mathcal{V}$ is congruence modular (CM) if and only if it has, for some $n \geq 0$, a sequence of terms $J_1(x,y,z), \ldots, J_{2n+1}(x,y,z)$, and $P(x,y,z)$ satisfying the equations G(n):

$$\begin{aligned} &J_1(x,x,y) = x, \qquad J_{2n+1}(x,y,y) = P(x,y,y), \qquad P(x,x,y) = y, && \\ &J_i(x,y,x) = x && \text{for } 0 \leq i \leq 2n+1, \\ &J_{2i+1}(x,y,y) = J_{2i+2}(x,y,y) && \text{for } 0 \leq i \leq n-1, \\ &J_{2i}(x,x,y) = J_{2i+1}(x,x,y) && \text{for } 1 \leq i \leq n. \end{aligned} \tag{G(n)}$$

Directed Gumm terms are terms $D_1(x,y,z), \ldots, D_n(x,y,z)$, and $Q(x,y,z)$ satisfying DG(n) for some $n \geq 1$:

$$\begin{aligned} &D_1(x,x,y) = x, \qquad D_n(x,y,y) = Q(x,y,y), \qquad Q(x,x,y) = y, && \\ &D_i(x,y,x) = x && \text{for } 1 \leq i \leq n, \\ &D_i(x,y,y) = D_{i+1}(x,x,y) && \text{for } 1 \leq i < n. \end{aligned} \tag{DG(n)}$$

Similarly to the congruence distributive case, we show that a variety has Gumm terms if and only if it has directed Gumm terms, and that given Gumm terms we can find terms satisfying G(n) and DG($2n+1$) for some n at the same time. These two results are contained in Theorem 6.1 and Observation 1.2.

Our context makes it natural to introduce another Maltsev condition that looks similar to directed Jónsson terms but is actually much stronger. The condition is that for some $n \geq 1$ there are terms $P_1(x,y,z), \ldots, P_n(x,y,z)$ satisfying P(n):

$$\begin{aligned} &P_1(x,y,y) = x, \qquad P_n(x,x,y) = y, && \\ &P_i(x,y,x) = x && \text{for } 1 \leq i \leq n, \\ &P_i(x,x,y) = P_{i+1}(x,y,y) && \text{for } 1 \leq i < n. \end{aligned} \tag{P(n)}$$

This condition, which we call *Pixley terms*, first appeared in P. Lipparini [7].

Observe that if we remove the equations "$J_i(x,y,x) = x$" from DJ(n), we obtain a Maltsev condition that is always trivially satisfied by taking $D_1(x,y,z) =$

y and $D_i(x,y,z) = z$ for all $1 < i \leq n$. For contrast, removing the equations $P_i(x,y,x) = x$ from P(n) produces the classical Hagemann-Mitschke terms [5], and these have highly nontrivial consequences. A variety has a chain of n Hagemann-Mitschke terms if and only if it has $(n+1)$-permuting congruences. The variety of lattices, for example, satisfies J(1) but does not have Hagemann-Mitschke terms.

A. Pixley [8] proved that a variety is congruence distributive and all its congruences permute if and only if it satisfies P(1). A term $P_1(x,y,z)$ for which

$$P_1(x,y,x) = P_1(x,y,y) = P_1(y,y,x) = x$$

holds has long been called a *Pixley term*. In this connection, note that the term $J_1(x,y,z)$ with the equations $J_1(x,y,x) = J_1(x,x,y) = J_1(y,x,x) = x$ constituting Jónsson terms J(0) is familiarly known as a *majority* term; and both J(0) and DJ(1) are just asserting that we have a majority term.

Here is our principal result about these Maltsev conditions.

Theorem 1.1. *Let $\mathcal{V}$ be any variety of algebras.*

1. *$\mathcal{V}$ is congruence distributive if and only if it has directed Jónsson terms. In such a case there is a sequence of terms satisfying $DJ(2n+1)$ and $J(n)$ at the same time (for some $n \geq 1$). See Corollary 4.1 and Observation 1.2.*

2. *For any integer $k \geq 1$, a variety $\mathcal{V}$ is congruence distributive and has $(k+1)$-permuting congruences if and only if it satisfies $P(k)$. See Theorem 5.1 for the "$\Rightarrow$" implication.*

3. *$\mathcal{V}$ is congruence modular if and only if it has directed Gumm terms. In such a case there is a sequence of terms satisfying $G(n)$ and $DG(2n+1)$ at the same time (for some $n \geq 1$). See Theorem 6.1 and Observation 1.2.*

Statement (2) is Proposition 5 in P. Lipparini [7]. However, our proof, given in Section 5, is new, and shows more.

Observation 1.2. *Let $\mathcal{V}$ be a variety that admits a chain of terms satisfying $DJ(n)$. Then $\mathcal{V}$ admits a chain of terms that satisfy $J(n-1)$ and $DJ(2n-1)$ at the same time. Similarly, $DG(n)$ implies the existence of a chain of terms that simultaneously satisfies $G(n-1)$ and $DG(2n-1)$.*

Proof. Given directed Jónsson terms $D_1, \ldots, D_n$, we produce the new terms by letting

$$J_1(x,y,z) = D_1(x,y,z), \qquad J_{2i}(x,y,z) = D_{i+1}(x,x,z),$$
$$J_{2i+1}(x,y,z) = D_{i+1}(x,y,z) \qquad \text{for } 1 \leq i \leq n-1.$$

We leave to the reader the easy proof that DG(n) implies appropriate terms for congruence modular varieties. •

Similarly, P(n) implies J(n): given some Pixley terms $P_1, \dots, P_n$, take

$$\begin{aligned} &J_1(x,y,z) = x, \qquad J_{2n+1}(x,y,z) = z, \\ &J_{2i}(x,y,z) = P_i(x,y,z) && \text{for } 0 \le i < n, \\ &J_{2i+1}(x,y,z) = P_{i+1}(x,z,z) && \text{for } 0 \le i < n. \end{aligned}$$

It is an easy exercise to show that P(k) implies $(k+1)$-permuting congruences.

Our proof of the converse implications, that is, J(n) implies DJ(k) for some k, and G(n) implies DG(k) for some k, will take some work and will be concluded in Sections 4 and 6. The fact that a $(k+1)$-permutable variety with Jónsson terms satisfies P(k) is demonstrated in Section 5.

2 Absorption

The notion of *absorption* was introduced by L. Barto and M. Kozik [2], who proved deep results about absorption in finite algebras and used this theory as a powerful tool for applying universal algebraic methods in the study of constraint satisfaction problems (this area where universal algebra and theoretical computer science meet has blossomed over the past decade).

