Commensurated Subgroups and Ends of Groups

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Abstract

If G is a group, then subgroups A and B are *commensurable* if $A \cap B$ has finite index in both A and B. The *commensurator* of A in G, denoted $Comm_G(A)$, is

$$\{g \in G | (gAg^{-1}) \cap A \text{ has finite index in both } A \text{ and } gAg^{-1}\}.$$

It is straightforward to check that $Comm_G(A)$ is a subgroup of G. A subgroup A is *commensurated* in G if $Comm_G(A) = G$. The centralizer of A in G is a subgroup of the normalizer of A in G which is a subgroup of $Comm_G(A)$. We develop geometric versions of commensurators in finitely generated groups. In particular, $g \in Comm_G(A)$ iff the Hausdorff distance between A and gA is finite. We show a commensurated subgroup of a group is the kernel of a certain map, and a subgroup of a finitely generated group is commensurated iff a Schreier (left) coset graph is locally finite. The ends of this coset graph correspond to the filtered ends of the pair (G, A). This last equivalence is particularly useful for deriving asymptotic results for finitely generated groups. Our primary goals in this paper are to develop and compare the basic theory of commensurated subgroups to that of normal subgroups, and to initiate the development of the asymptotic theory of commensurated subgroups.

1 Introduction

Classically (1966), A. Borel proved several results about irreducible lattices in semisimple Lie groups that cemented commensurators as critical to the theory [4]. In 1975, G. A. Margulis extended these results [17].

The concept of a commensurated subgroup shows up in a variety of settings in the literature. In his Memoirs article on Hecke algebras [13], A. Krieg defines the pair (Q, G) to be "Hecke" if Q is commensurated in G. He shows $GL_n(\mathbb{Z})$ is commensurated in $GL_n(\mathbb{Q})$, for all n.

Margulis' Normal Subgroup Theorem. *Let G be a connected semi-simple Lie group with finite center with $rk_{\mathbb{R}}(G) > 2$, and let Γ be an irreducible lattice. If N is a normal subgroup of Γ, then either N lies in the center of G (and hence N is finite) or the quotient G/N is finite.*

In the preprint [25], Y. Shalom and G. A. Willis use the term "commensurated" and we follow that terminology. They show that for $n > 2$, all commensurated subgroups of $SL_n(\mathbb{Z})$, are finite or of finite index. They also show $SL_n(\mathbb{Z}[1/p])$ satisfies Margulis' theorem but its subgroup $SL_n(\mathbb{Z})$ is commensurated. Still under certain hypotheses, there are commensurated versions of Margulis' theorem (see [25]).

One goal of this paper is to develop the basic theory of commensurated subgroups of groups in analogy with the theory of normal subgroups of groups. There are significant parallels and subtle differences between the two theories. We include a focus on finitely generated groups as a means to examine the geometric group theory of commensurated subgroups and to uncover basic geometric intuition in the subject.

Commensurators seem increasingly important in the study of the geometry and topology of finitely generated groups. In [2] and [3], J. Behrstock and W. Neumann, and J. Behrstock, W. Neumann and T. Januszkiewicz respectively, show quasi-isometry classes of 3-manifold groups and free products of abelian groups are tightly connected to commensurability classes. In [15] and [21], C. Leininger, D. Long, and A. Reid, and M. Mj (respectively) analyze when commensurators of finitely generated Kleinian groups in $PSL(2, \mathbb{C})$ are discrete. In [14], P. Kropholler defines a subgroup Q to be "near normal" in G if Q is commensurated in G. Kropholler proves a generalization of the Lyndon-Hochschild-Serre spectral sequence for group extensions, which replaces the normal subgroup by a commensurated subgroup.

In section 2, we derive several rather technical results. Corollaries 2.4 and 2.5 give geometric interpretations of commensurators in a finitely generated group and provide geometric motivation for what follows. In particular, we show that a subgroup Q of a finitely generated group G is commensurated iff the Hausdorff distance between Q and gQ is finite for every $g \in G$.

Section 3 contains the bulk of the basic theory of commensurated subgroups. We show the intersection of two commensurated subgroups is commensurated, but the intersection of a countable number of commensurated subgroups may not be commensurated. The union of two commensurated subgroups may not generate a commensurated subgroup, but the union of a commensurated subgroup and a normal subgroup generate a commensurated subgroup. The ascending union of commensurated subgroups may not be commensurated. The image and inverse image of a commensurated subgroup under an epimorphism is commensurated. We examine commensurated subgroups in amalgamated products and HNN extensions of groups and show that commensurated subgroups of a word hyperbolic group behave like normal subgroups with respect to limit sets and quasi-convexity.

In section 4, we produce two characterizations of commensurated subgroups of finitely generated groups. First we show that a subgroup of a finitely generated group G is commensurated iff it is the kernel of a certain map of G to a set. In a second characterization of commensurated subgroups, we show a subgroup H of a finitely generated group G is commensurated iff a corresponding Schreier (left) coset graph is locally finite.

In section 5, we show that if Q is a commensurated subgroup of G then G acts transitively on a left coset graph and this graph has either 0, 1, 2 or an uncountable number of ends, in direct analogy with H. Hopf's theorem for finitely generated groups. When Q is finitely generated, we produce a strong equivalence (theorem 5.4) between the ends of the coset graph and the filtered ends of the pair (G, Q) (see Chapter 14 of [8] for a study of filtered ends of a pair of groups). When Q is not finitely generated the filtered ends of the pair (G, Q) still naturally map onto the ends of the coset graph. We give an example of a finitely generated group G and commensurated subgroup Q where the number of ends of $Q\backslash\Gamma$ is countably infinite (for Γ a Cayley graph of G) but the left coset graph has an uncountable number of ends.

In section 6, we examine a connection between a left Schreier coset graph and the bounded packing ideas of C. Hruska and D. Wise [10].

In Section 7, we list some of our asymptotic results that will appear in a separate paper. These are semistability and simple connectivity at infinity results that generalize fundamental results in the subject. We point out results about L^2-Betti numbers of groups where normal subgroups can be replaced by commensurated subgroups, providing generalizations of important theorems in the literature. An analysis of Higman's simple group also appears in this section as well as a series of natural questions associated to this group.

2 Basic Technical Results

If H is a subgroup of G and $g \in G$, then g is an element of the *commensurator* of H in G if $gHg^{-1} \cap H$ has finite index in both gHg^{-1} and H. For H a subgroup of G we write $Comm_H(G)$ for the commensurator of H in G. We include proofs of the next two well known results for completeness.

Lemma 2.1 *Suppose A, B, C and D are subgroups of a group G, A has finite index in B, and C has finite index in D. Then $A \cap C$ has finite index in $B \cap D$.*

Proof: First we show $A \cap D$ has finite index in $B \cap D$. Write $B = \cup_{i=1}^{n} Ab_i$ for $b_i \in B$. If Ab_i contains an element of D then we assume that $b_i \in D$. Reordering, $B = (\cup_{i=1}^{m} Ad_i) \cup (\cup_{i=m+1}^{n} Ab_i)$, where $d_i \in D$ for $i \leq m$ and $(Ab_i) \cap D = \emptyset$ for $i > m$. Then $D \cap B = \cup_{i=1}^{m} D \cap (Ad_i)$. Since $d_i \in D$ we have $D \cap (Ad_i) = (D \cap A)d_i$ and so $A \cap D$ has finite index in $B \cap D$.

The same proof shows $A \cap C$ has finite index in $A \cap D$ (which has finite index in $B \cap D$)) and so $A \cap C$ has finite index in $B \cap D$. $\square$

Lemma 2.2 *If H is a subgroup of a group G, then $Comm_H(G)$ is a subgroup of G.*

Proof: If $g \in Comm_H(G)$ then $H \cap gHg^{-1}$ has finite index in H and gHg^{-1} so $g^{-1}(H \cap gHg^{-1})g \equiv g^{-1}Hg \cap H$ has finite index in $g^{-1}Hg$ and H, so $g^{-1} \in Comm_H(G)$.

If $g, k \in Comm_H(G)$ then

1) $H \cap gHg^{-1}$ has finite index in H and gHg^{-1}, and

2) $H \cap kHk^{-1}$ has finite index in H and kHk^{-1}.

Conjugating 1) by k we have:

3) $kHk^{-1} \cap kgHg^{-1}k^{-1}$ has finite index in $kgHg^{-1}k^{-1}$.

By lemma 2.1, and equations 2) and 3) we have:

4) $H \cap kHk^{-1} \cap kgHg^{-1}k^{-1}$ has finite index in $kHk^{-1} \cap kgHg^{-1}k^{-1}$.

By equations 3) and 4):

5) $H \cap kHk^{-1} \cap kgHg^{-1}k^{-1}$ has finite index in $kgHg^{-1}k^{-1}$.

As $H \cap kgHg^{-1}k^{-1}$ sits between the two groups of 5) we have:

$*$) $H \cap kgHg^{-1}k^{-1}$ has finite index in $kgHg^{-1}k^{-1}$.

Since both k^{-1} and g^{-1} are in $Comm_H(G)$, equation $*$) implies:

$*'$) $H \cap g^{-1}k^{-1}Hkg$ has finite index in $g^{-1}k^{-1}Hkg$.

Conjugating by kg gives:

$**)$ $kgHg^{-1}k^{-1} \cap H$ has finite index in H.
Equations $*)$ and $**)$ imply $kg \in Comm_H(G)$. $\square$

The next result "morally" says that if there is any sort of metric on a group G, and $g \in Comm_H(G)$ then the "Hausdorff" distance between H and gH in G is bounded. Corollary 2.4 is a geometric version of this theorem when G is finitely generated.

Theorem 2.3 *If H is a subgroup of G and $g \in G$ then $g \in Comm_H(G)$ iff there are finite subsets A and B of G such that for each $h \in H$ there is an $a \in A$ and $b \in B$ such that $ha \in gH$, and $ghb \in H$. (Equivalently, there is $a \in A$ such that $h(ag^{-1}) \in gHg^{-1}$ and $b \in B$ such that $ghg^{-1}(gb) \in H$.)*

Proof: Suppose $g \in Comm_H(G)$. Choose $h_i \in H$ such that

$$\cup_{i=1}^{n}(gHg^{-1} \cap H)h_i = H$$

For $h \in H$, say $h = xh_i$ for some $i \in \{1, \ldots, n\}$ and some $x \in (gHg^{-1} \cap H)$. Then $x = hh_i^{-1} \in (gHg^{-1} \cap H)$ and $hh_i^{-1}g \in gH$. So, we can let A be the finite set $\{h_1^{-1}g, \ldots, h_n^{-1}g\}$. Since $Comm_H(G)$ is a subgroup of G, $g^{-1} \in Comm_H(G)$. By the preceding argument, there is a finite subset B of G such that for each $h \in H$ there is a $b \in B$ such that $hb \in g^{-1}H$. Equivalently, $ghb \in H$.

