Commensurated Subgroups, Semistability and Simple Connectivity at Infinity

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A subgroup Q of a group G is *commensurated* if the commensurator of Q in G is the entire group G. Our main result is that a finitely generated group G containing an infinite, finitely generated, commensurated subgroup H, of infinite index in G, is one-ended and semistable at ∞. Furthermore, if Q and G are finitely presented and either Q is one-ended or the pair (G, Q) has one filtered end, then G is simply connected at ∞. A normal subgroup of a group is commensurated, so this result is a generalization of M. Mihalik's result [17] and of B. Jackson's result [11]. As a corollary, we give an alternate proof of V. M. Lew's theorem that a finitely generated group G containing an infinite, finitely generated, subnormal subgroup of infinite index is semistable at ∞. So, several previously known semistability and simple connectivity at ∞ results for group extensions follow from the results in this paper. If $\phi : H \to H$ is a monomorphism of a finitely generated group and $\phi(H)$ has finite index in H, then H is commensurated in the corresponding ascending HNN extension, which in turn is semistable at ∞.

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1 Introduction

Given a group G and a subgroup H of G, the element g of G is in the *commensurator* of H in G (denoted $Comm(H, G)$) if $gHg^{-1} \cap H$ has finite index in both H and gHg^{-1}. In the mid-1960's, A. Borel [1], proved a series of results that highlight the critical nature of commensurators in the structure of semisimple Lie groups. These results were extended by G. A. Margulis [16], in 1975. If G is the commensurator of Q in G, then Q is *commensurated* in G. In particular, if H is normal in G, then H is commensurated in G. The authors [3], develop the basic theory of commensurated subgroups and showed this theory closely parallels the theory of normal subgroups of a group, but with subtle differences.

A locally-finite, connected CW-complex X is *semistable at* ∞ if any two proper maps $r, s : [0, \infty) \to X$ which converge to the same end are properly homotopic. The early ideas of R. Lee and F. Raymond [14], and F.E.A. Johnson [12], on the 'fundamental group of an end' were instrumental in extending the idea of semistability at ∞ of a space to the notion of the semistability at ∞ for a finitely presented group. R. Geoghegan [7], introduced the idea of using shape theory as the correct setting to study the homology of ends of groups. The best reference for the fundamentals of the subject of semistability at ∞ is R. Geoghegan's book [8]. Many classes of finitely generated groups are known to be semistable at ∞ (see [17], [19], [20], [21] and [22], for instance). It is unknown if all finitely presented groups are semistable at ∞. If a finitely presented group G is semistable at ∞, then one can define invariants for G, such as the fundamental group at an end of G, independent of choice of basepoint ray in some associated space. The idea of semistability at ∞ is also of interest in the study of cohomology of groups. R. Geoghegan and M. Mihalik [9], have shown that if the group G is finitely presented and semistable at ∞, then $H^2(G;\mathbb{Z}G)$ is free abelian. It should be noted that a basic unsolved problem in the study of group cohomology is whether or not $H^2(G;\mathbb{Z}G)$ is free abelian for all finitely presented groups G.

The study of ends of groups was started by H. Freudenthal [6] and H. Hopf [10]. A finitely generated group G has either 0, 1, 2, or an infinite number of ends. It is elementary to see that finitely presented groups with either 0 or 2 ends are semistable at ∞. By Mihalik [20], and Dunwoody's accessibility theorem [5], the semistability question for finitely presented groups reduces to the question of whether or not all one-ended finitely presented groups are semistable at ∞.

The strongest result to date in this subject is the following combination result of M. Mihalik and S. Tschantz [23].

Theorem 1.1 **(M. Mihalik, S. Tschantz)** *If $G = A *_H B$ is an amalgamated product where A and B are finitely presented and semistable at ∞, and H is finitely generated, then G is semistable at ∞. If $G = A*_H$ is an HNN-extension where A is finitely presented and semistable at ∞ and H is finitely generated, then G is semistable at ∞.*

This result generalizes to the obvious statement about graphs of groups and was used by Mihalik and Tschantz [24], to prove that all one relator groups are semistable at ∞. It should be noted that this result is non-trivial when A and B are free groups.