If **C** and **D** are subalgebras of an algebra **A** we say that **C** *absorbs* **D** if $\emptyset \neq C \subseteq D$ and there is a term operation $s(x_1, \dots, x_n)$ of the algebra **A** such that $\mathbf{A} \models s(x, \dots, x) = x$ (i.e. s is idempotent) and whenever $\bar{d} \in D^n$ with $d_i \in D \setminus C$ for at most one $i \in \{1, \dots, n\}$, then $s(\bar{d}) \in C$. We denote the fact that **C** absorbs **D** in this sense by $\mathbf{C} \lhd \mathbf{D}$, or $\mathbf{C} \lhd_s \mathbf{D}$ where s is the term operation that witnesses the absorption.

In this paper, however, a different variant of absorption is needed. We will say that a sequence $J_1, \dots, J_{2n+1}$ of terms is a chain of *weak Jónsson terms* if $J_1, \dots, J_{2n+1}$ satisfy all of the equations J(n) except perhaps $J_i(x,y,x) = x$. We define *weak directed Jónsson chains*, *weak Gumm chains*, and *weak directed Gumm chains* similarly, always dropping the requirement that $J_i(x,y,x) = x$.

If **C** and **D** are subalgebras of **A**, $\emptyset \neq C \subseteq D$, and $t(x,y,z)$ is a ternary idempotent term operation of **A**, then we write $\mathbf{C} \lhd^m_t \mathbf{D}$ and say that **C** *middle absorbs* ***D*** *with respect to* t if $t(a,b,c) \in C$ whenever $a, c \in C$ and $b \in D$. If $\mathcal{T}$ is a set of ternary idempotent term operations of **A**, we say that **C** middle absorbs **D** with respect to $\mathcal{T}$, written $\mathbf{C} \lhd^m_{\mathcal{T}} \mathbf{D}$, provided that $\mathbf{C} \lhd^m_t \mathbf{D}$ for every $t \in \mathcal{T}$.

We are interested in four special cases of middle absorption: Jónsson absorption, Gumm absorption, and directed versions thereof. We save Gumm absorption for the end of this paper and concentrate on Jónsson absorption for now.

We say that **C** *Jónsson absorbs* **D** if $\mathbf{C} \lhd^m_{\mathcal{J}} \mathbf{D}$, where $\mathcal{J}$ is a sequence of weak Jónsson terms. Directed Jónsson absorption is defined analogously, with weak directed Jónsson terms. We shall write $\mathbf{C} \lhd_J \mathbf{D}$ (in words, C Jónsson absorbs D) to indicate either that $\mathbf{C} \lhd^m_{\mathcal{J}} \mathbf{D}$ for some chain $\mathcal{J}$ of weak Jónsson terms, or that $\mathbf{C} \lhd^m_{\mathcal{J}} \mathbf{D}$ for a specific system of terms that is being held fixed. The context will make clear which is meant. Our use of the notation $\mathbf{C} \lhd_{DJ} \mathbf{D}$ (directed Jónsson absorption) is analogous.

One can show that if $\mathbf{A}$ is a finite idempotent algebra—equivalently, every one-element subset of A is a subuniverse—then $\mathbf{A}$ admits a chain of Jónsson terms (respectively, directed Jónsson terms) if and only if for every $a \in A$ we have $\{a\} \lhd_J \mathbf{A}$ (respectively, $\{a\} \lhd_{DJ} \mathbf{A}$). Moreover, it is immediate that standard absorption, $\mathbf{C} \lhd \mathbf{D}$, implies $\mathbf{C} \lhd_{DJ} \mathbf{D}$, which in turn implies $\mathbf{C} \lhd_J \mathbf{D}$. Indeed, suppose that $\mathbf{C} \lhd_t \mathbf{D}$ for $t = t(x_1, \ldots, x_n)$. Take $Q_1(x, y, z) = t(x, \ldots, x, y)$,

$$Q_j(x, y, z) = t(x, \ldots, x, y, z, \ldots, z) \qquad \text{with } y \text{ in the } (n-j+1)\text{-th place},$$

for $1 < j < n$, and $Q_n(x, y, z) = t(y, z, \ldots, z)$. This is a system of directed Jónsson operations with respect to which $\mathbf{C}$ middle absorbs $\mathbf{D}$. The proof that $\mathbf{C} \lhd_{DJ} \mathbf{D}$ implies $\mathbf{C} \lhd_J \mathbf{D}$ is similar to the argument that if $\mathcal{V}$ is a variety with a chain of terms that satisfy $\mathrm{DJ}(n)$, then $\mathcal{V}$ has a chain of terms satisfying $\mathrm{J}(n-1)$.

The second principal result of our paper is included in Theorem 2.2. Before introducing it we present a proof of the same result for finite algebras. The result was motivated by Barto [1] and the proof essentially follows the argument presented there.

Theorem 2.1. *Suppose that E and F are admissible preorders on $\boldsymbol{A}$ (that is, they are subalgebras of $\boldsymbol{A}^2$ that are reflexive and transitive over A). If $E \lhd_J F$, then $E = F$.*

Proof (assuming $\boldsymbol{A}$ is finite). Suppose that E and F are admissible preorders of the finite algebra $\mathbf{A}$ and $E \lhd_J F$. Let $J_1, \ldots, J_{2n+1}$ be the terms that witness the Jónsson absorption, and let $(a, b) \in F$. We must show that $(a, b) \in E$. For ease of notation, we will write $x \dashrightarrow y$ for $(x, y) \in F$ and $x \to y$ for $(x, y) \in E$ (so we want to show $(a \dashrightarrow b) \Rightarrow (a \to b)$).

Without loss of generality, we can assume that $\mathbf{A}$ is idempotent, and is generated by $\{a, b\}$. Thus, b is a top element in the order $\dashrightarrow$, since if $c \in A$ then we can write $c = t(a, b)$ for some term t, and then $c = t(a, b) \dashrightarrow t(b, b) = b$ because $\dashrightarrow$ respects all term operations. Since $\mathbf{A}$ is finite, we can also assume that a is $\to$-maximal in $\mathbf{A}$. (If there was a c strictly $\to$-larger than a in the algebra generated by $\{a, b\}$, we could replace a by c.) Using $J_1(\to, \dashrightarrow, \to) \subset \to$ and a Jónsson equation, we have

$$a = J_1(a, a, b) \to J_1(a, b, b).$$

Now we prove by induction on i that $a \to J_{2i+1}(a, b, b)$ for all $0 \leq i \leq n$. Suppose that $a \to J_{2i+1}(a, b, b) = J_{2i+2}(a, b, b) = q$. Let $p = J_{2i+2}(a, a, b)$. Absorption gives that $p \to q$, and that $p = J_{2i+3}(a, a, b) \to J_{2i+3}(a, b, b)$, so all we need to show is that $a \to p$.