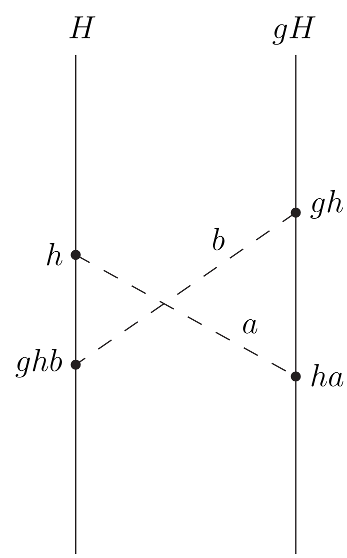

Figure 1

Assume A and B are finite subsets of G satisfying the conclusion of the theorem. Define a function $\alpha : H \to A$ such that $h\alpha(h) \in gH$. Suppose $h_1, h_2 \in H$ and $\alpha(h_1) = \alpha(h_2)$. As $h_1\alpha(h_1)$ and $h_2\alpha(h_2)$ are elements of gH, we have $h_1\alpha(h_1)Hg^{-1} = h_2\alpha(h_2)Hg^{-1} = gHg^{-1}$. Then

$$h_2h_1^{-1}gHg^{-1} = h_2h_1^{-1}h_1\alpha(h_1)Hg^{-1} = h_2\alpha(h_2)Hg^{-1} = gHg^{-1}$$

In particular:

$$\text{If } \alpha(h_1) = \alpha(h_2) \text{ then } h_2h_1^{-1} \in gHg^{-1} \cap H.$$

Say $im(\alpha) = \{a_1, \ldots, a_n\}$ and select $h_i \in H$ such that $\alpha(h_i) = a_i$. If $h \in H$ and $\alpha(h) = a_i$, then $hh_i^{-1} \in H \cap gHg^{-1}$ and $h \in (H \cap gHg^{-1})h_i$. We have $H = \cup_{i=1}^n (H \cap gHg^{-1})h_i$ and $H \cap gHg^{-1}$ has finite index in H.

For each $h \in H$ there is $b \in B$ such that $ghb \in H$, and so $hb \in g^{-1}H$. The preceding argument implies $H \cap g^{-1}Hg$ has finite index in H. Conjugating (by g^{-1}) gives $gHg^{-1} \cap H$ has finite index in gHg^{-1}. $\square$

If S is a finite generating set for a group G, $\Gamma(G, S)$ the Cayley graph of G with respect to S, and H a subgroup of G, then for any $g_1, g_2 \in G$, the *Hausdorff* distance between g_1H and g_2H, denoted $D_S(g_1H, g_2H)$, is the smallest integer K such that for each element h of H the edge path distance from g_1h to g_2H in Γ is $\leq K$ and the edge path distance from g_2h to g_1H in Γ is $\leq K$. If no such K exists, then $D_S(g_1H, g_2H) = \infty$.

As a direct consequence of theorem 2.3 we have:

Corollary 2.4 *Suppose S is a finite generating set for a group G and H is a subgroup of G, then $g \in G$ is in $Comm_H(G)$ iff the Hausdorff distance $D_S(H, gH) < \infty$ iff $D_S(H, gHg^{-1}) < \infty$.*

In particular, a subgroup Q of a finitely generated group G is commensurated in G iff the Hausdorff distance $D(Q, gQ)$ is finite for all $g \in G$ iff $D(Q, gQg^{-1})$ is finite for all $g \in G$.

Corollary 2.5 *Suppose H is a subgroup of a group G and $g \in Comm_H(G)$ then there are finite sets $U(g)$ and $V(g)$ such that $gH \subset \cup_{v \in V(g)} Hv$ and $Hg \subset \cup_{u \in U(g)} uH$.*

Proof: By theorem 2.3, there is a finite set B so that for each $h \in H$ there is $b \in B$ such that $ghb \in H$. Then $gh \in Hb^{-1}$ and $gH \subset H \cdot B^{-1}$.

Define $V(g) \equiv B^{-1}$. Since $g^{-1} \in Comm_H(G)$ we have $g^{-1}H \subset H \cdot V(g^{-1})$. Inverting, $Hg \subset (V(g^{-1}))^{-1} \cdot H \equiv U(g) \cdot H$. $\square$

In [13], A. Krieg has an elementary proof that a subgroup H of a group G is commensurated iff for each $g \in G$, HgH is contained in a finite number of left (right) H cosets. We prove a slightly more general result.

Lemma 2.6 *Suppose H is a subgroup of G and $g \in Comm_H(G)$ then HgH is contained in the union of a finite number of left (right) H cosets. If $gH \subset \cup_{i=1}^n Hx_i$ and $g^{-1}H \subset \cup_{i=1}^m Hy_i$ then $g \in Comm_H(G)$.*

Proof: Note that $Hg \subset Comm_H(G)$ and for each $h \in H$, the finite sets B of theorem 2.3 for g and hg can be selected to be the same (if $h_1 \in H$ and $gh_1b \in H$ for some $b \in B$ then for any $h \in H$, $hgh_1b \in H$). If $x \in HgH$ there is $b \in B$ such that $xb \in H$. I.e. $x \in bH$ and $HgH \subset BH$. If B' is the corresponding finite set for $g^{-1} \in Comm_H(G)$, then $Hg^{-1}H \subset B'H$. Inverting we have $HgH \subset (B')^{-1}H$.

For the second part, if $gH \subset \cap_{i=1}^n Hx_i$ then for $h \in H$ there is an $i \in \{1, \ldots, n\}$ such that $ghx_i \in H$. If $g^{-1}H \subset \cup_{i+1}^m Hy_i$ then for $h \in H$ there is an $i \in \{1, \ldots, m\}$ such that $g^{-1}h \in Hy_i$ and $hy_i^{-1} \in gH$. Set $A = \{x_1, \ldots, x_n\}$ and $B = \{y_1^{-1}, \ldots, y_m^{-1}\}$, and apply theorem 2.3. $\square$

The next two results are independent from the rest of the paper, except the next corollary is used to construct an interesting transversal in lemma 3.6.

Corollary 2.7 *Suppose H is a subgroup of G and $g \in Comm_H(G)$. There is a finite subset $A(g, H, G)$ of $HgH \subset Comm_H(G)$ and functions $\alpha_{(g,H,G)}$ and $\beta_{(g,H,G)}$ (written $A(g)$, α_g, and β_g respectively, when H and G are unambiguous) such that:*

1) $\alpha_g : H \to A(g)$ and $\beta_g : H \to (A(g))^{-1}$,

2) for each $h \in H$, $h\alpha_g(h) \in gH$ and $gh\beta_g(h) \in H$,

3) $[image(\alpha_g) \cup (image(\beta_g))^{-1}] = A(g)$, and

4) for each $a \in A(g)$, there is $h_1 \in H$ such that $h_1a \in gH$, and $h_2 \in H$ such that $gh_2a^{-1} \in H$. (It is not possible to make $A(g)$ symmetric.)

Proof: Consider the sets A and B of theorem 2.3. There are functions $\alpha : H \to A$ and $\beta : H \to B$ such that for each $h \in H$, $h\alpha(h) \in gH$ and $gh\beta(h) \in H$. Without loss, we assume α and β are onto. Define $A(g) \equiv$

$A \cup B^{-1}$. Define $\alpha_g : H \to A(g)$ to agree with α and $\beta_g : H \to (A(g))^{-1}$ to agree with β.

If $a \in A \subset A(g)$ then any element in $\alpha^{-1}(a)$ will play the role of h_1 in our result. As $h_1 a \in gH$, we write $h_1 a = gh_2$ and $gh_2 a^{-1} \in H$. If $b \in B^{-1} \subset A(g)$, let $h_2 \in H$ be such that $\beta_g(h_2) = b^{-1}$. Then $gh_2\beta_g(h_2) = gh_2 b^{-1} \in H$. Say $gh_2 b^{-1} = h_1$. Then $gh_2 = h_1 b \in gH$. $\square$

Remark 1. The next result applies locally. It would be interesting to extend this to a more general global result.

Corollary 2.8 *Suppose $g \in Comm_H(G)$.*

1) $A(g^{-1})$ may be selected to be $(A(g))^{-1}$ with $\alpha_{g^{-1}} \equiv \beta_g$ and $\beta_{g^{-1}} \equiv \alpha_g$.

2) If $k \in A(g)$ then $A(k)$ may be selected to be $A(g)$ with $\alpha_k(h) \equiv \alpha_g(h_1 h)$ where $h_1 k = gh_1'$ for some $h_1, h_1' \in H$ and $\beta_k(h) \equiv \beta_g(h_2 h)$ where $gh_2 = h_2' k$ for some $h_2, h_2' \in H$.

Proof: Suppose $h \in H$ then $h\alpha_g(h) \in gH$, and $g^{-1}h\alpha_g(h) \in H$. So, we may define $\beta_{g^{-1}}(h) \equiv \alpha_g(h) \in A(g)$. As $gh\beta_g(h) \in H$, $h\beta_g(h) \in g^{-1}H$ and we may define $\alpha_{g^{-1}}(h) \equiv \beta_g(h) \in (A(g))^{-1}$.

Suppose $k \in A(g)$ then $h_1 k = gh_1'$ for some $h_1, h_1' \in H$. Define $\alpha_k : H \to A(g)$ by $\alpha_k(h) \equiv \alpha_g(h_1 h)$. Then $h_1 h\alpha_k(h) = h_1 h\alpha_g(h_1 h) \in gH = h_1 kH$ and so $h\alpha_k(h) \in kH$, as required.

The equality $gh_2 k^{-1} = h_2'$ is valid for some $h_2, h_2' \in H$. Define $\beta_k : H \to (A(g))^{-1}$ by $\beta_k(h) = \beta_g(h_2 h)$. As $gh_2 h\beta_k(h) = gh_2 h\beta_g(h_2 h) \in H$, substituting for g shows $(h_2' k h_2^{-1})h_2 h\beta_k(h) \in H$ and $kh\beta_k(h) \in H$. $\square$

Remark 2. As one might expect, there is some overlap in the literature with the elementary results in this section. As pointed out earlier, one of Krieg's results in [13], is basically the same as our lemma 2.6, In the preprint [22], L. Mosher, M. Sageev and K. Whyte show that two subgroups A and B of G are commensurable in G if and only if the Hausdorff distance between A and B is finite (compare with corollary 2.4).

3 Examples and Basic Facts for Commensurated Subgroups

In order to check that a subgroup Q of a group G is commensurated it suffices to show that a set of generators of G is contained in $Comm_Q(G)$. This is particularly useful when G is finitely generated.

Example 1. We show that the subgroup $\langle x \rangle$ is commensurated in the Baumslag-Solitar group $BS(m,n) \equiv \langle t, x : t^{-1}x^m t = x^n \rangle$.

As $(x\langle x \rangle x^{-1}) \cap \langle x \rangle = \langle x \rangle$, $x \in Comm_{\langle x \rangle}(BS(m,n))$. Next observe that $(t^{-1}\langle x \rangle t) \cap \langle x \rangle = \langle x^n \rangle$. Certainly $\langle x^n \rangle$ has finite index in $\langle x \rangle$. As $\langle x^m \rangle$ has finite index in $\langle x \rangle$, $t^{-1}\langle x^m \rangle t = \langle x^n \rangle$ has finite index in $t^{-1}\langle x \rangle t$ and $t^{-1} \in Comm_{\langle x \rangle}(BS(m,n))$. As t^{-1} and x generate $BS(m,n)$, $\langle x \rangle$ is commensurated in $BS(m,n)$.

Note that the normal closure of x in $BS(1,2)$ is the commutator subgroup of $BS(1,2)$ and is isomorphic to the dyadic rationals.