All word hyperbolic groups are semistable at ∞ (see G. Swarup [28]). R. Geoghegan [7], has shown that a one-ended CAT(0) group G is semistable at ∞ if and only if some (equivalently any) visual boundary for G has the shape of a locally connected continuum. It is elementary to construct a semistable at ∞, one-ended CAT(0) group with non-locally connected boundary. For instance, the direct product of the integers with the free group of rank 2 has visual boundary homeomorphic to the suspension of a Cantor set. While the suspension of a Cantor set is non-locally connected, it has the same shape as the Hawaiian earring, which is a locally connected space. Mihalik [21], defined a notion of semistability at ∞ for a finitely generated group that generalizes the original definition (i.e., a finitely presented group is semistable at ∞ with respect to the alternative definition if and only if it is semistable at ∞ with respect to the original definition). With this more general definition, the finitely generated analogs to the main results obtained in [17] and [19] are quite apparent. In fact, this more general definition is used to show certain finitely presented groups are semistable at ∞ (see [21]). In his Ph.D dissertation, Lew [15], proved that if G is a finitely generated group containing an infinite, finitely generated, subnormal subgroup

H of infinite index in G, then G is one-ended and semistable at ∞. Lew's proof of this theorem generalized arguments used in the proofs in [17] and [19]. Our main theorem is used in §5 to produce an alternative proof of Lew's theorem.

Theorem 1.2 (Main Theorem) *If a finitely generated group G has an infinite, finitely generated, commensurated subgroup Q, and Q has infinite index in G, then G is one-ended and semistable at ∞. Furthermore, if G and Q are finitely presented and either Q is one-ended or the pair (G,Q) has one filtered end, then G is simply connected at ∞.*

As an example, the cyclic subgroup $\langle x\rangle$ of the Baumslag-Solitar group

$$B(m,n) \equiv \langle x,t : t^{-1}x^m t = x^n\rangle \text{ (for non-zero integers } m,n)$$

is commensurated in $B(m,n)$.

A connected CW-complex X is *simply connected at* ∞ if for each compact set C in X there is a compact set D in X such that loops in $X-D$ are homotopically trivial in $X-C$. Simple connectivity at ∞ implies semistability at ∞. As with semistability at ∞, the idea of simple connectivity at ∞ can be extended from spaces to finitely presented groups and if G is finitely presented and simply connected at ∞, then $H^2(G;\mathbb{Z}G)$ is trivial. L. Siebenmann [27], developed the idea of simple connectivity at ∞ to give an obstruction to finding a boundary for an open manifold. R. Lee and F. Raymond [14], used the idea of the simple connectivity at ∞ of a group in order to analyze manifolds covered by Euclidean space. B. Jackson [11], proves:

Theorem 1.3 (B. Jackson) *Suppose $1\to H\to G\to K\to 1$ is a short exact sequence of infinite, finitely presented groups and either H or K is one-ended, then G is simply connected at ∞.*

In his thesis, J. Profio [25], improved Jackson's result:

Theorem 1.4 (J. Profio) *Suppose $H\lhd N\lhd G$ where G and H are finitely presented, H is 1-ended, and H has infinite index in G. Then G is simply connected at ∞.*

M. Davis [4], constructs examples of aspherical closed n-manifolds for $n\geq 4$, that are not covered by $\mathbb{R}^n$. In fact, Davis argues that the fundamental groups of his manifolds are semistable at ∞, but not simply connected at ∞ (and hence not covered by $\mathbb{R}^n$). All of Davis' group are subgroups of finite index in finitely generated Coxeter groups. Mihalik [22], showed all Artin and Coxeter groups are semistable at ∞.