The maximality of a yields $q \to a$. Since p lies in the subalgebra generated by $\{a, b\}$, we have $a \dashrightarrow p$. Putting it together, we have $q \to a \dashrightarrow p \to q$.

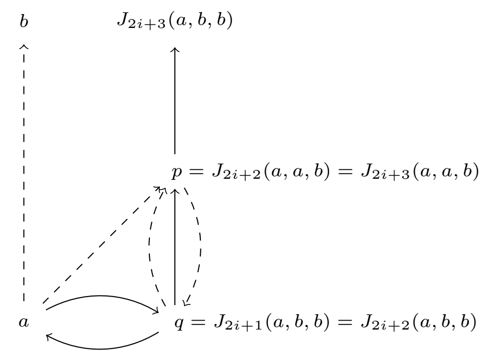


Figure 1: The elements a, b, p, q in the finite case of Theorem 2.1.

We have obtained $q \dashrightarrow p \dashrightarrow q$. Absorption now allows us to prove that $q \to p$:

$$\begin{aligned} q = J_1(q,q,p) \to J_1(q,p,p) &= J_2(q,p,p) \to J_2(q,q,p) \\ &= J_3(q,q,p) \to \cdots \to J_{2n+1}(q,q,p) \to J_{2n+1}(q,p,p) = p. \end{aligned}$$

Therefore, $a \to q \to p \to J_{2i+3}(a,b,b)$ (see Figure 1 as a reference to what we did) and we have $a \to J_{2i+1}(a,b,b)$ for all i. In particular, $a \to J_{2n+1}(a,b,b) = b$, and we are done. •

Note that there is a straightforward proof of the conclusion of the above Theorem if we assume that $E \lhd_{DJ} F$ instead of $E \lhd_J F$.

Using Theorem 2.1, we will now prove part 1 of Theorem 1.1 in the finite case. Let $\mathcal{V}$ be an idempotent CD variety, and let $\mathbf{F}_2(x,z)$ and $\mathbf{F}_3(x,y,z)$ be the free two and three generated algebras in $\mathcal{V}$. Let

$$\mathcal{G} = \big\{ t(x,y,z) \in \mathbf{F}_3 \colon t(x,y,x) = x \text{ holds in } \mathcal{V} \big\},$$
$$F = \{(t(x,x,z), t(x,z,z)) \colon t \in \mathbf{F}_3\}, \quad \text{and} \quad E = \{(t(x,x,z), t(x,z,z)) \colon t \in \mathcal{G}\}.$$

Denote by $\to$ (resp. $\dashrightarrow$) the transitive closures of E (resp. F). It is straightforward to show that E, F, $\to$, and $\dashrightarrow$ are admissible relations on $\mathbf{F}_2$ (using reflexivity). Since E, F are reflexive, the relations $\to$ and $\dashrightarrow$ are preorders on $\mathbf{F}_2$.

Observe that $(x,z) \in F$ (we can choose t to be the projection to the second coordinate). Let $\mathcal{J}$ be a chain of Jónsson terms in $\mathcal{V}$. One can easily verify that then $E \lhd^m_{\mathcal{J}} F$, from which it follows that $\to \lhd^m_{\mathcal{J}} \dashrightarrow$. Using Theorem 2.1, we then have that $\to$ and $\dashrightarrow$ are the same. In particular, $x \to z$, and there is a sequence of terms $D_1, \dots, D_m \in \mathcal{G}$ witnessing this fact. Examining the terms $D_1, \dots, D_m$, we get the following system of equalities in $\mathcal{V}$:

$$\begin{aligned} &D_1(x,x,z) = x, \qquad D_m(x,z,z) = z, \\ &D_i(x,y,x) = x && \text{for } 1 \le i \le m, \\ &D_i(x,z,z) = D_{i+1}(x,x,z) && \text{for } 1 \le i < m, \end{aligned}$$

which means that $D_1, \ldots, D_m$ are directed Jónsson terms.

Of course, the sequence of proofs presented so far only works when $\mathbf{F}_2$ is finite, but we will improve that. In fact, we will show that one can always make Jónsson absorption into directed Jónsson absorption.

Theorem 2.2. *Let $\mathcal{V}$ be a variety, and $\mathcal{J}$ be a chain of weak Jónsson terms of $\mathcal{V}$. Then there exists a chain $\mathcal{D}$ of weak* directed *Jónsson terms of $\mathcal{V}$ such that for all $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{V}$ we have $\boldsymbol{B} \lhd^m_{\mathcal{J}} \boldsymbol{A} \Rightarrow \boldsymbol{B} \lhd^m_{\mathcal{D}} \boldsymbol{A}$.*

The proof of Theorem 2.2 will have to wait until Section 4, after we have constructed suitable tools.

3 Paths in the free algebra

This section contains the core of this paper – a proof of a somewhat technical result from which Theorem 2.2 follows.

We choose and fix a variety $\mathcal{W}$ whose only basic operations are $J_1 \ldots, J_{2k+1}$, which satisfy the equations

$$\begin{aligned} &J_1(x,x,y) = x, \\ &J_{2i+1}(x,y,y) = J_{2i+2}(x,y,y) \qquad \text{for } 0 \leq i \leq k-1, \\ &J_{2i}(x,x,y) = J_{2i+1}(x,x,y) \qquad \text{for } 1 \leq i \leq k. \end{aligned} \tag{3.1}$$

By adding more equations and operations, we could make $\mathcal{W}$ congruence distributive or congruence modular. Our aim is to turn the chain $J_1, \ldots, J_{2k+1}$ into a longer chain of directed terms that ends at something like $J_{2k+1}(x,z,z)$.

Notice that the operations of $\mathcal{W}$ are idempotent. Let $\mathbf{F}_3$ be the free algebra on three generators in $\mathcal{W}$, freely generated (relative to $\mathcal{W}$) by the elements x, y, z. Let $\mathbf{F}_2 \leq \mathbf{F}_3$ be the subalgebra of $\mathbf{F}_3$ freely generated by x and z.

We shall be working with two binary relations E, F on $\mathbf{F}_2$. Define $\mathbf{F}$ to be the subalgebra of $\mathbf{F}_2^2$ generated by the pairs (x,x), (x,z) and (z,z), that is

$$F = \big\{(t(x,x,z), t(x,z,z))\colon t \text{ is a ternary term of } \mathcal{W}\big\}.$$

Let $\mathcal{J} = \{J_1, \ldots, J_{2k+1}\}$ and define $\mathcal{G}$ to be the set of all $\mathcal{W}$-terms $t(x,y,z)$ such that whenever $\mathbf{A}, \mathbf{B} \in \mathcal{W}$ are algebras such that $\mathbf{B} \lhd^m_{\mathcal{J}} \mathbf{A}$, then $\mathbf{B} \lhd^m_t \mathbf{A}$. While the set $\mathcal{G}$ is hard to describe explicitly, one can easily see that $\mathcal{J} \subseteq \mathcal{G}$ and that $\mathcal{G}$ is a subalgebra of $\mathbf{F}_3$.