Lemma 3.1 *Suppose Q is a commensurated subgroup of a group G and H is a subgroup of G, then $Q \cap H$ is commensurated in H.*

Proof: For each $h \in H$, $(h^{-1}Qh) \cap Q$ has finite index in both Q and $h^{-1}Qh$. Then $(h^{-1}Qh) \cap Q \cap H \equiv [h^{-1}(Q \cap H)h] \cap (Q \cap H)$ has finite index in both $Q \cap H$ and $(h^{-1}Qh) \cap H \equiv h^{-1}(Q \cap H)h$. $\square$

Lemma 3.2 *If Q is a normal, finite or a finite index subgroup of a group G, then Q is commensurated in G. If Q is commensurated in G then for any automorphism α of G, $\alpha(Q)$ is commensurated in G.*

Proposition 3.3 *Suppose A and B are commensurated subgroups of a group G. Then $A \cap B$ is commensurated in G.*

Proof: For $g \in G$, $A \cap gAg^{-1}$ has finite index in both A and gAg^{-1}, and $B \cap gBg^{-1}$ has finite index in both B and gBg^{-1}. Hence by lemma 2.1, $(A \cap B) \cap g(A \cap B)g^{-1} \equiv (A \cap gAg^{-1}) \cap (B \cap gBg^{-1})$ has finite index in both $A \cap B$ and $gAg^{-1} \cap gBg^{-1} \equiv g(A \cap B)g^{-1}$. $\square$

Example 2. The arbitrary intersection of commensurated subgroups need not be commensurated. In 1949, M. Hall Jr. proved [11] that free groups are subgroup separable. A group G is *subgroup separable* if any finitely generated subgroup of G is the intersection of subgroups of finite index in G. In particular, any infinite cyclic subgroup A of $F_2 \equiv \langle x, y \rangle$, the free group of rank 2, is the intersection of subgroups of finite index in F_2. By lemma 3.2, each subgroup of finite index in F_2 is commensurated in F_2, but if $A = \langle x \rangle$, then

$\langle x\rangle \cap y\langle x\rangle y^{-1} = \{1\}$. So A is the intersection of commensurated subgroups (of finite index in F_2), but A is not commensurated in F_2. $\Box$

The next example shows that the ascending union of commensurated subgroups is not necessarily commensurated.

Example 3. Let

$$H \equiv \langle x_0, x_1, \ldots : x_0^{2^k} = x_k^2, \text{ for } k \geq 1, [x_i, x_j] = 1 \text{ for } i, j \geq 0\rangle \text{ and}$$

$$H_n \equiv \langle x_0, \ldots, x_n : x_0^{2^k} = x_k^2 \text{ for } 1 \leq k \leq n, [x_i, x_j] = 1 \text{ for } 0 \leq i, j \leq n\rangle.$$

The map $i_n(x_k) = x_k$ for $0 \leq k \leq n$, and the map $q_n(x_k) = x_k$ for $0 \leq k \leq n$ and $q_n(x_k) = x_0^{2^{k-1}}$ for $k > n$ extend to homomorphisms $i_n : H_n \to H$ and $q_n : H \to H_n$. The composition $q_n i_n$ is the identity on H_n and so the subgroup of H generated by $\{x_0, \ldots, x_n\}$ is isomorphic to H_n (and a retract of H). We identify H_n with this subgroup. As $H_0 \equiv \langle x_0\rangle$ is infinite cyclic, x_n is of infinite order in H for all n. Consider the monomorphism of H_0 determined by $x_0 \to x_0^2$. If G is the resulting HNN-extenssion, then G has presentation:

$$G \equiv \langle t, x_0, x_1, \ldots : t^{-k} x_0 t^k = x_0^{2^k} = x_k^2, \text{ for } k \geq 1, [x_i, x_j] = 1 \text{ for } i, j \geq 0\rangle.$$

Note that each generator in this presentation of G has infinite order. Now

$$(x_i^{-1}\langle x_0\rangle x_i) \cap \langle x_0\rangle = (x_i\langle x_0\rangle x_i^{-1}) \cap \langle x_0\rangle = \langle x_0\rangle$$

and

$$(t^{-1}\langle x_0\rangle t) \cap \langle x_0\rangle = \langle x_0^2\rangle \text{ and } (t\langle x_0\rangle t^{-1}) \cap \langle x_0\rangle = \langle x_0\rangle.$$

Hence the infinite cyclic group $\langle x_0\rangle$ is commensurated in G.

The group $\langle x_0\rangle$ has finite index in the abelian group $H_n \equiv \langle x_0, \ldots, x_n\rangle$. In fact $H_n/\langle x_0\rangle$ is isomorphic to $\oplus_{i=1}^n \mathbb{Z}_2$. By lemma 3.7 (below), H_n is commensurated in G for all $n \geq 0$.

The group H is the ascending union of the nested groups H_n. We prove H is not commensurated in G, by showing $t^{-1}Ht \cap H = \langle x_0^2\rangle$ (which has infinite index in H).

Suppose $g \in t^{-1}Ht \cap H$. Let $g = t^{-1}ht \in H$ for some $h \in H$. By lengths of normal forms for the HNN extension G, it must be that h is an element of the associated subgroup $\langle x_0\rangle$ (i.e. elements of the base group H of the HNN extension G, have length 1, but $t^{-1}ht$ has length 3 unless $h \in \langle x_0 \rangle$). Now $g = t^{-1}x_0^k t = x_0^{2k}$, and so H is not commensurated in G.

In lemma 3.5, we show that the inverse image of a commensurated subgroup under an epimorphism is commensurated. In our example, consider the epimorphism $q_0' : G \to \langle t, x_0 : t^{-1}x_0 t = x_0^2 \rangle$ where $q_0'(t) = t$, $q_0'(x_0) = x_0$ and $q_0'(x_k) = x_0^{2^{k-1}}$, for $k > 0$. The subgroup $(q_0')^{-1}(\langle x_0 \rangle)$ is commensurated in G and has generating set $\langle x_0, x_1, tx_2t^{-1}, \ldots, t^{k-1}x_k t^{-(k-1)}, \ldots \rangle$. $\square$

Lemma 3.4 *Suppose $f : G_1 \to G_2$ is an epimorphism and Q is commensurated in G_1 then $f(Q)$ is commensurated in G_2.*

Proof: For $g_2 \in G_2$ let $g_1 \in G_1$ be such that $f(g_1) = g_2$. As $Q \cap g_1 Q g_1^{-1}$ has finite index in Q and $g_1 Q g_1^{-1}$ we have $Q = \cup_{i=1}^n (Q \cap g_1 Q g_1^{-1}) q_i$ for some $q_i \in Q$. Then $f(Q) = \cup_{i=1}^n (f(Q) \cap g_2 f(Q) g_2^{-1}) f(q_i)$ (since f is an epimorphism) and $f(Q) \cap g_2 f(Q) g_2^{-1})$ has finite index in Q. Similarly for $g_2 f(Q) g_2^{-1}$. $\square$

Lemma 3.5 *Suppose $f : G_1 \to G_2$ is a homomorphism and Q is commensurated in G_2 then $f^{-1}(Q)$ is commensurated in G_1.*

Proof: For $g \in G$ we first show that $f^{-1}[f(g)Qf(g^{-1})] = gf^{-1}(Q)g^{-1}$: Note that $x \in f^{-1}[f(g)Qf(g^{-1})]$ iff $f(x) = f(g)qf(g^{-1})$ for some $q \in Q$ iff $f(g^{-1})f(x)f(g) \in Q$ iff $f(g^{-1}xg) \in Q$ iff $g^{-1}xg \in f^{-1}(Q)$ iff $x \in gf^{-1}(Q)g$.

Since $Q \cap f(g)Qf(g^{-1})$ has finite index in both Q and $f(g)Qf(g^{-1})$, $f^{-1}[Q \cap f(g)Qf(g^{-1})] = f^{-1}(Q) \cap gf^{-1}(Q)g^{-1}$ has finite index in both $f^{-1}(Q)$ and $gf^{-1}(Q)g^{-1}$. $\square$

Lemma 3.6 *If Q is commensurated in G and N is a normal subgroup of G then the subgroup of G generated by Q and N is commensurated in G. Furthermore, one may arrange things so that: There is a transversal T for N in $\langle Q \cup N \rangle$ such that $T \subset Q$, and for any $g \in G$, $t \in T$ and $n \in N$:*

$$\alpha_{(g,\langle Q \cup N \rangle, G)}(tn) = \alpha_{(g,Q,G)}(t) \text{ and } \beta_{(g,\langle Q \cup N \rangle, G)}(tn) = \beta_{(g,Q,G)}(t)$$

Proof: Let $q : G \to G/N$ be the quotient map. By lemmas 3.5 and 3.6, $q^{-1}(q(Q))(\equiv \langle Q, N \rangle)$ is commensurated in G.

For the second part of the lemma, we write $A(g)$ for $A(g, Q, G)$ and α_g for $\alpha_{(g,Q,G)}$ for all $g \in G$. As N is normal in G, each element $f \in \langle Q \cup N \rangle$ can be written as qn for some $q \in Q$ and some $n \in N$. Hence there is a transversal

$T \subset Q$ for N in $\langle Q \cup N \rangle$. Suppose $g \in G$, $t \in T$ and $n \in Q$. By corollary 2.7 there is $q' \in Q$ such that $t\alpha_g(t) = gq'$. Let $\alpha_g(t)^{-1} n \alpha_g(t) = n' \in N$. Then

$$tn\alpha_g(t) = t\alpha_g(t)\alpha_g(t)^{-1}n\alpha_g(t) = gq'n'.$$

So we define $\alpha_{(g,\langle Q\cup N\rangle,G)}(tn) = \alpha_{(g,Q,G)}(t)$ for all $n \in N$.

By corollary 2.7, there is $\hat{q} \in Q$ such that $gq\beta_g(t) = \hat{q}$. Let $\hat{n} = \beta_g(t)^{-1}n\beta_g(t) \in N$. Then

$$gtn\beta_g(t) = gt\beta_g(t)\beta_g(t)^{-1}n\beta_g(t) = \hat{q}\hat{n}.$$

So we define $\beta_{(g,\langle Q\cup N\rangle,G)}(tn) = \beta_{(g,Q,G)}(t)$ for all $n \in N$. $\square$

In [24], E. Rips, shows that for any finitely presented group G there is a word hyperbolic group H and epimorphism $f : H \twoheadrightarrow G$ with finitely generated kernel. By lemmas 3.5 and 3.6, if (in this setting) Q is a finitely generated commensurated subgroup of G, then $f^{-1}(Q)$ is finitely generated and commensurated in H.

Lemma 3.7 *Suppose Q is a commensurated subgroup of G and Q' is a subgroup of G such that either Q' has finite index in Q, or Q has finite index in Q', then Q' is commensurated in G.*

Proof: Suppose Q' has finite index in Q and $g \in G$. Then lemma 2.1 implies $Q' \cap gQ'g^{-1}$ has finite index in $Q \cap gQg^{-1}$. Since $Q \cap gQg^{-1}$ has finite index in both Q and gQg^{-1}, so does $Q' \cap gQ'g^{-1}$. But then $Q' \cap gQ'g^{-1}$ has finite index in Q' and $gQ'g^{-1}$. Similarly if Q has finite index in Q'. $\square$

In the next example we show that the union of two commensurated subgroups may not generate a commensurated subgroup, in contrast to lemma 3.6 which shows the union of a commensurated subgroup and a normal subgroup generates a commensurated subgroup.