The remainder of the paper is organized as follows. In Section 2 we give the basics of commensurated subgroups of groups. This includes a geometric characterization of commensurated groups and a result connecting the filtered ends of a Cayley graph of a group to the ends of a Schreier coset graph. In Section 3, we say what it means for a finitely generated group to be semistable at ∞ and list a number of equivalent formulations of this concept. The proof of the main theorem takes up most of Section 4. Section 5 is devoted to an alternative proof of a theorem of V. M. Lew. Section 6 contains a generalization of a result of B. Jackson about simply connected at ∞ group extensions. The techniques of proof in sections 4, 5 and 6 are geometric. The working definition of commensurated subgroup is used to construct proper homotopies between proper edge path rays in certain Cayley 2-complexes and to homotopically kill certain loops by homotopies that avoid prescribed compact sets.

2 Commensurable Preliminaries

If S is a finite generating set for a group G, $\Gamma(G,S)$ the Cayley graph of G with respect to S, and H a subgroup of G, then for any $g_1, g_2 \in G$, the *Hausdorff* distance between g_1H and g_2H, denoted $D_S(g_1H, g_2H)$, is the smallest integer K such that for each element h of H the edge path distance from g_1h to g_2H in Γ is $\leq K$ and the edge path distance from g_2h to g_1H in Γ is $\leq K$. If no such K exists, then $D_S(g_1H, g_2H) = \infty$. Conner and Mihalik [3], prove the following geometric characterization of commensurated subgroups of finitely generated groups. This characterization is the working definition of commensurated subgroup in this paper.

Proposition 2.1 (G. Conner, M. Mihalik) *Suppose S is a finite generating set for a group G and H is a subgroup of G, then $g \in G$ is in $Comm(H,G)$ if and only if the Hausdorff distance $D_S(H, gH) < \infty$ if and only if $D_S(H, gHg^{-1}) < \infty$.*

In particular, a subgroup Q of a finitely generated group G is commensurated in G if and only if the Hausdorff distance $D_S(Q, gQ)$ is finite for all $g \in G$ if and only if $D_S(Q, gQg^{-1})$ is finite for all $g \in G$.

Suppose G is a group with finite generating set S and H is a subgroup of G. Let $\Lambda(S,H,G)$ be the graph with vertices the left cosets gH of G and a directed edge (labeled s) from gH to fH if for some $s \in S$ and $h_1, h_2 \in H$, we have $gh_1sh_2 = f$. (Equivalently, in the Cayley graph $\Gamma(S,G)$, there is an edge labeled s with initial point in gH and end point in fH.) Basically, Λ is a (left) *Schreier* coset graph. Note that Λ may have several edges labeled s at a vertex.

The following result appears in [3] and is used in the proof of our main theorem.

Proposition 2.2 (G. Conner, M. Mihalik) *Suppose G is a group with finite generating set S and Q is commensurated in G. Then the graph $\Lambda(S,Q,G)$ is locally finite and G acts (on the left) transitively on the vertices of Λ and by isometries (using the edge path metric) on Λ. For $\Gamma(S,G)$ the Cayley graph of G, the projection map $p : \Gamma(S,G) \to \Lambda(S,Q,G)$ respects the action of G and induces a bijection from the filtered ends of $\Gamma(S,G)$ to the ends of $\Lambda(S,Q,G)$. The graph $\Lambda(S,Q,G)$ has 0, 1, 2 or infinitely many ends.*

3 Semistability Preliminaries

Much of the groundwork for studying the notion of semistability for a finitely presented group has appeared in [11], [12], [13], [14], and [17] and is well organized in [7]. We will recall some of the ideas presented in these papers to set the notation for future use.

A continuous function $f : X \to Y$ is *proper* if for each compact subset C of Y, $f^{-1}(C)$ is compact in X. A proper map $r : [0, \infty) \to X$ is called a *ray* in X. If K is a locally finite, connected CW-complex, then one can define an equivalence relation $\sim$ on the set A of all rays in K by setting $r \sim s$ if and only if for each compact set $C \subset K$, there exists an integer $N(C)$ such that $r([N(C), \infty))$ and $s([N(C), \infty))$ are contained in the same unbounded path component of $K - C$ (a path component of $K - C$ is *unbounded* if it is not contained in any compact subset of K). An equivalence class of $A/\sim$ is called *an end of* K, the set of equivalence classes of $A/\sim$ is called *the set of ends of* K and two rays in K, in the same equivalence class, are said to *converge to the same end*. The cardinality of $A/\sim$, denoted by $e(K)$, is the *number of ends of* K.