From this it immediately follows that

$$E = \big\{(t(x,x,z), t(x,z,z))\colon t(x,y,z) \in \mathcal{G}\big\}$$

is an admissible relation over $\mathbf{F}_2$. Moreover, it is straightforward to verify from the definition of absorption that $\mathcal{G} \lhd^m_{\mathcal{J}} \mathbf{F}_3$, from which it follows that $E \lhd^m_{\mathcal{J}} F$. We will view the pair $E \lhd^m_{\mathcal{J}} F$ as a generic instance of absorption in $\mathcal{W}$. Notice that $(x,x), (z,z) \in E$ since the projections x, z belong to $\mathcal{G}$. Thus, since all operations are idempotent, we have that the relations E and F are reflexive over F_2. That is, $(a, a) \in E$ for all $a \in F_2$.

It is important to notice that for every $a \in F_2$ we have $(x, a), (a, z) \in F$. To see this, write $a = t(x, z)$ for a term t, and apply the term operation $t^{\mathbf{F}_2}$ of $\mathbf{F}_2$ to the pairs (x, x) and (x, z) and use that $t(x, x) = x$, yielding $(x, a) \in F$; and for $(a, z) \in F$ apply $t^{\mathbf{F}_2}$ to (x, z) and (z, z).

We shall write $p \dashrightarrow q$ to indicate that the pair (p, q) belongs to the transitive closure of F and $p \to q$ to indicate that (p, q) belongs to the transitive closure of E. Both relations $\dashrightarrow$ and $\to$ are admissible preorders of $\mathbf{F}_2$ (i.e. they are transitive and reflexive). We leave it to the reader to verify that $\to \lhd^m_{\mathcal{J}} \dashrightarrow$.

We now introduce *left powers of elements* of $\mathbf{F}_2$: for any $a = a(x, z) \in \mathbf{F}_2$ define $a^0 = z$ and, inductively,

$$a^{k+1}(x, z) = a(x, a^k).$$

In more complicated expressions, we evaluate powers first, so for example $a^2(b, c)$ means "take $a^2(x, z)$ and substitute $x = b, z = c$", giving us $a(b, a(b, c))$. Observe that thus defined, exponentiation satisfies the equalities $(a^k)^\ell = a^{k\ell}$ and $z^k = z$ for any $a \in \mathbf{F}_2$ and any k, ℓ nonnegative integers.

Letting $J = J(x, z) = J_{2k+1}(x, z, z)$, we can state the core result of this paper, whose proof takes up the remainder of this section.

Theorem 3.1. *There exists $b \in \boldsymbol{F}_2$ such that $x \to J^{2^k}(b, J^{2^k-1})$.*

The next lemma is essential for our proof of Theorem 3.1. Every endomorphism of $\mathbf{F}_2$ is uniquely determined by the elements to which it sends x and z, and, conversely, for any pair $a, b \in \mathbf{F}_2$ there is an endomorphism σ of $\mathbf{F}_2$ that sends each $c(x, z) \in \mathbf{F}_2$ to $c(a, b) = c(a(x, z), b(x, z))$ (in particular $\sigma(x) = a$ and $\sigma(z) = b$). An endomorphism σ of $\mathbf{F}_2$ will be called *special* if $\sigma(x) \dashrightarrow \sigma(z)$.

Lemma 3.2. *Every special endomorphism of $\boldsymbol{F}_2$ respects $\dashrightarrow$ and $\to$. That is, given $a \dashrightarrow b$,*

- *if $c = c(x, z) \dashrightarrow d(x, z) = d$ then $c(a, b) \dashrightarrow d(a, b)$; and*
- *if $c = c(x, z) \to d(x, z) = d$ then $c(a, b) \to d(a, b)$.*

Proof. To show that σ, moving x to a and z to b with $a \dashrightarrow b$, respects $\dashrightarrow$, it suffices to show that $c\,F\,d$ implies $c(a, b) \dashrightarrow d(a, b)$. Let $c(x, z)\,F\,d(x, z)$. Thus there is a term $s(u, v, w)$ so that

$$c(x, z) = s(x, x, z) \qquad \text{and} \qquad d(x, z) = s(x, z, z).$$

Applying σ to these equations, we have that

$$c(a, b) = s(a, a, b) \qquad \text{and} \qquad d(a, b) = s(a, b, b),$$

or in a more suggestive matrix form:

$$\begin{pmatrix} c(a, b) \\ d(a, b) \end{pmatrix} = s \begin{pmatrix} a & a & b \\ a & b & b \end{pmatrix}.$$

Now observe that in each of the three columns of the matrix on the right hand side, the rows are related by $\dashrightarrow$. Since s preserves $\dashrightarrow$, we have $c(a,b) \dashrightarrow d(a,b)$, as required.

To show that σ respects $\rightarrow$, it again suffices to show that $c\,E\,d$ implies $c(a,b) \rightarrow d(a,b)$. Let $c(x,z)\,E\,d(x,z)$. As before, there is a term $s(u,v,w)$ such that

$$c(x,z) = s(x,x,z) \qquad \text{and} \qquad d(x,z) = s(x,z,z),$$

but this time we also know that $s(x,y,z) \in \mathcal{G}$. We again apply σ and write the result in a matrix form:

$$\begin{pmatrix} c(a,b) \\ d(a,b) \end{pmatrix} = s \begin{pmatrix} a & a & b \\ a & b & b \end{pmatrix}.$$

Observe that in the first and third columns on the right hand side, the rows are $\rightarrow$-related, while the middle column is $\dashrightarrow$-related. Since $s \in \mathcal{G}$ it follows that $\rightarrow \lhd_s^m \dashrightarrow$ and hence the pair on the left hand side must be $\rightarrow$-related. Therefore $c(a,b) \rightarrow d(a,b)$, as required. •

Using Lemma 3.2, it is an easy exercise to show that if $a \rightarrow b$, then $a^n \rightarrow b^n$ for any positive integer n. Figures 2 and 3 illustrate the next definition.