Example 4. Let H be the group $\langle x, y : x^2 = y^2 \rangle$ and G the HNN extension with base H and associates subgroups $\langle x^2 \rangle$ and $\langle x^4 \rangle$ with presentation:

$$G \equiv \langle x, y, t : x^2 = y^2, t^{-1}x^2t = x^4 \rangle.$$

To see that $\langle x^2 \rangle = \langle y^2 \rangle$ is commensurated in G, simply observe that:

$$x\langle x^2 \rangle x^{-1} = y\langle x^2 \rangle y^{-1} = \langle x^2 \rangle \text{ and } t^{-1}\langle x^2 \rangle t = \langle x^4 \rangle$$

so that x, y and t^{-1} are in $Comm_{\langle x^2\rangle}(G)$. By lemma 3.7, $\langle x\rangle$ and $\langle y\rangle$ are commensurated in G. We show that $\langle x, y\rangle$ is not commensurated in G. Note that

$$\langle x, y\rangle/\langle x^2\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2.$$

It suffices to show $t^{-1}\langle x, y\rangle t \cap \langle x, y\rangle = \langle x^4\rangle$. Suppose $z \in t^{-1}\langle x, y\rangle t \cap \langle x, y\rangle$. Write $z = t^{-1}ut$ for some $u \in \langle x, y\rangle$. As z is in $\langle x, y\rangle$, the base group of the HNN extension G, lengths of normal forms implies that u is in the domain associated subgroup $\langle x^2\rangle$ (elements of the base group have length 1, and $t^{-1}ut$ has length 3 unless $u \in \langle x^2\rangle$). Then $z = t^{-1}x^{2k}t = x^{4k}$. $\square$

Lemma 3.8 *Suppose Q_1 is a commensurated subgroup of the group H, Q_1 and Q_2 are commensurable in H and $h : Q_1 \to Q_2$ is an isomorphism, then Q_1 is commensurated in the HNN-extension $G \equiv \langle t, H : t^{-1}qt = h(q)$ for all $q \in Q_1\rangle$.*

Proof: By hypothesis $t^{-1}Q_1t \cap Q_1 = Q_2 \cap Q_1$ has finite index in Q_2 and Q_1. Since t^{-1} and H generate G, Q_1 is commensurated in G. $\square$

As a direct consequence of the previous lemma we have:

Lemma 3.9 *If $f : H \to H$ is a monomorphism and $f(H)$ has finite index in H, then H (and by the previous fact $f(H)$) is a commensurated subgroup of the (ascending) HNN extension $G = H*_f$.*

Lemma 3.10 *If $G = G_1 *_Q G_2$ and Q is commensurated in G_i for $i \in \{1, 2\}$ then Q is commensurated in G.*

Proof: For each $g \in \{G_1 \cup G_2\}$, $g^{-1}Qg \cap Q$ has finite index in both Q and $g^{-1}Qg$, by hypothesis. As $G_1 \cup G_2$ generates G and is a subset of the subgroup $Comm_Q(G)$ of G (see corollary 2.2), $G = Comm_Q(G)$. $\square$

In [20], M. Mihalik and W. Towle proved that an infinite quasi-convex subgroup of a word hyperbolic group has finite index in its normalizer. The same proof shows:

Theorem 3.11 *Suppose H is an infinite quasi-convex subgroup of a word hyperbolic group G then H has finite index in its commensurator.*

Proof: (Outline) Let a be an element of $Q \equiv Comm_H(G)$. It suffices to bound the distance from a to H in Γ, a Cayley graph of G with respect to some finite generating set (containing a set of generators for H). Let $\alpha \equiv (\ldots, h_{-1}, h_0, h_1, \ldots)$ be a bi-infinite geodesic in the generators of H (so that α is quasi-geodesic in Γ). Assume the initial vertex of h_0 is $a \equiv x_0$ and the initial point of h_n is x_n. Choose N large, with respect to the Hausdorff distance $D \equiv D(H, aH)$ in Γ. Let x be a point of H within D of x_N. Consider the geodesic rectangle $([1,a],[a,x_N],[x_N,x],[x,1])$. By thin geodesic triangles, some x_i (for $1 \leq i \leq N$) is within D_1 of $x'_i \in H$ (where D_1 only depends on δ, the thin triangle constant and the quasi-convexity constants for α and H). Similarly there is a j such that $-N \leq j \leq -1$ such that x_j is within D_1 of $x'_j \in H$.

The geodesic quadrilateral $([x_j,x_i],[x_i,x'_i],[x'_i,x'_j],[x'_j,x_j])$ has (opposite) sides of length $\leq D_1$, implying each point of $[x_j,x_i]$ is close to each point of $[x'_j,x'_i]$. As a is close to $[x_j,x_i]$, a is close to $[x'_j,x'_i]$ and so a is close to a point of H. $\Box$

Theorem 3.12 *The limit set of an infinite commensurated subgroup of a word hyperbolic group H is the entire boundary of H.*

Proof: (This proof is basically the same as the standard one for normal subgroups.) Let Q be an infinite commensurated subgroup of a word hyperbolic group H. Let Γ be a Cayley graph for H (on a finite generating set).

$(*)$ By the definition of commensurated, the limit set ∂Q is the same as $\partial(hQ)$ in Γ for all $h \in H$.

As word hyperbolic groups have only finitely many conjugacy classes of finite subgroups, Q contains an element a, of infinite order. Let $a^{\pm\infty} = \partial\langle a\rangle$ in Γ. Let $\alpha = (\ldots, a_{-1}, a_0, a_1, \ldots)$ be a bi-infinite geodesic edge path in Γ with $\partial(\alpha) = a^{\pm\infty}$. As elements of infinite order determine quasi-geodesics in Γ, α is of bounded distance D from $Q \subset \Gamma$. In particular, $a^{\pm\infty} \in \partial Q$.

Let x_0 be the initial vertex of the edge a_0, $\beta = (b_1, b_2, \ldots)$ a geodesic edge path in Γ beginning at x_0, b^∞ the boundary point of β, y_i the initial point of b_i, and let $g_i \in H$ be the group element such that $g_i x_0 = y_i$. Consider the ideal triangle with sides $g_i\alpha$, $[x_0, g_i(a^\infty))$ and $[x_0, g_i(a^{-\infty}))$. Since y_i is a vertex of $g(\alpha)$, one of the two sides of the ideal triangle, $[x_0, g_i(a^\infty))$ or $[x_0, g_i(a^{-\infty}))$, passes within δ (the hyperbolic constant for thin triangles in Γ) of y_i. Hence b^∞ is a limit point of the boundary points of $\{g_i(\alpha)\}_{i=1}^\infty$. As $g_i(\alpha)$ is within D of $g_i(Q)$, $\partial(g_i(\alpha)) \subset \partial(g_iQ) = \partial Q$ (see $(*)$). Hence $b \in \partial Q$. As β was arbitrary $\partial Q = \partial H$. $\Box$

We end this section by showing there is not an analogue to the notion of regular covering for commensurated subgroups. If G is a group and H a subgroup of G then the *normalizer* of H in G, denoted $N(H,G)$ is $\{g \in G : gHg^{-1} = H\}$. Suppose X is a connected finite complex and $\pi_1(X) = G$. Let $p : \tilde{X} \to X$ be the universal covering of X. If H is a subgroup of G then the deck transformations of $H\backslash\tilde{X}$ are isomorphic to $N(H,G)/H$, and H is normal in G iff the deck transformations of $H\backslash\tilde{X}$ act transitively on each fiber (over X). When H is commensurated in G one might expect the deck transformations to act co-compactly on $H\backslash\tilde{X}$. This is not the case unless $N(H,G)$ has finite index in G.

Simply consider the quotient map $r : H\backslash\tilde{X} \to (H\backslash N(H,G))\backslash(H\backslash\tilde{X}) = N(H,G)\backslash\tilde{X}$, and recall that $N(H,G)\backslash\tilde{X}$ is compact iff $N(H,G)$ has finite index in G. The subgroup $\langle x\rangle$ is commensurated in $BS(1,2) = \langle x, t : t^{-1}xt = x^2\rangle \equiv G$, and $N(\langle x\rangle, G)$ is the same as the normal closure of $\langle x\rangle$ in G. So $G/N(\langle x\rangle) \approx \mathbb{Z}$.

4 Characterizations of commensurated subgroups of finitely generated groups

In this section we produce two characterizations of commensurated subgroups of finitely generated groups that connect the theory to both well developed and emerging ideas in group theory.

Lemma 4.1 *Suppose Q is a subgroup of the finitely generated group G. Fix a finite generating set, S, for G, and let $|\cdot|$ be the corresponding word-length norm on G, let d be the induced left invariant word metric on G where $d(a,b) = |b^{-1}a|$, and D be the corresponding Hausdorff metric on subsets of G.*

Suppose Q is commensurated in G. Let $k = \max_{s\in S}(D(sQ,Q)) + 1$. Then for all $a, b \in G$ we have the following:

1. *$D(bQ, Q) \le k|b|$*
2. *$D(bQb^{-1}, Q) \le (k+1)|b|$*
3. *$D(QbQ, Q) \le k|b|$.*

4. $D(aQbQ, abQ) = D(QbQ, bQ) \leq 2k|b|$

Proof: Let $b = b_1 b_2 \cdots b_n$ where each $b_i \in S$. Then

$$D(bQ, Q) \leq D(bQ, b_1 \cdots b_{n-1}Q) + D(b_1 \cdots b_{n-1}Q, b_1 \cdots b_{n-2}Q) + \cdots + D(b_1Q, Q)$$

$$= D(b_nQ, Q) + D(b_{n-1}Q, Q) + \cdots + D(b_1Q, Q) \leq k|b|$$

by left invariance. Then

$$D(bQb^{-1}, Q) \leq D(bQb^{-1}, bQ) + D(bQ, Q) \leq |b| + k|b| = (k+1)|b|.$$

Also, for any $q, q' \in Q$,

$$d(q'bq, Q) = d(bq, Q) \leq D(bQ, Q) \leq k|b| \text{ and}$$

$$d(QbQ, q) = d(QbQ, 1) \leq d(QbQ, b) + d(b, 1) = 0 + |b| \leq k|b|.$$

Next,

$$D(aQbQ, abQ) = D(QbQ, bQ) \leq D(QbQ, Q) + D(Q, bQ) \leq k|b| + k|b| = 2k|b|.$$

□

If $f : G \to A$ is a function from a group to a set then define the *neutral set* of f to be:

$$N(f) = \{x \in G : f(gx) = f(g) \text{ for all } g \in G\}.$$

If A is also a group, and f is a homomorphism, then the neutral set of f is $ker(f)$. It is easy to check that the neutral set for any function is a group:

If $x, y \in N(f)$ then $f(gxy) = f(gx) = f(g)$ (so $xy \in N(f)$), and $f(gx^{-1}) = f(gx^{-1}x) = f(g)$ for all $g \in G$ (so $x^{-1} \in N(f)$).

A function $\phi : G \to L$ from a group to a set is defined to be a *commensurated homomorphism* if there is an integer k such that for all $a, b \in G$,

$$D(\phi^{-1}(\phi(a)) \cdot \phi^{-1}(\phi(b)), \phi^{-1}(\phi(ab))) \leq 2k|b|,$$

Define $\ker(\phi) \equiv \phi^{-1}(\phi(1_G))$.

Theorem 4.2 *A subset Q of a finitely generated group G is a commensurated subgroup of G iff there is a set L and Q is the* kernel *of a commensurated homomorphism $\phi : G \to L$ iff Q is the neutral set of ϕ.*

Proof: Fix a generating set, S, for G, use the notation of lemma 4.1, and suppose Q is a commensurated subgroup of G and let L be the set of left cosets $\{gQ \mid g \in G\}$ of Q in G. Let ϕ be the natural map from G to L taking $g \in G$ to the left coset gQ (so $Q = \ker(\phi)$). Let $k = \max_{s\in S}(D(sQ, Q)) + 1$ as in lemma 4.1. Then $D(\phi^{-1}(\phi(a)) \cdot \phi^{-1}(\phi(b)), \phi^{-1}(\phi(ab))) = d(aQbQ, abQ) \leq 2k|b|$ again by lemma 4.1.