If G is a finitely generated group with generating set S, then denote the *Cayley graph of* G *with respect to* S by $\Gamma(G, S)$. We define the *number of ends of* G, denoted by $e(G)$, to be the number of ends of the Cayley graph of G with respect to a finite generating set. (In particular, $e(G) = e(\Gamma(G, S))$. This definition is independent of the choice of finite generating set for G. If G is finitely generated, then $e(G)$ is either 0, 1, 2, or is infinite (in which case it has the cardinality of the real numbers). We let $*$ denote the basepoint of $\Gamma(G, S)$, which corresponds to the identity of G.

If f and g are rays in K, then one says that f and g are *properly homotopic* if there is a proper map $H : [0, 1] \times [0, \infty) \to K$ such that $H|_{\{0\}\times[0,\infty)} = f$ and $H|_{\{1\}\times[0,\infty)} = g$. If $f(0) = g(0) = v$, one says f and g are *properly homotopic relative to* v (or *rel*$\{v\}$) if additionally $H|_{[0,1]\times\{0\}} = v$.

Definition 3.1 A locally finite, connected CW-complex K is *semistable at* ∞ if any two rays in K converging to the same end are properly homotopic.

Theorem 2.1 of Mihalik [17], and Lemma 9 of Mihalik [19], provide several equivalent notions of semistability. The space considered in [17] is simply connected, but simple connectivity is not important in that argument. A slight modification of proofs give the following result.

Theorem 3.2 *Suppose* K *is a locally finite, connected and one-ended CW-complex. Then the following are equivalent:*

(1) K *is semistable at* ∞.

(2) *For any ray* $r : [0, \infty) \to K$ *and compact set* C, *there is a compact set* D *such that for any third compact set* E *and loop* α *based on* r *and with image in* $K - D$, α *is homotopic rel*$\{r\}$ *to a loop in* $K - E$, *by a homotopy with image in* $K - C$.

(3) *For any compact set C there is a compact set D such that if r and s are rays based at v and with image in $K-D$, then r and s are properly homotopic $rel\{v\}$, by a proper homotopy in $K-C$.*

If K is simply connected, then a fourth equivalent condition can be added to this list:

4. If r and s are rays based at v, then r and s are properly homotopic $rel\{v\}$.

Example 1. Note that the one-ended CW-complex obtained by attaching a loop at 0 to the interval $[0,\infty)$ is semistable at ∞. Consider a ray r which maps $[0,\infty)$ homeomorphically to $[0,\infty)$ and a ray s which maps $[0,1]$ once around the loop and then maps $[1,\infty)$ homeomorphically to $[0,\infty)$. Clearly r and s are properly homotopic, but not by a proper homotopy $rel\{0\}$.

The following fact is proved by F. E. A. Johnson [12] and [13].

Theorem 3.3 *Suppose X and Y are finite, connected CW-complexes with $\pi_1(X)$ isomorphic to $\pi_1(Y)$. Then the universal cover of X is semistable at ∞ if and only if the universal cover of Y is semistable at ∞.*

Definition 3.4 If G is a one-ended, finitely presented group and, X is some (equivalently any) finite, two-dimensional CW-complex with fundamental group G, then we say *G is semistable at ∞ if the universal cover of X is semistable at ∞.*

We now define the notion of semistabilty for a finitely generated group following Mihalik [21]. We give the definition for one-ended groups since this is the case that concerns us. Suppose G is a one-ended finitely generated group with generating set $S \equiv \{g_1, g_2, \ldots, g_n\}$ and let $\Gamma(G,S)$ be the Cayley graph of G with respect to this generating set. Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a finite set of relations in G written in the letters $\{g_1^{\pm}, g_2^{\pm}, \ldots, g_n^{\pm}\}$. For any vertex $v \in \Gamma(G,S)$, there is an edge path cycle labeled α_i at v. The two dimensional CW-complex $\Gamma_{(G,S)}(\alpha_1, \ldots, \alpha_m)$ is obtained by attaching to each vertex of $\Gamma(G,S)$, 2-cells corresponding to the relations $\alpha_1, \ldots, \alpha_n$.