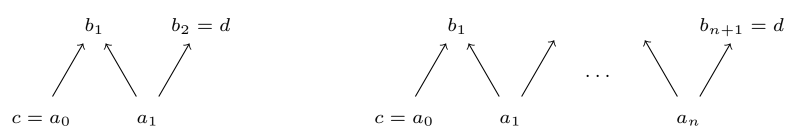
Definition 3.3. Let n be a nonnegative integer. An *n-fence* from c to d, denoted by $F(c,d)$, is a sequence of elements of $\mathbf{F}_2$ satisfying

$$c = a_0 \rightarrow b_1 \leftarrow a_1 \rightarrow b_2 \leftarrow a_2 \rightarrow \cdots \leftarrow a_n \rightarrow b_{n+1} = d.$$

Let n be a positive integer. An *n-box* B is a sequence $q_1 \dashrightarrow p_1 \dashrightarrow q_2 \dashrightarrow p_2 \dashrightarrow q_3 \dashrightarrow \cdots \dashrightarrow q_n \dashrightarrow p_n$ such that

$$p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_n \qquad \text{and} \qquad q_1 \rightarrow q_2 \rightarrow \cdots \rightarrow q_n.$$

An *n-box* from c to b and d, denoted by $B(c;b,d)$, is an n-box with $c = q_1$, $q_n \rightarrow b$, and $p_n \rightarrow d$. Note that a 0-fence from c to d is simply $c \rightarrow d$.



(a) A 1-fence $F(c,d)$. (b) An n-fence $F(c,d)$.

Figure 2: Pictures of fences.

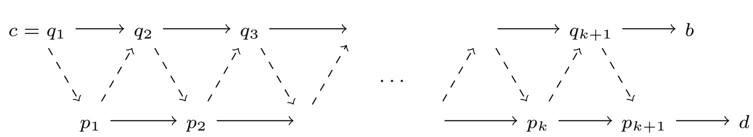


Figure 3: A $(k+1)$-box $B(c;b,d)$.

The next three lemmas contain the heart of the proof of Theorem 3.1.

Lemma 3.4. *Suppose that $B(c; b, d)$ is a $(k+1)$-box. Then $c \to J_{2k+1}(b, d, d)$.*

Proof. Label the vertices of the box from left to right according to Figure 3 as $q_1, p_1, q_2, p_2, \ldots, q_{k+1}, p_{k+1}, b, d$.

Observe that since $q_1 \to q_2$, $p_1 \to p_2$, and $q_1 \dashrightarrow p_1 \dashrightarrow q_2$ (and $\to \lhd^m_{\mathcal{J}} \dashrightarrow$), we have the sequence:

$$\begin{aligned} c = J_1(q_1, q_1, p_1) \to J_1(q_2, p_1, p_1) = J_2(q_2, p_1, p_1) \\ \to J_2(q_2, q_2, p_2) = J_3(q_2, q_2, p_2). \end{aligned}$$

Continuing in this vein, we obtain for i ranging from 1 to k the sequence:

$$\begin{aligned} c \to J_{2i-1}(q_i, q_i, p_i) \to J_{2i-1}(q_{i+1}, p_i, p_i) = J_{2i}(q_{i+1}, p_i, p_i) \\ \to J_{2i}(q_{i+1}, q_{i+1}, p_{i+1}) = J_{2i+1}(q_{i+1}, q_{i+1}, p_{i+1}). \end{aligned}$$

Letting $i = k$ (and thus $2i + 1 = n$), we conclude that

$$c \to J_{2k+1}(q_{k+1}, q_{k+1}, p_{k+1}) \to J_{2k+1}(q_{k+1}, p_{k+1}, p_{k+1}).$$

Finally, using $q_{k+1} \to b$ and $p_{k+1} \to d$, we get $c \to J_{2k+1}(b, d, d)$. •

Lemma 3.5. *Assume that there is a 1-fence $x \to b \leftarrow a \to d$. Then for every $\ell > 1$ there is an ℓ-box $B(x; b, d(b, d))$.*

Proof. We put $q_1 = x$ and $p_1 = a(x, a)$. For $2 \le i \le \ell$, let

$$q_i = b(q_{i-1}, a) \qquad \text{and} \qquad p_i = a(q_i, a).$$

We claim that the result is an ℓ-box $B(x; b, d(b, d))$. The rest of the proof consists of verifying the various $\dashrightarrow$ and $\to$ relations involved. We invite the reader to use Figure 4 for a reference (note that some diagonal edges are solid where the definition of a box required only dashed edges – this is all right since $\to$ is a subset of $\dashrightarrow$).

$$\begin{tikzcd}
x = q_1 \arrow[rr] \arrow[dr, dashed] & & q_2 = b(x,a) \arrow[rr] \arrow[dr, dashed] & & q_3 = b(q_2,a) & & q_\ell \arrow[rr] \arrow[dr, dashed] & & b \\
& p_1 = a(x,a) \arrow[rr] \arrow[ur] & & p_2 = a(q_2,a) \arrow[ur] & \cdots & p_{\ell-1} \arrow[rr] \arrow[ur] & & p_\ell \arrow[r] & d(b,d)
\end{tikzcd}$$

Figure 4: The ℓ-box $B(x; b, d(b, d))$.

Observe that $x \dashrightarrow a$, so the endomorphism σ sending x to x and z to a is special. It is easy to see that $\sigma(b) = b(x, a) = q_2$ and $\sigma(a) = a(x, a) = p_1$. Since $x \to b \leftarrow a$, it follows by Lemma 3.2 that $x \to q_2 \leftarrow p_1$.

We now proceed by induction to prove that $q_i \to q_{i+1}$ and $p_i \to p_{i+1}$ for all $i = 1, \ldots, \ell - 1$. We already know the arrows for $i = 1$, and from $q_{i-1} \to q_i$, we easily get both $q_i = b(q_{i-1}, a) \to b(q_i, a) = q_{i+1}$ and $p_i \to p_{i+1}$ for all applicable values of i.

Observe that $q_1 = x \dashrightarrow a$. Since $q_i = b(q_{i-1}, a)$, induction gives us that $q_i \dashrightarrow a$ for all i. Repeated use of this set of dashed arrows allows us to prove that $p_i \to q_{i+1}$ and $q_i \dashrightarrow p_i$ for all i in the following way. Consider first the endomorphism σ sending x to q_i and z to a. Since $q_i \dashrightarrow a$, this is a special endomorphism. Since $a \to b$, we have $p_i = \sigma(a) \to \sigma(b) = q_{i+1}$ for all i. To see $q_i \dashrightarrow p_i$, observe that $q_i = a(q_i, q_i) \dashrightarrow a(q_i, a) = p_i$.