Conversely, suppose $\phi : G \to L$ is a commensurated homomorphism. We proceed to show that $\ker(\phi)$ is the neutral set of ϕ and a commensurated subgroup of G. For convenience, let $Q \equiv \ker(\phi)$. Let $g \in G$, then

$$D(\phi^{-1}(\phi(g)) \cdot \phi^{-1}(\phi(1_G)), \phi^{-1}(\phi(g \cdot 1_G))) \leq 2k|1_G| = 0 \text{ implying}$$

$$(*) \quad \phi^{-1}(\phi(g)) \cdot Q = \phi^{-1}(\phi(g)) \text{ for all } g \in G$$

If $q \in Q$ then by $(*)$, $gq \in \phi^{-1}(\phi(g))$ for all $g \in G$. Then $\phi(gq) = \phi(g)$ for all $g \in G$, and $Q \subset N(\phi)$.

If $y \in N(\phi)$, then $\phi(gy) = \phi(g)$ for all $g \in G$. In particular for $g = 1_G$ we have $\phi(y) = \phi(1_G)$ and $y \in Q$. Thus Q is equal to the neutral set of f. In particular, Q is a subgroup of G.

Finally, by $(*)$ we see that each set $\phi^{-1}(\phi(a))$ is a union of left cosets of Q and since it contains a it contains aQ.

By hypothesis we have

$$D(\phi^{-1}(\phi(a)) \cdot \phi^{-1}(\phi(a^{-1})), Q) \leq 2k|a^{-1}| \text{ for all } a \in G, \text{ and so}$$

$$\sup_{q,q' \in Q} d(aqa^{-1}q', Q) \leq 2k|a|.$$

For $q' = 1_G$ we obtain $\sup_{q\in Q} d(aqa^{-1}, Q) \leq 2k|a|$, and

$$\sup_{q\in Q} d(aq, Q) \leq (2k+1)|a|.$$

Conversely, left invariance yields

$$\sup_{q\in Q} d(q, a^{-1}Q) \leq (2k+1)|a^{-1}| \text{ for all } a^{-1} \in G.$$

Thus $D(aQ, Q) \leq (2k+1)|a|$ for all $a \in G$. $\square$

Remark 3. This result should be compared to the result of Krieg in [13] showing: If S is commensurated in G, there is a natural map of G into the positive rationals, Q^+. The map is a homomorphism of groups with S in the kernel.

Our second characterization of commensurated subgroups is based on the following lemma.

Lemma 4.3 *Suppose S is a finite generating set for the group G and Q is a subgroup of G. Then Q is commensurated in G iff there are only finitely many cosets qsQ where $q \in Q$ and $s \in S^{\pm 1}$. (Equivalently, Q is commensurated in G iff there are only finitely many cosets gQ such that, in the Cayley graph $\Gamma(G, S)$, an edge connects a vertex of Q to a vertex of gQ.)*

Proof: If Q is a commensurated subgroup of G then by lemma 2.6, QsQ is contained in a finite number of cosets gQ for each $s \in S^{\pm 1}$.

For the converse, suppose for each $s \in S$, QsQ and $Qs^{-1}Q$ are contained in the union of finitely many cosets gQ. Inverting, $Qs^{-1}Q$ and QsQ are each contained in the union of finitely many cosets Qg^{-1}. By lemma 2.6, $S \subset Comm_Q(G)$ and Q is commensurated in G. $\square$

Suppose G is a group with finite generating set S and H is a subgroup of G. Let $\Lambda(S, H, G)$ be the (left) Schreier coset graph with vertices the left cosets gH of G and a directed edge (labeled s) from gH to fH ($gH \neq fH$) if for some $s \in S$ and $h_1, h_2 \in H$, we have $gh_1sh_2 = f$. (Equivalently, in the Cayley graph $\Gamma(S, G)$, there is an edge labeled s with initial point in gH and end point in fH.) The following result is a direct consequence of lemma 4.3.

Theorem 4.4 *Suppose G is a group with finite generating set S. Then Q is commensurated in G iff $\Lambda(S, Q, G)$ is locally finite. The group G acts (on the left) transitively on the vertices of $\Lambda(S, Q, G)$ and by isometries (using the edge path metric) on $\Lambda(S, Q, G)$. The stabilizer of gQ is gQg^{-1} and the quotient map $p : \Gamma(S, G) \to \Lambda(S, Q, G)$, defined by $p(gq) = gQ$ for all $g \in G$ and $q \in Q$, commutes with the left action of G.*

Remark 4. Observe that if Q is a commensurated subgroup of G, S is a finite generating set for G and v is a vertex of $\Lambda(S, Q, G)$ then there may be more than one edge emanating from v with label $s \in S$. If S contains n elements, then there at most $2n$ labeled (and directed) edges connecting two given vertices of Λ.

5 Ends

H. Hopf [9] and H. Freudenthal [6] developed the theory of ends of a finitely generated group. In particular, if G is a group with finite generating set S, then the Cayley graph $\Gamma(G,S)$ has either 0, 1, 2 or an uncountable number of ends. R. Geoghegan's book [8] gives a complete analysis of the proper homotopy theory of ends of groups and it is our standard reference for this subject. A continuous function $f : X \to Y$ is *proper* if for each compact set C in Y, $f^{-1}(C)$ is compact in X. An *end* of a connected graph Γ is an equivalence class of proper rays $r : [0,\infty) \to \Gamma$ where r and s are equivalent if for any compact set C in Γ there is a path α_C in Γ, beginning on r and ending on s such that α_C avoids C. The *number of ends* of a connected locally finite CW-complex X is the largest integer N such that for some compact subset C of X, $X - C$ has N unbounded components (unbounded here means not contained in a compact set). If no such integer N exists, then X has an infinite number of ends. The cardinality of the set of ends of X agrees with the number of ends of X.

Suppose S is a finite set of generators for a group G, Γ the Cayley graph of G with respect to S, Q a subgroup of G and $\pi : \Gamma \to Q\backslash\Gamma$ the quotient map. The cardinality of the set of ends of Γ and $Q\backslash\Gamma$ does not depend on the generating set S and is called the *number of ends* of G and of the pair (G,Q), respectively.

A ray $r : [0,\infty) \to \Gamma$ is *Q-filtered* if $\pi r : [0,\infty) \to Q\backslash\Gamma$ is proper. If r and s are Q-filtered in Γ then they *converge to the same Q-filtered end* of Γ if for any compact set C in $Q\backslash\Gamma$ there is a path α_C in Γ, beginning on r and ending on s such that $\pi\alpha_C$ avoids C. A *Q-filtered end* of Γ is an equivalence class of Q-filtered rays converging to the same Q-filtered end of Γ.

Each Q-filtered ray in Γ is proper and each Q-filtered end of Γ (respectively, end of $Q\backslash\Gamma$) contains an edge path ray that begins at the identity vertex $*$ (respectively, $\pi(*)$). If r and s are Q-filtered rays in Γ that converge to the same Q-filtered end, then πr and πs converge to the same end of $Q\backslash\Gamma$, so π induces a map from the Q-filtered ends of Γ to the ends of $Q\backslash\Gamma$. Any proper edge path ray at $\pi(*)$ in $Q\backslash\Gamma$ lifts to a proper edge path ray at $*$ so there is a natural map from the set of Q-filtered ends of Γ onto the set of ends of $Q\backslash\Gamma$. (For more details, see proposition 14.5.3. of [8].)

In section 4 we defined the (left Schreier) coset graph of a group G with finite generating set S and subgroup H and denoted it by $\Lambda(S,H,G)$. There is a quotient map from the Cayley graph to this coset graph $p : \Gamma(S,G) \to$

$\Lambda(S, H, G)$ that takes each left coset gH to the corresponding vertex, each edge of Γ with label $s \in S \cap H$ to a vertex, and each directed edge in Γ with label $s \notin S \cap H$ to a directed edge with the same label. Let $\pi : \Gamma(S, G) \to H\backslash\Gamma(S, G)$ be the quotient map. The main result of this section is theorem 5.4 that establishes a proper continuous $m : \Lambda(S, Q, G) \to \Gamma(S, G)$ when Q is a finitely generated commensurated subgroup of G, such that πm is proper, pm is properly homotopic to the identity and p and m induce bijections between the ends of Λ and the Q-filtered ends of Γ.

If S is a finite generating set for the group G and N is a normal subgroup of G, then the number of ends of the group $N\backslash G$ is the same as the number of ends of $N\backslash\Gamma(S, G)$ and hence is 0, 1, 2 or uncountable. This need not be the case for a commensurated subgroup Q of a finitely generated group G (see example 6). Instead, the coset graph $\Lambda(S, Q, G)$ seems a more appropriate object of analysis than $Q\backslash\Gamma(S, G)$. This line of reasoning is verified in our paper [5] where the graph $\Lambda(S, Q, G)$ is fundamental in developing the results of that paper.

First we need three elementary technical results.

Lemma 5.1 *Suppose A is a subgroup of a group G and $q : G \to A\backslash G$ is the quotient map ($g \to Ag$).*

A) If $x, y, g \in G$, then $yx^{-1} \in A$ iff $q(x) = q(y)$ iff $q(xg) = q(yg)$,

B) For $g \in G$ and $a_1, a_2 \in A$, $q(ga_1) = q(ga_2)$ iff $a_1 \in (A \cap (g^{-1}Ag))a_2$.

Also, the following are equivalent:

1) $q(xA) \cap q(yA) \neq \emptyset$

2) $q(xA) = q(yA)$

3) for some $a \in A$, $yax^{-1} \in A$.

Proof: Note that $q(x) \equiv Ax = Ay \equiv q(y)$ iff $yx^{-1} \in A$ iff $ygg^{-1}x^{-1} \in A$ iff $q(xg) = q(yg)$ and the proof of part A) is complete.

For part B), note that $q(ga_1) = q(ga_2)$ iff $ga_1a_2^{-1}g^{-1} \in A$ iff $a_1a_2^{-1} \in g^{-1}Ag$ iff $a_1a_2^{-1} \in A \cap (g^{-1}Ag)$ iff $a_1 \in (A \cap (g^{-1}Ag))a_2$, and part B) is proved.

If $q(xA) \cap q(yA) \neq \emptyset$ there is $a_1, a_2 \in A$ such that $q(xa_1) = q(ya_2)$. By part A), $q(xa_1a) = q(ya_2a)$ for all $a \in A$ and so $q(xA) = q(yA)$.

If $q(xA) = q(yA)$ then $q(x) = q(ya)$ for some $a \in A$, and $yax^{-1} \in A$.

If $yax^{-1} \in A$ and $a \in A$ then $ya = a'x$ for some $a' \in A$. Then $q(ya) = q(a'x) = q(x)$ and $q(xA) \cap q(yA) \neq \emptyset$. $\square$

Lemma 5.2 *Suppose G is a group with finite generating set S, Q is a commensurated subgroup of G, Γ is the Cayley graph of G with respect to S, and $\pi : \Gamma \to Q\backslash\Gamma$ is the quotient map, then*

1) for each $g \in G$, the set $\pi(gQ)$ is finite and has cardinality the index of $Q \cap g^{-1}Qg$ in Q,

2) for any $g \in G$ there are only finitely many cosets hQ of G such that $\pi(hQ) = \pi(gQ)$, and

3) for any compact set C in $Q\backslash\Gamma$ there are only finitely many distinct cosets gQ ($g \in G$) such that $\pi(gQ) \cap C \neq \emptyset$.

Proof: Since Q is commensurated in G, $Q \cap g^{-1}Qg$ has finite index in Q for all $g \in G$. By lemma 5.1 part B), if $\{(Q \cap g^{-1}Qg)q_i\}_{i=1}^n$ are the $Q \cap g^{-1}Qg$ cosets of Q then π is constant and distinct on each $g(Q \cap g^{-1}Qg)q_i$, and 1) is proved.

Suppose g_iQ, for $i \in \{1, 2, \ldots\}$, are distinct cosets in G such that $\pi(g_iQ) = \pi(gQ)$ for all i. Let α be an edge path from g to $*$. By lemma 5.1 part 3), there is $q_i \in Q$ such that $g_iq_ig^{-1} \in Q$. The path $g_iq_ig^{-1}\alpha$ begins at $g_iq_i \in g_iQ$ and ends at $g_iq_ig^{-1} \in Q$. But if $N = |\alpha|$ then each coset g_iQ is within N of Q in $\Lambda(S, Q, G)$. This contradicts the fact that $\Lambda(S, Q, G)$ is locally finite, and 2) is proved.

Suppose g_iQ, for $i \in \{1, 2, \ldots\}$, are distinct cosets in G such that $\pi(g_iQ) \cap C \neq \emptyset$ for some compact set $C \subset Q\backslash\Gamma$. Then there is a vertex $v \in C$ such that for infinitely many i, $v \in \pi(g_iQ)$. By lemma 5.1, all these $\pi(g_iQ)$ are the same, which is impossible by part 2). $\square$

Lemma 5.3 *Suppose G is a group with finite generating set S, Q is a commensurated subgroup of G, Γ is the Cayley graph of G with respect to S, and $\pi : \Gamma \to Q\backslash\Gamma$ the quotient map. Then an edge path r (with consecutive vertices $v_0, v_1, \ldots$) in Γ is Q-filtered iff for each $g \in G$, there are only finitely many indices i such that $v_i \in gQ$.*

Proof: Suppose r is Q-filtered. If for infinitely many i, $v_i \in gQ$, then for each such i, $\pi(v_i) \in \pi(gQ)$. By lemma 5.2, $\pi(gQ)$ is finite. But then πr is not proper, contrary to our assumption that r is Q-filtered.

Suppose that for each $g \in G$ there are only finitely many indices such that $v_i \in gQ$. If r is not Q-filtered, then πr is not proper and there is a vertex $v \in Q\backslash\Gamma$ such that for infinitely many i, $\pi(v_i) = v$. But if $\pi(v_i) = \pi(v_j)$ then by lemma 5.1, $v_iQ = v_jQ$. This is contrary to our assumption on the v_i. $\square$

Theorem 5.4 *Suppose G is a group with finite generating set S, Q is a finitely generated commensurated subgroup of G with generating set a subset of S, and Γ is the Cayley graph of G with respect to S. Let $p : \Gamma \to \Lambda(S, Q, G)$ and $\pi : \Gamma \to Q\backslash\Gamma$ be the quotient maps.*

1) There is a proper continuous map $m : \Lambda(S, Q, G) \to \Gamma$ such that $pm : \Lambda \to \Lambda$ is properly homotopic to the identity, and $\pi m : \Lambda \to Q\backslash\Gamma$ is proper. In particular, m induces a function from the set of ends of Λ to the set of Q-filtered ends of Γ.

2) The map $p : \Gamma \to \Lambda$ induces a map from the Q-filtered ends of Γ to the ends of Λ.

3) The maps $pm : \Lambda(S, Q, G) \to \Lambda(S, Q, G)$ and $mp : \Gamma \to \Gamma$ induce the identity on the set of ends of $\Lambda(S, Q, G)$ and the Q-filtered ends of Γ, respectively. In particular, m and p induce bijections between the set of ends of $\Lambda(S, Q, G)$ and the Q-filtered ends of Γ.

Proof: Let $m : \Lambda(S, Q, G) \to \Gamma$ be defined as follows: First pick a vertex a_{gQ} in each coset gQ. Assume that $a_Q = *$. If $V = vQ$ is a vertex of $\Lambda(S, Q, G)$, define $m(V) = a_V$. If e is an edge of $\Lambda(S, Q, G)$ with initial vertex V and terminal vertex W, map e (linearly) to an edge path with initial vertex a_V followed by a Q-path from a_V to a vertex of u of V such that the edge at u with the same label as e ends in W. The next edge of $m(e)$ is the edge at u with the same label as e. Follow this edge by a Q-path to a_W. Clearly $pm(e)$ is homotopic rel.$\{0, 1\}$, to e by a homotopy with image in the edge e and so pm is properly homotopic to the identity.

$$\begin{tikzcd} & \Gamma \arrow[dl, "\pi"'] \arrow[d, shift right, "p"'] \\ Q\backslash\Gamma & \Lambda(S,Q,G) \arrow[u, shift right, "m"'] \end{tikzcd}$$

Figure 2

Next we prove m is proper. Otherwise, there are infinitely many distinct edges $e_1, e_2, \ldots$ of $\Lambda(S, Q, G)$ such that $m(e_i)$ intersect some finite complex K of Γ. The set $m(e_i)$ intersects only two cosets, V_i and W_i the initial and terminal vertex of e_i, respectively. Hence for infinitely many i, V_i intersects K, or for infinitely many i, W_i intersects K. As only finitely many Q-cosets intersect any finite subcomplex of Γ, infinitely many V_i or W_i are identical. This contradicts the fact that $\Lambda(S,Q,G)$ is locally finite, and so we conclude that m is proper.

If πm is not proper, then there is a compact C in $Q\backslash\Gamma$ and mutually disjoint edges $e_1, e_2, \ldots$ of $\Lambda(S,Q,G)$ such that $\pi(m(e_i)) \cap C \neq \emptyset$ for all i. If g_iQ and t_iQ are the initial and terminal vertex of e_i respectively, then either $\pi(g_iQ)$ or $\pi(t_iQ)$ intersects C non-trivially. But this contradicts part 3) of lemma 5.2. We conclude that πm is proper and part 1) is proved.

Suppose s is a Q-filtered edge path ray in Γ. Given any coset gQ of G, only finitely many vertices of s belong to gQ by lemma 5.3). Hence $p(s)$ is proper in $\Lambda(S,Q,G)$. If r and s are Q-filtered edge path rays in Γ that determine the same filtered end of (G,Q), there is a Q-filtered edge path t that intersects both r and s in infinitely many vertices. The proper edge path $p(t)$ intersects both $p(r)$ and $p(s)$ in infinitely many vertices, so $p(r)$ and $p(s)$ determine the same end of $\Lambda(S,Q,G)$. Hence p determines a map of the filtered ends of (G,Q) to the ends of $\Lambda(S,Q,G)$ and 2) is proved.

By part 1), pm induces the identity on the set of ends of $\Lambda(S,Q,G)$. Suppose s is a Q-filtered edge path ray in Γ. The proper edge path ray $mp(s)$ intersects the exact same set of gQ cosets as does s and in the same order (by the definitions of m and p). Connect each vertex v_i of s to the corresponding vertex $mp(v_i)$ of $mp(s)$ by a Q-edge path α_i. Let C be a compact subcomplex of $Q\backslash\Lambda$. We need only show that for some i, $\pi(\alpha_i)$ avoids C. Note that v_iQ contains the vertices of α_i. For any i there are only finitely many j such that $v_iQ = v_jQ$ (since by part 2), ps is proper and $p(v_i) = v_iQ \in \Lambda(S,Q,G)$). Now if for infinitely many i, $\pi(\alpha_i)$ intersects C non-trivially, then there would be infinitely many distinct v_iQ such that $\pi(v_iQ)$ intersects C non-trivially. But that is impossible by part 3) of lemma 5.2. Instead $mp(s)$ and s determine the same Q-filtered end in Γ. $\square$

When a commensurated subgroup Q of G is not finitely generated, there is still an induced map from the set of filtered ends of (G,Q) onto the set of ends of the coset graph.

Theorem 5.5 *Suppose Q is a commensurated subgroup of the group G, S is a finite generating set for G, and $p : \Gamma \to \Lambda(S,Q,G)$ is projection. Then p induces a map of the set of filtered ends of (G,Q) onto the set of ends of $\Lambda(S,Q,G)$.*

Proof: The proof of part 2) of theorem 5.4 does not use the hypothesis that

Q is finitely generated and so p induces a map of the set of Q-filtered ends of (G, Q) to the set of ends of $\Lambda(S, Q, G)$.

For any proper edge path ray $r : [0, \infty) \to \Lambda(S, Q, G)$ at $*$, we define a proper edge path ray $\hat{r} : [0, \infty) \to \Gamma$ at $*$. First observe that there is an integer N such that if cosets gQ and hQ of G are adjacent in $\Lambda(S, Q, G)$, then gQ and hQ are of Hausdorff distance $\leq N$ apart in Γ. Let the consecutive vertices of r be $Q, g_1Q, g_2Q, \ldots$. Let α_1 be an edge path of length $\leq N$ from $* \in \Gamma$ to a vertex v_1 of g_1Q. Inductively, for $i > 1$, let α_i be an edge path of length $\leq N$ from v_{i-1} to a vertex $v_i \in g_iQ$.

Suppose the edge path $\hat{r} \equiv (\alpha_1, \alpha_2, \ldots)$ is not Q-filtered in Γ. Then by lemma 5.3, there is a coset gQ and infinitely many indices i such that α_i intersects gQ non-trivially. But then for each such i, g_iQ is within N of gQ in $\Lambda(S, Q, G)$. This contradicts the hypothesis that r is proper. Instead, we have $\hat{r}$ is Q-filtered.

Now $p(\hat{r})$ is a proper edge path in $\Lambda(S, Q, G)$ that passes through each vertex that r passes through. Hence r and $p(\hat{r})$ determine the same end of $\Lambda(S, Q, G)$. This implies that the map from filtered ends of (G, Q) to ends of $\Lambda(S, Q, G)$ induced by p is onto. $\square$

Remark 5. When Q is not finitely generated, it seems unlikely there is a proper continuous map $m : \Lambda(S, Q, G) \to \Gamma$ (in analogy with the map m of theorem 5.4) inducing a bijection of filtered ends of (G, Q) and ends of $\Lambda(S, Q, G)$. In fact, it seems unlikely there is a proper continuous map $m' : \Lambda(S, Q, G) \to Q\backslash\Gamma$ such that π and $m'p$ induce the same map on filtered ends of (G, Q). If such an m' could be found, then it would induces a map from the set of ends of $\Lambda(S, Q, G)$ onto the set of ends of $Q\backslash\Gamma$. As an alternative we next introduce a refinement of $\Lambda(S, Q, G)$, that can be mapped to $Q\backslash\Gamma$.