All that remains now is to get the two arrows at the rightmost end of the box. Similarly to the previous paragraph, it is easy to prove by induction on i that $q_i \to b$ for all i, so in particular $q_\ell \to b$. To obtain $p_\ell \to d(b, d)$, observe that $p_\ell = a(q_\ell, a) \to d(q_\ell, a) \to d(b, a) \to d(b, d)$ (we have used first Lemma 3.2, then $q_\ell \to b$, and finally $a \to d$). •

Lemma 3.6. *For each $0 \le i < k$, there exists a $(k-i)$-fence from x to $J^{2^{i+1}-1}$. (Recall that $J = J(x, z) = J_{2k+1}(x, z, z)$.)*

Proof. We proceed by induction on i. For $i = 0$, we get a k-fence from x to J by putting $b_\ell = J_{2\ell-1}(x, z, z)$ and $a_\ell = J_{2\ell}(x, x, z)$, for $1 \le \ell \le k$.

Suppose now that $1 \le i < k$ and we have a $(k - i + 1)$-fence

$$x \to b_1 \leftarrow a_1 \to b_2 \leftarrow a_2 \to \cdots \leftarrow a_{k-i} \to b_{k-i+1} \leftarrow a_{k-i+1} \to J^{2^i-1}. \tag{3.2}$$

We proceed to construct a $(k - i)$-fence from x to $J^{2^{i+1}-1}$.

Applying first Lemma 3.5 and then Lemma 3.4 to the 1-fence with vertices x, b_1, a_1, b_2 above, we get

$$x \to J_{2k+1}(b_1, b_2(b_1, b_2), b_2(b_1, b_2)) = J(b_1, b_2(b_1, b_2)).$$

Denote the term on the right hand side of the above arrow by b_1'. Using $b_1 \leftarrow x$, we get

$$b_1' = J(b_1, b_2(b_1, b_2)) \leftarrow J(x, b_2(x, b_2)) = J(x, b_2^2).$$

Since $b_2^2 \leftarrow a_2^2$, we obtain $b_1' \leftarrow J(x, a_2^2)$. Consider the sequence $b_1', a_1' = J(x, a_2^2)$, and

$$a_\ell' = J(x, a_{\ell+1}^2) \qquad \text{and} \qquad b_\ell' = J(x, b_{\ell+1}^2)$$

for $2 \le \ell \le k - i$. It is easy to verify that

$$x \to b_1' \leftarrow a_1' \to b_2' \leftarrow a_3' \to \cdots \leftarrow a_{k-i}'.$$

Let us look at the element a_{k-i}' in this fence. We have

$$a_{k-i}' = J(x, a_{k-i+1}^2) \to J\left(x, \left(J^{2^i-1}\right)^2\right) = J(x, J^{2^{i+1}-2}) = J^{2^{i+1}-1}$$

(we use $a_{k-i+1} \to J^{2^i-1}$ from (3.2) above). We have therefore found a $(k - i)$-fence from x to $J^{2^{i+1}-1}$, as was needed. •

We are now ready to prove Theorem 3.1.

Theorem (Theorem 3.1)**.** *There exists $b \in \boldsymbol{F}_2$ such that $x \to J^{2^k}(b, J^{2^k-1})$.*

Proof. By taking $i = k-1$ in Lemma 3.6, we obtain a 1-fence $x \to b \leftarrow a \to J^{2^k-1}$. Applying Lemmas 3.5 and 3.4, and observing that

$$J(b, J^{2^k-1}(b, J^{2^k-1})) = J^{2^k}(b, J^{2^k-1}),$$

we get $x \to J^{2^k}(b, J^{2^k-1})$. •

4 Directed Jónsson terms

Theorem (Theorem 2.2)**.** *Let $\mathcal{V}$ be a variety, and $\mathcal{J}$ be a chain of weak Jónsson terms of $\mathcal{V}$. Then there exists a chain $\mathcal{D}$ of weak directed Jónsson terms of $\mathcal{V}$ such that for all $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{V}$ we have $\boldsymbol{B} \lhd^m_{\mathcal{J}} \boldsymbol{A}$ implies $\boldsymbol{B} \lhd^m_{\mathcal{D}} \boldsymbol{A}$.*

Proof. Let $J_1, \ldots, J_{2k+1}$ be a chain of weak Jónsson terms in $\mathcal{V}$. By taking an inessential expansion of $\mathcal{V}$, we can assume that J_i are basic operations of $\mathcal{V}$. Consider the variety $\mathcal{W}$ from the previous chapter. Since the equational basis of $\mathcal{W}$ is a subset of the identities true in $\mathcal{V}$, the variety $\mathcal{W}$ interprets into $\mathcal{V}$.

Theorem 3.1 gives us that there is a chain $\mathcal{D} = \{D_1, \ldots, D_m\} \subseteq \mathcal{G}$ such that the system of equalities

$$\begin{aligned}
&D_1(x, x, z) = x, \\
&D_i(x, z, z) = D_{i+1}(x, x, z) \quad \text{for each } i = 1, \ldots, m-1, \\
&D_m(x, z, z) = J^{2^k}(b, J^{2^k-1})
\end{aligned}$$

holds in $\mathcal{W}$. Since $\mathcal{W}$ interprets into $\mathcal{V}$, these equalities must also hold in $\mathcal{V}$. Moreover, in $\mathcal{V}$ we have the equality $J(x, z) = J_{2k+1}(x, z, z) = z$, so $J^{2^k}(b, J^{2^k-1}) = z$.

Finally, let $\mathbf{B} \leq \mathbf{A}$ be algebras in $\mathcal{V}$. By removing all of the basic operations except $J_1, \ldots, J_{2k+1}$, we obtain a pair of reducts $\mathbf{B}^\star \leq \mathbf{A}^\star$ which both lie in $\mathcal{W}$. If $\mathbf{B} \lhd^m_{\mathcal{J}} \mathbf{A}$, then trivially $\mathbf{B}^\star \lhd^m_{\mathcal{J}} \mathbf{A}^\star$, and $D_1, \ldots, D_m \in \mathcal{G}$ gives us $\mathbf{B}^\star \lhd^m_{\mathcal{D}} \mathbf{A}^\star$. Since $\mathbf{A}^\star$ is a reduct of $\mathbf{A}$, we immediately have $\mathbf{B} \lhd^m_{\mathcal{D}} \mathbf{A}$.