Suppose S is a finite generating set for a group G and Q is a commensurated subgroup of G. While our focus in this section is on the case when Q is finitely generated, part B) of lemma 5.1 motivates the definition of another locally finite graph that is useful in studying the filtered ends of the pair (Q, G) when Q is not finitely generated. Define $\tilde{\Lambda}(S, Q, G)$ to be the graph with vertices the elements of the set $\{g(Q \cap g^{-1}Qg)q : g \in G,\ q \in Q\}$. There is a directed edge (labeled s) from $g(Q \cap g^{-1}Qg)q_1$ to the distinct vertex $f(Q \cap f^{-1}Qf)q_2$ if there is an edge labeled s in the Cayley graph $\Gamma(S, G)$ with initial point in $g(Q \cap g^{-1}Qg)q_1$ and end point in $f(Q \cap f^{-1}Qf)q_2$. Unfortunately, G does not act nicely on $\tilde{\Lambda}(S,Q,G)$ (as it does on $\Lambda(S,Q,G)$). Still, we can show that $N(Q,G)$, the normalizer of Q in G, acts on $\tilde{\Lambda}(S,Q,G)$ (see proposition 5.7). The quotient maps $r_1 : \Gamma(S,G) \to \tilde{\Lambda}(S,Q,G)$ and $r_2 : \tilde{\Lambda}(S,Q,G) \to \Lambda(S,Q,G)$, defined by $r_1(gxq) = g(Q \cap g^{-1}Qg)q$ and $r_2(g(Q \cap g^{-1}Qg)q) = gQ$ for all $g \in G$, $x \in Q \cap g^{-1}Qg$ and $q \in Q$ are such that $r_2 r_1 = p$ (recall $p : \Gamma(S,G) \to \Lambda(S,Q,G)$ by $p(g) = gQ$ for each vertex $g \in \Gamma(S,G)$). Note that r_2 is finite to 1 in the sense that the preimage of the vertex gQ under the map r_2 is $\{g(Q \cap g^{-1}Qg)q : q \in Q\}$ (a finite set since $Q \cap g^{-1}Qg$ has finite index in Q). Basically we obtain $\tilde{\Lambda}(S,Q,G)$ from $\Lambda(S,Q,G)$ by splitting the vertex gQ of $\Lambda(S,Q,G)$ into the finitely many vertices $\{g(Q \cap g^{-1}Qg)q : q \in Q\}$ determined by the finitely many $Q \cap g^{-1}Qg$ cosets of Q.

Theorem 5.6 *Suppose Q is a commensurated subgroup of the group G, S is a finite generating set for G, and $r_1 : \Gamma \to \tilde{\Lambda}(S,Q,G)$ and $r_2 : \tilde{\Lambda}(S,Q,G) \to \Lambda(S,Q,G)$ are the projections defined by $r_1(gxq) = g(Q \cap g^{-1}Qg)q$ and $r_2(g(Q \cap g^{-1}Qg)q) = gQ$ for all $g \in G$, $x \in Q \cap g^{-1}Qg$ and $q \in Q$ (so that $r_2 r_1 = p$).*

Then r_1 induces a map of the set of filtered ends of (G,Q) into the set of ends of $\tilde{\Lambda}(S,Q,G)$ and r_2 is a proper onto map. There is a continuous map $\pi' : \tilde{\Lambda}(S,Q,G) \to Q\backslash\Gamma$ such that $\pi = \pi' r_1 : \Gamma \to Q\backslash\Gamma$. In particular, the induced image under r_1 of the set of filtered ends of (G,Q) in $\tilde{\Lambda}(S,Q,G)$ is mapped onto the set of ends of $Q\backslash\Gamma$ by π' and onto the set of ends of $\Lambda(S,Q,G)$ by r_2.

Proof: An argument completely analogous to that in part 2) of theorem 5.4 shows that r_1 induces a map of the set of filtered ends of (G,Q) into the set of ends of $\tilde{\Lambda}(S,Q,G)$.

$$\begin{tikzcd} \Gamma \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[dr, "r_1"] & \Lambda(S,Q,G) \\ Q\backslash\Gamma & \tilde{\Lambda}(S,Q,G) \arrow[l, "\pi'"'] \arrow[u, "r_2"'] \end{tikzcd}$$

Figure 3

The map r_2 is proper since the preimage of a vertex in $\Lambda(S,Q,G)$ is finite. By theorem 5.5, $p(=r_2r_1)$ induces a map of the filtered ends of (G,Q) onto the ends of $\Lambda(S,Q,G)$ so r_2 induces a map from the induced image under r_1 of the filtered ends of (G,Q) onto the set of ends of $\Lambda(S,Q,G)$. By lemma 5.1, part B), π is constant on the sets $g(Q\cap g^{-1}Qg)q$ for $g\in G$ and $q\in Q$. Hence we define $\pi'(g(Q\cap g^{-1}Qg)q)$ to be $\pi(g(Q\cap g^{-1}Qg)q)(\equiv \pi(gq))$ and obtain $\pi=\pi' r_1$. $\square$

Proposition 5.7 *Suppose Q is a commensurated subgroup of the group G, and S is a finite generating set for G then $N(Q,G)$ the normalizer of Q in G acts by isometries on $\tilde\Lambda(S,Q,G)$ so that each $n\in N(Q,G)$ maps the set of vertices/cosets $g(Q\cap g^{-1}Qg)q_i$ of gQ bijectively to the set of cosets $ng(Q\cap n^{-1}g^{-1}Qgn)n^{-1}q_in$ of ngQ.*

Proof: Suppose $Q=\cup_{i=1}^m(Q\cap g^{-1}Qg)q_i$. For each $n\in N(Q,G)$, we have $n^{-1}(Q\cap g^{-1}Qg)q_in=(Q\cap n^{-1}g^{-1}Qgn)n^{-1}q_in$ (defining a bijection between the $Q\cap g^{-1}Qg$ cosets of Q and the $Q\cap n^{-1}g^{-1}Qgn$ cosets of Q). Simply define $n:\tilde\Lambda(S,Q,G)\to\tilde\Lambda(S,Q,G)$ by $n(g(Q\cap g^{-1}Qg)q_i)=ng(Q\cap n^{-1}g^{-1}Qgn)n^{-1}q_in$, for each $n\in N(Q,G)$ and $g\in G$ $\square$

Suppose X is a finite connected CW-complex with $\pi_1(X)=G$ and $\tilde X$ the universal cover of X. If Q is subgroup G, then $N(Q,G)$ the normalizer of Q in G is the set of deck transformations of the intermediate covering space $Q\backslash\tilde X$. The next example is meant to show how a small change in Q can lead to a dramatic change in $N(Q,G)$.

Example 5. Let G be the direct product of the infinite cyclic group $\langle z\rangle$ with the free product $\langle t,x:x^2\rangle\approx\mathbb{Z}*\mathbb{Z}_2$. So

$$G\equiv\mathbb{Z}_z\times(\mathbb{Z}_t*\mathbb{Z}_2)\equiv\langle t,x,z:x^2,[z,t],[z,x]\rangle$$

Since the infinite cyclic group $\langle z\rangle$ is central in G, $Q\equiv\langle z,x\rangle\approx\mathbb{Z}\times\mathbb{Z}_2$ is commensurated in G. If $g\in G$ is in the normalizer of Q then $g^{-1}zg=z$ and since x is the only order 2 element of Q, $g^{-1}xg=x$, so g is in the centralizer of Q. Now, g has the form z^nw for $w\in\langle t,x\rangle$ and $g^{-1}xg=w^{-1}xw=x$. Certainly the centralizer of x in $\langle t,x\rangle$ is $\langle x\rangle$ so that $w\in\langle x\rangle$. Hence while the normalizer of $\mathbb{Z}_z$ is all of G (large as possible), and $\mathbb{Z}_z$ has index 2 in Q, the normalizer of Q is Q (small as possible).

The next result should be compared to theorem 14.5.10 of [8] (when Q is finitely generated).

Theorem 5.8 *If G is a group with finite generating set S, and Q is a commensurated subgroup of G then $\Lambda(S,Q,G)$ has 0, 1, 2 or an uncountable number of ends, and this number is independent of the finite generating set S.*

If Q has infinite index in $N(Q,G)$ then $\tilde{\Lambda}(S,Q,G)$ has 0, 1, 2 or an infinite number of ends, also independent of the finite generating set S.

Proof: As G acts transitively on the vertices of $\Lambda(S,Q,G)$ and by isometries on $\Lambda(S,Q,G)$ the standard proof that a Cayley graph of a group has 0, 1, 2 or an uncountable number of ends can be modified to show that $\Lambda(S,Q,G)$ has 0, 1, 2 or an uncountable number of ends. I.e. if $\Lambda(S,Q,G)$ has at least 3 ends, let K be a finite subgraph of $\Lambda(S,Q,G)$ such that $\Lambda(S,Q,G)-K$ has $n \geq 3$ unbounded components. Choose one of the unbounded components A of $\Lambda(S,Q,G)-K$ and an element $g \in G$ so that $gK \subset A$ and gK is far from K. Then $\Lambda(S,Q,G)-(K \cup gK)$ has at least $2(n-1)$ unbounded components and showing $\Lambda(S,Q,G)$ has an infinite number of ends. Since the same can be done in each unbounded component of $\Lambda(S,Q,G)-K$, a standard argument continuing this line of reasoning shows $\Lambda(S,Q,G)$ has an uncountable number of ends.

Suppose Q has infinite index in $N(Q,G)$. Let $n_1Q, n_2Q, \ldots$ be distinct cosets in $N(Q,G)$ then the n_iQ are distinct vertices of $\tilde{\Lambda}(S,Q,G)$. For any compact set C in $\tilde{\Lambda}(S,Q,G)$ there is an $i > 0$ such that $n_i(C) \cap C = \emptyset$ and an argument as above shows $\tilde{\Lambda}(S,Q,G)$ has 0, 1, 2 or ∞ ends, but does not show infinitely many ends implies an uncountable number of ends (If $\tilde{\Lambda}(S,Q,G)-C$ has n unbounded components there in no guarantee that for each such component there is an n_i that translates C into that component.)

If T is another finite generating set for G, then the graph $\Lambda(T,Q,G)$ has the same set of vertices as does $\Lambda(S,Q,G)$ (the left cosets gQ). Let $f_S : \Lambda(S,Q,G) \to \Lambda(T,Q,G)$ be defined as follows: f_S restricted to the vertices of $\Lambda(S,Q,G)$ is the identity. Suppose $s \in S$. Choose a T-word w_s such that in G, $s = w_s$. If e is any directed edge of $\Lambda(S,Q,G)$ with label $s \in S$ and initial vertex g_1Q and terminal vertex g_2Q, let $\tilde{e}$ be an edge of $\Gamma(S,G)$ with label s, initial vertex $v_1 \in g_1Q$ and terminal vertex $v_2 \in g_2Q$. The edge path $\tau_{\tilde{e}}$ at v_1 with labeling defined by w_s ends at v_2. Define f_S to linearly map e to the edge path $\tau_e \equiv p(\tau_{\tilde{e}})$ of $\Lambda(T,Q,G)$ (where $p : \Gamma(S,G) \to \Lambda(S,Q,G)$

is the quotient map). Note that τ_e is an edge path from g_1Q to g_2Q. Both f_T and f_S are proper and the compositions $f_S f_T$ and $f_T f_S$ are the identity on vertices so f_T and f_S induce isomorphisms between the set of ends of $\Lambda(S,Q,G)$ and $\Lambda(T,Q,G)$. Similarly for $\tilde{\Lambda}(S,Q,G)$. $\square$

Suppose S is a finite generating set for the group G and Q is a commensurated subgroup of G. Corollary 2.5 suggests the graphs $\Lambda(S,Q,G)$ and $Q\backslash\Gamma(S,G)$ are quasi-isometric. This need not be the case. In the following example, the set of ends of $Q\backslash\Gamma$ and Λ have different cardinality.

Example 6. If $G = \langle t, x : t^{-1}xt = x^2\rangle$ and $Q = \langle x\rangle$, then $\Lambda(\{x,t\},Q,G)$ is a tri-valent tree. The graph of $Q\backslash\Gamma(S,G)$ is obtained as follows: Begin with a ray, with vertices labeled v_i for $i \leq 0$. Assume the directed edge from v_{i-1} to v_i is labeled t. There is a loop labeled x at each v_i. Call this graph A_0. Attach a directed edge labeled t to v_0 with end vertex v_1 and a loop of length 2 to v_1 with each edge labeled x. Let w_1 label the vertex of this loop opposite v_1. Call this graph $\hat{B}_1$. Let $\hat{B}_1'$ be another copy of $\hat{B}_1$ and attach $\hat{B}_1$ to $\hat{B}_1'$ along the respective loops of length 2 with a half twist (so that v_1 is identified with w_1' and w_1 is identified with v_1'). Call the resulting graph A_1. Note that A_1 has 2-ends.

Next, attach an edge at v_1 labeled t with end vertex v_2 and attach a loop of length 4 to v_2 such that each edge of the loop is labeled x. Let w_2 label the vertex of this loop opposite v_2. Attach to this graph an edge labeled t beginning at w_1 and ending at w_2. Call the resulting graph $\hat{B}_2$. Let $\hat{B}_2'$ be another copy of $\hat{B}_2$ and attach $\hat{B}_2$ to $\hat{B}_2'$ along the respective loops of length 4 with a one quarter twist. Call the resulting graph A_2. Note that A_2 has 4-ends.

Next, attach an edge at v_2 labeled t with end vertex v_3 and attach a loop of length 8 to v_2 such that each edge of the loop is labeled x. Attach to this graph three additional edges, each labeled t and each beginning at a vertex of the loop at v_2 and ending at a vertex at the loop at v_3 so that the relations $t^{-1}xt = x^2$ is satisfied. Call the resulting graph $\hat{B}_3$. Let $\hat{B}_3'$ be another copy of $\hat{B}_3$ and attach $\hat{B}_3$ to $\hat{B}_3'$ along the respective loops of length 8 with a one eighth twist. Call the resulting graph A_3. Note that A_3 has 8-ends. Continue to construct $Q\backslash\Gamma(S,G)$.

The number of ends of $Q\backslash\Gamma(S,G)$ is countable, while the number of ends of $\Lambda(\{x,t\},Q,G)$ is uncountable. As the cardinality of the set of ends of a graph is a quasi-isometry invariant, the graphs $\Lambda(\{x,t\},Q,G)$ and $Q\backslash\Gamma(S,G)$

are not quasi-isometric. □

6 Connections to Bounded Packing

In [10], C. Hruska and D. Wise make the following definition:
(Bounded packing) Let G be a discrete group with a left invariant metric d. Suppose also that d is proper in the sense that every metric ball is finite. A subgroup H has *bounded packing* in G (with respect to d) if, for each constant D, there is a number $N = N(G, H, D)$ so that for any collection of N distinct cosets gH in G, at least two are separated by a distance of at least D. (Here $d(g_1H, g_2H)$ is the infimum of $d(g_1h_1, g_2h_2)$ for all $h_1, h_2 \in H$.)

The main theorem of [10] is the following bounded packing result (which is more general and more sophisticated than theorem 3.11):

Theorem (Hruska-Wise) Let H be a relatively quasi-convex subgroup of a relatively hyperbolic group G. Suppose $H \cap gPg^{-1}$ has bounded packing in gPg^{-1} for each conjugate of each peripheral subgroup P. Then H has bounded packing in G.

Consider the coset graph $\Lambda(S, Q, G)$ where S is a finite generating set for the group G and Q is commensurated in G. For $\Gamma(S, G)$ the Cayley graph of G with respect to S, the natural projection map $p : \Gamma \to \Lambda$ respects the left action of G on Γ and Λ. If d and D are the edge path metrics on Γ and Λ respectively, and $d(g_1Q, g_2Q)$ is the infimum of $d(g_1q_1, g_2q_2)$ for all $q_1, q_2 \in Q$, then $D(g_1Q, g_2Q) \leq d(g_1Q, g_2Q)$ for all $g_1, g_2 \in G$. As G acts transitively on Λ, and since Λ is locally finite (theorem 4.4) we have:

Lemma 6.1 *If G is finitely generated and Q is commensurated in G, then Q has bounded packing in G.*

For $F(x, y)$, the free group on $\{x, y\}$, it is elementary to show that $\langle x \rangle$ has bounded packing in $F(x, y)$. Certainly $\langle x \rangle$ is not commensurated in $F(x, y)$.

If the wording is slightly changed in the bounded packing definition for finitely generated groups, then one gets commensurated.

Lemma 6.2 *Let G be a finitely generated group with word metric d. A subgroup H is commensurated in G iff, for each constant D, there is a number $N = N(G, H, D)$ so that for any collection of N distinct cosets gH in G, at least one is separated from H by a distance of at least D. (Here again, $d(H, gH)$ is the infimum of $d(h_1, gh_2)$ for all $h_1, h_2 \in H$.)*

Proof: Assume S is a finite generating set for G. If H is commensurated in G, let $N-1$ be the number of vertices gH of the (locally finite) coset graph $\Lambda(S,H,G)$, within D of the vertex H. Then for N distinct cosets $g_1H,\ldots,g_NH$ of G at least one, g_iH, is of distance greater than D from H in $\Lambda(S,H,G)$ and hence $d(H,g_iH)>D$.

For the converse, let $g\in G$ and say $d(1,g)=D$. By hypothesis, there are only finitely many distinct cosets $g_1H,\ldots,g_NH$ such that $d(H,g_iH)\le D$. For each $h\in H$, $d(H,hgH)\le D$ and so $hgH=g_iH$ for some i. Without loss assume $g_1=g$. Let $H_i=\{h\in H : hg\in g_iH\}$ so that $\{H_i\}_{i=1}^N$ partitions H (reindexing, we may assume that $H_i\neq\emptyset$). Note that $h\in H_1$ iff $hg\in gH$ iff $g^{-1}hg\in H$, so $H_1=H\cap gHg^{-1}$.

Suppose $a_1,a_2\in H_i$. Then $a_1g=g_ih_1$ and $a_2g=g_ih_2$ for some $h_1,h_2\in H$. Now, $a_1^{-1}a_2=gh_1^{-1}h_2g^{-1}\in gHg^{-1}$ and $a_1^{-1}a_2\in H\cap gHg^{-1}\equiv H_1$. Hence

$$H_i\subset aH_1 \text{ for all } a\in H_i$$

For $a\in H_i$ and $h\in H_1$, write $h=gh'g^{-1}$ for some $h'\in H$. Then, $ag\in g_iH$ and $ahg=agh'\in agH=g_iH$. So $ah\in H_i$ and

$$aH_1\subset H_i \text{ for all } a\in H_i$$

Hence $H_i=aH_1$ for all $a\in H_i$

Suppose $h_i\in H_i$, then $H=\cup_{i=1}^N h_iH_1$ and H_1 has finite index in H. This implies that there is an integer D_g such that $d(h,H_1)\le D_g$ for each $h\in H$. As each point of H_1 is within $|g|=D$ of gH, each point of H is within $D+D_g$ of gH. Now, each point of H is within $D+D_{g^{-1}}$ of $g^{-1}H$, so each point of gH is with in $D+D_{g^{-1}}$ of H, and H is commensurated in G. $\square$

7 Closing Remarks: Semistability, L^2-Betti Numbers, and Simple Groups

In a separate paper [5], we show that certain asymptotic aspects of groups with particular normal subgroups agree with those having corresponding commensurated subgroups. See theorems 7.1, 7.2 and 7.3.

Theorem 7.1 *Suppose G is a finitely generated group, and Q is an infinite, finitely generated, commensurated subgroup of G of infinite index in G, then G is one-ended and semistable at infinity.*

The corresponding result for normal subgroups is the main result of Mihalik's paper [18]. Using theorem 7.1, we obtain a relatively short proof of Vee Ming Lew's theorem on semistability of groups with infinite finitely generated subnormal subgroup of infinite index.

Example 7. An infinite finitely generated subnormal subgroup of a finitely generated group need not be commensurated. Let $\{\ldots, x_{-1}, x_0, x_1, \ldots\}$ be a set of free generators for an infinite rank free abelian group N. Consider G, the group extension of N obtained by adding a generator t and relations $t^{-1}x_i t = x_{i+1}$ for all i (so G is the wreath product $\mathbb{Z} \rtimes \mathbb{Z}$). Then t and $x_0 \equiv x$ generate G. Now N is normal in G and $\langle x \rangle$ is normal in N (since N is abelian). But $\langle x \rangle$ is not normal in G. In fact, we show $\langle x \rangle$ is not commensurated in G.

If $\langle x \rangle$ is commensurated in G then $t^{-1}\langle x \rangle t \cap \langle x \rangle$ has finite index in both terms. That implies for some non-zero m and n, $t^{-1}x^m t = x^n$. This implies $x_1^m = x_0^n$, but that only happens (in N) when $m = n = 0$.

Remark 6. If a finitely generated group has infinitely many ends then it has non-zero first L^2-Betti number (see [1]). D. Gaboriau has proven [7]: If a countable group Γ acts freely, ergodically, in a measure-preserving way, on some standard probability space and Γ contains a finitely generated, infinite, normal subgroup N with infinite index in Γ, then the first L^2-Betti number of Γ vanishes. This result (answering in the affirmative a question of Gromov) was previously proved, under the additional assumption that Γ/N is not a torsion group, by W. Lück ([16]). Theorem 5.12 [23] (J. Peterson and A. Thom) implies that normal can be replaced by commensurated in Gaboriau's result.

The main theorem of M. Mihalik's paper [19] implies that if H is an infinite finitely presented group and $\phi : H \to H$ is a monomorphism then the resulting HNN extension, $G \equiv \langle t, H : t^{-1}ht = \phi(h) \text{ for all } h \in H\rangle$, is semistable at infinity (and simply connected at infinity when H is 1-ended). The technique for this proof breaks down when H is only finitely generated and the corresponding semistability problem is currently unresolved.

As a direct corollary to theorem 7.1 and lemma 3.9, we obtain:

Theorem 7.2 *Suppose H is a finitely generated group, $\phi : H \to H$ is a monomorphism and $\phi(H)$ has finite index in H. The resulting HNN exten-*

sion:

$$G \equiv \langle t, H : t^{-1}ht = \phi(h) \text{ for all } h \in H\rangle$$

is semistable at infinity.

Theorem 7.3 *Suppose G is a finitely presented group and Q is a subgroup of G that is infinite, finitely presented, commensurated in G, and of infinite index in G. If either Q has one end or the pair (G, Q) has one filtered end, then G is simply connected at infinity.*

Theorem 7.3 generalizes B. Jackson's corresponding result for normal subgroups in [12].

Remark 7. Higman's group

$$G \equiv \langle a_1, \ldots, a_4 : a_i^{-1}a_{i+1}a_i = a_{i+1}^2 \text{ cyclically for all } i\rangle$$

is an infinite finitely presented group with no proper subgroups of finite index. A proper normal subgroup N of G is *maximal* if it is not contained in any other proper normal subgroup. As the ascending union of normal subgroups is normal, any proper normal subgroup of G is contained in a maximal proper normal subgroup of G. (To see this, list the elements of G as $g_1, g_2, \ldots$. If N is a proper normal subgroup of G, let $N_0 = N$ and N_i be the normal closure of $N_{i-1} \cup \{g_i\}$ if this group is not G and otherwise let $N_i = N_{i-1}$. Now $\{N_i\}_{i=0}^{\infty}$ is an ascending sequence of normal subgroups in G. Hence $M \equiv \cup_{i=0}^{\infty} N_i$ is normal in G. The group M is a proper normal subgroup of G since otherwise, the generators, a_i are elements of M for all i, and so for some j, a_i is in N_j for all i (this is impossible since $N_j \neq G$). Now M is maximal since if $g \in G$ is not in M then the normal closure of $M \cup \{g\}$ is G. As G has no proper subgroups of finite index, G/M is an infinite finitely generated simple group.

a) The ascending union of commensurated subgroups is not necessarily a commensurated subgroup. (See example 3)

b) Does Higman's group contain interesting commensurated subgroups?

c) Is G/M nearly simple (i.e. does it contain non-trivial commensurated subgroups)?

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