The chain $D_1, \ldots, D_m$ middle absorbs anything that $\mathcal{J}$ absorbs, and satisfies in $\mathcal{V}$ the system of equalities

$$\begin{aligned}
&D_1(x, x, z) = x, \\
&D_i(x, z, z) = D_{i+1}(x, x, z) \quad \text{for each } i = 1, \ldots, k-1, \\
&D_m(x, z, z) = z.
\end{aligned}$$

Therefore, $D_1, \ldots, D_m$ is the weak directed Jónsson chain $\mathcal{D}$ we were looking for. •

Corollary 4.1. *Let $\mathcal{V}$ be a variety with a system of Jónsson terms $\mathcal{J}$. Then $\mathcal{V}$ has a system of directed Jónsson terms.*

Proof. Let $\mathbf{F}_3^{id}$ be idempotent reduct of the free three generated algebra in $\mathcal{V}$. Then $\mathbf{F}_3^{id}$ contains a chain of Jónsson terms $\mathcal{J}$ such that $\{x\} \lhd_{\mathcal{J}}^m F_3^{id}$. Applying Theorem 2.2 with $\mathbf{B} = \{x\}$ and $\mathbf{A} = \mathbf{F}_3^{id}$ gives us that there is a chain of directed weak Jónsson terms $\mathcal{D}$ such that $\{x\} \lhd_{\mathcal{D}}^m F_3^{id}$. Every D_i in $\mathcal{D}$ satisfies $D_i(x, y, x) = x$, making $\mathcal{D}$ a chain of directed Jónsson terms for $\mathcal{V}$. •

We are now ready to give a full proof of Theorem 2.1.

Theorem (Theorem 2.1)**.** *Suppose that E and F are admissible preorders on $\boldsymbol{A}$ (that is, they are subalgebras of $\boldsymbol{A}^2$ that are reflexive and transitive). If $E \lhd_J F$ then $E = F$.*

Proof. Let $\mathbf{A} \in \mathcal{V}$, where $\mathcal{V}$ has a weak Jónsson system of terms $\mathcal{J}$ and suppose that E, F are admissible preorders of $\mathbf{A}$ with $E \lhd_{\mathcal{J}} F$. Let $\mathcal{D} = \{D_1, \ldots, D_m\}$ be the system of weak directed Jónsson terms for $\mathbf{A}$ supplied by Theorem 2.2. Then $E \lhd_{\mathcal{D}}^m F$ and so for every $(a, b) \in F$ we have:

$$a = D_1(a, a, b) \, E \, D_1(a, b, b) = D_2(a, a, b) \, E \, D_2(a, b, b) = \cdots$$
$$\cdots = D_m(a, a, b) \, E \, D_m(a, b, b) = b,$$

yielding $(a, b) \in E$. •

5 Pixley terms

We now proceed to prove the statement (2) of Theorem 1.1 (Lipparini's Proposition 5 in [7]).

Theorem 5.1. *Let k be any positive integer and let $\mathcal{V}$ be a $(k+1)$-permutable variety with a system of Jónsson terms $\mathcal{J}$. Then $\mathcal{V}$ has a system of Pixley terms $\mathcal{P} = \{P_1, \ldots, P_k\}$ such that whenever $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{V}$ and $\boldsymbol{B} \lhd_{\mathcal{J}}^m \boldsymbol{A}$, then $\boldsymbol{B} \lhd_{\mathcal{P}}^m \boldsymbol{A}$.*

Proof. The proof is a variant of the proof of Theorem 3.1. Choose and fix an arbitrary idempotent variety $\mathcal{V}$ that has a system $\mathcal{J}$ of Jónsson terms and a system $H_1, \ldots, H_k$ of Hagemann-Mitschke terms, i.e. terms that satisfy the equations

$$H_1(x, z, z) = x, \qquad H_k(x, x, z) = z,$$
$$H_i(x, x, z) = H_{i+1}(x, z, z) \qquad \text{for } 1 \le i < k.$$

Starting as in Section 3, we let $\mathbf{F}_2$ be the free algebra of rank two in $\mathcal{V}$ freely generated by x and z. Let F be the subalgebra of $\mathbf{F}_2^2$ generated by the pairs (x, x), (x, z), and (z, z), that is

$$F = \big\{(t(x, x, z), t(x, z, z)) \colon t \text{ a term of } \mathcal{V}\big\}.$$

As before, we define $\mathcal{G}$ to be the set of all $\mathcal{V}$-terms $t(x, y, z)$ such that whenever $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ and $\mathbf{B} \lhd_{\mathcal{J}}^m \mathbf{A}$, then also $\mathbf{B} \lhd_t^m \mathbf{A}$, and let

$$E = \big\{(t(x, x, z), t(x, z, z)) \colon t(x, y, z) \in \mathcal{G}\big\}.$$

As before, E and F are idempotent admissible relations over $\mathbf{F}_2$ and we have $E \lhd^m_{\mathcal{J}} F$.

Using $p \to q$ to denote that (p, q) belongs to the transitive closure of E, we proved in Sections 3 and 4 that $x \to z$. Since the operations H_i respect E and $\to$, we have that $z \to x$. This is a classical observation, but the proof is easy and so we give it in the following paragraph.

Since $x \to x$, $z \to z$, and $x \to z$, we have

$$\begin{aligned} z = H_1(z,x,x) \to H_1(z,z,x) &= H_2(z,x,x) \\ &\to H_2(z,z,x) = H_3(z,x,x) \to \cdots \to H_k(z,z,x) = x. \end{aligned}$$

Transitivity of $\to$ gives $z \to x$.

We now demonstrate the classical fact that $E^{k+1} = E^k$, which gives us that E^k is the transitive closure of E (and in particular $(z, x) \in E^k$). Since E is reflexive, we have $E^k \subseteq E^{k+1}$. Suppose that we have $(a, b) \in E^{k+1}$. Then there are a_i for $i \leq k + 1$ such that

$$a = a_0 \, E \, a_1 \, E \cdots E \, a_i \, E \, a_{i+1} \, E \cdots E \, a_{k+1} = b.$$

Letting $c_i = H_{i+1}(a_i, a_{i+1}, a_{i+1})$ for $0 \leq i < k$, it is easy to verify that

$$a \, E \, c_1 \, E \, c_2 \, E \cdots E \, c_{k-1} \, E \, b,$$

so $(a, b) \in E^k$.

Continuing with the main proof, we have $(z, x) \in E^k$. This means that there are $\mathcal{V}$-terms

$$D_1(x,y,z), \ldots, D_k(x,y,z) \in \mathcal{G}$$

satisfying $z = D_1(x,x,z)$, $D_i(x,z,z) = D_{i+1}(x,x,z)$ for $1 \leq i < k$, and $D_k(x,z,z) = x$. As before, whenever $\mathbf{A} \lhd^m_{\mathcal{J}} \mathbf{B}$, we also have $\mathbf{A} \lhd^m_{D_1,\ldots,D_k} \mathbf{B}$, so in particular $D_i(x,z,x) = x$. The terms $P_i = D_{k-i+1}$ for $1 \leq i \leq k$ then satisfy $\mathrm{P}(k)$ in $\mathcal{V}$. •

6 Directed Gumm terms

To conclude the proof of Theorem 1.1, we focus on Gumm terms and introduce Gumm absorption. Gumm terms $\mathrm{G}(n)$, directed Gumm terms $\mathrm{DG}(n)$, and weak versions of both were defined in the introduction. (Recall that weak versions drop the conditions $J_i(x,y,x) = x$.)

When a variety $\mathcal{V}$ has a chain of weak Gumm terms (respectively, weak directed Gumm terms) $J_1, \ldots, J_n, P$, and $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ are such that $\mathbf{B} \leq \mathbf{A}$, we say that $\mathbf{B}$ *Gumm absorbs* $\mathbf{A}$ (respectively, *directed Gumm absorbs* $\mathbf{A}$) with respect to these chains if $\mathbf{B} \lhd^m_{J_1,\ldots,J_n} \mathbf{A}$. We next state and prove a variant of Theorem 2.2 for Gumm terms.

Theorem 6.1. *Let $\mathcal{V}$ be a variety, and $J_1, \ldots, J_{2k+1}, P$ be a chain of weak Gumm terms of $\mathcal{V}$. Then there exists a chain $D_1, \ldots, D_m, Q$ of weak directed*

Gumm terms of $\mathcal{V}$ such that whenever $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{V}$ and $\boldsymbol{B} \lhd^m_{\mathcal{J}_1,\ldots,J_{2k+1}} \boldsymbol{A}$, then $\boldsymbol{B} \lhd^m_{\mathcal{D}_1,\ldots,D_m} \boldsymbol{A}$.

In particular, if $J_1, \ldots, J_{2k+1}, P$ is a chain of Gumm terms, then it follows that $D_1, \ldots, D_m, Q$ is a chain of directed Gumm terms.

Note that Theorem 6.1 immediately gives us the third assertion of Theorem 1.1.

Proof. The argument follows the same pattern as our proof of Theorem 2.2. We consider the variety $\mathcal{W}$ and use Theorem 3.1 to obtain terms $D_1, \ldots, D_m$ in $\mathcal{V}$ such that

$$\begin{aligned} &D_1(x, x, z) = x, \\ &D_i(x, z, z) = D_{i+1}(x, z, z) \quad \text{for each } i = 1, \ldots, m-1, \\ &D_m(x, z, z) = J^{2^k}(b, J^{2^k-1}), \end{aligned}$$

where b is some term composed from $J_1, \ldots, J_{2k+1}$, and

$$J(x, y) = J_{2k+1}(x, y, y) = P(x, y, y) \text{ in } \mathcal{V}.$$

The term $J^{2^k}(b, J^{2^k-1})$ can be expressed as

$$\underbrace{J(b, J(b, \ldots, J}_{2^k\text{-many } J\text{'s}}(b, \underbrace{J(x, J(x, \ldots, J}_{(2^k-1)\text{-many } J\text{'s}}(x, z)) \ldots)) \ldots)),$$

More formally, if we let $d_0(x, z) = z$ and

$$\begin{aligned} d_i(x, z) &= J_{2k+1}(x, d_{i-1}(x, z), d_{i-1}(x, z)) &&\text{for } 1 \le i < 2^k, \\ d_i(x, z) &= J_{2k+1}(b(x, z), d_{i-1}(x, z), d_{i-1}(x, z)) &&\text{for } 2^k \le i < 2^{k+1}, \end{aligned}$$

then we will have $d_{2^{k+1}-1}(x, z) = D_m(xzz)$.

Now we systematically rewrite $J^{2^k}(b, J^{2^k-1})$, replacing all but the rightmost occurrence of z by y, and replacing all occurrences of J_{2k+1} by P, to obtain a term $Q(xyz)$.

More formally, we let $Q_0(xyz) = z$, $Q_1(x, y, z) = P(x, y, z)$, and

$$Q_i(x, y, z) = P(x, Q_{i-1}(x, y, y), Q_{i-1}(x, y, z))$$

for $2 \le i < 2^k$, and

$$Q_i(x, y, z) = P(b(x, y), Q_{i-1}(x, y, y), Q_{i-1}(x, y, z))$$

for $2^k \le i < 2^{k+1}$. Having done that, we let $Q(x, y, z) = Q_{2^{k+1}-1}(x, y, z)$.

Using the equality $J_{2k+1}(x, z, z) = P(x, z, z)$, one can easily prove that $Q(x, z, z) = J^{2^k}(b, J^{2^k-1})$ in $\mathbf{F}_2$. Idempotence of b together with $P(x, x, z) = z$ then gives us that $Q(x, x, z) = z$.

Thus we have a chain of weak directed Gumm terms $D_1, \ldots, D_m, Q$. Since $D_1, \ldots, D_m \in \mathcal{G}$, the middle absorption property follows as in Theorem 2.2. Showing that ordinary Gumm terms imply the existence of a chain of directed Gumm terms is analogous to the proof of Corollary 4.1. •

We can now also state and prove a version of Theorem 2.1 for Gumm terms.

Theorem 6.2. *Suppose that $\boldsymbol{E}$ and $\boldsymbol{F}$ are reflexive subalgebras of $\boldsymbol{A}^2$ and that $\boldsymbol{E}$ Gumm absorbs $\boldsymbol{F}$. Whenever $(a, b) \in F$, there is $c \in A$ such that $(b, c) \in F$ and (a, c) belongs to the transitive closure of E.*

Proof. Apply Theorem 6.1 to get weak directed Gumm terms $D_1, \ldots, D_m, Q$ for the variety generated by $\mathbf{A}$ so that $\mathbf{E} \lhd_{D_1,\ldots,D_m} \mathbf{F}$. Then

$$a\, E\, D_1(a, b, b)\, E\, D_2(a, b, b)\, E\, \cdots\, E\, D_m(a, b, b) = Q(a, b, b) = c,$$

where $b = Q(a, a, b)\, F\, Q(a, b, b) = c$. •

7 Final Remarks

We have worked through the various parts of the proof of Theorem 3.1, calculating the precise lengths of the E-chains produced. The final formula for the length of the E-chain connecting x to $J^{2^k}(b, J^{2^k-1})$ simplifies to

$$\frac{(2k+1)(k+1)((k+1)^{k-2}-1)}{k}.$$

Thus, to be precise, we have proved that $J(k)$ implies $DJ(m)$ with m equal to the displayed number. This is our best value for m. It would be interesting to know if a different approach, or the introduction of some new tricks, can lower this value of m substantially. We close the paper by posing a problem stemming from our work here.

Problem 7.1. Does there exist a sequence of algebras $\mathbf{A}_1, \mathbf{A}_2, \ldots$ such that each $\mathbf{A}_n$ is J(n), but the least m such that $\mathbf{A}_n$ is DJ(m) grows at least exponentially in n?

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