Mihalik [21], shows that if S and T are finite generating sets for the group G and there are finitely many S-relations P such that $\Gamma_{(G,S)}(P)$ is semistable at ∞, then there are finitely many T-relations Q such that $\Gamma_{(G,T)}(Q)$ is semistable at ∞. Hence the following definition:

Definition 3.5 We say *G is semistable at* ∞ if for some finite generating set S for G and finite set of S-relations P the complex $\Gamma_{(G,S)}(P)$ is semistable at ∞.

Note that if G has finite presentation $\langle S : P\rangle$, then G is semistable at ∞ with respect to Definition 3.4 if and only if G is semistable at ∞ with respect to Definition 3.5 if and only if $\Gamma_{(G,S)}(P)$ is semistable at ∞.

Lemma 2 [21], is as follows:

Lemma 3.6 *Suppose the finitely generated group G is one-ended and semistable at ∞. If S is a finite generating set for G and P is a finite set of S-relations in G such that $\Gamma_{(G,S)}(P)$ is semistable at ∞, then there is a finite set Q of S-relations such that: if r and s are rays in $\Gamma_{(G,S)}(P\cup Q)$, with $r(0)=s(0)$, then r is properly homotopic to s $rel\{r(0)\}$.*

Remark 1. Using the third equivalent notion of semistability in Theorem 3.2, it can be shown that in fact the set of relations Q in the previous lemma are unnecessary in order to draw the same conclusion. If $\Gamma_{(G,S)}(P)$ is semistable at ∞, and r and s are rays in $\Gamma_{(G,S)}(P)$ with $r(0)=s(0)$, then r is properly homotopic to s $rel\{r(0)\}$.

By an *edge path ray* in a locally finite, connected CW complex K, we mean a proper map $r:[0,\infty)\to K$ such that for each positive integer n, $r|_{[n-1,n]}$ is a homeomorphism to an edge of K. If G is finitely generated with finite generating set S, then any edge path ray, $r:([0,\infty),\{0\})\to(\Gamma(G,S),*)$, can be represented as $(e_1,e_2,\ldots)$ at $*$ with $e_i\in S^{\pm}$, and e_i the label of the i^{th} edge of r. Any edge path $(e_1,e_2,\ldots,e_k)$ of $\Gamma(G,S)$ corresponds to some group element $e'_1e'_2\ldots e'_k$ where $e'_i\in S^{\pm}$. But determining an edge path in $\Gamma(G,S)$ from some word $e'_1e'_2\ldots e'_k$ requires a specified basepoint, since the path $(e'_1,e'_2,\ldots,e'_k)$ at a vertex v determines a different edge path than $(e'_1,e'_2,\ldots,e'_k)$ based at another vertex w. The action of the group element wv^{-1} on $\Gamma(G,S)$ maps the edge path at v to the edge path at w with the same labeling. If A is a subcomplex of a locally finite, connected CW-complex K, let $St(A)$ denote the subcomplex of K consisting of the union of all 1-cells of K that intersect A along with any n-cell all of whose vertices lie in $St(A)$. Note then that $A\subseteq St(A)$ and if A is a finite subcomplex, then $St(A)$ is a finite subcomplex by the local finiteness of K. We recursively define the *Nth Star of A* for $N=1,2,3,\ldots$ by $St^N(A)=St(St^{N-1}(A))$ where $St^0(A)=A$. When it is not clear what the over-complex might be we use the notation $St(A,K)$ to denote the *Star of A in K*.

Since any ray $r:[0,\infty)\to K$ is properly homotopic to an edge path ray, we may concentrate on edge path rays when dealing with the semistability of a complex.

If e is an edge in K and $(e_1,e_2,e_3,\ldots)$ is an edge path in K based at the terminal point of e, then one denotes by $e*(e_1,e_2,e_3,\ldots)$ the edge path given by e followed by $(e_1,e_2,e_3,\ldots)$.

Definition 3.7 For a group G with finite generating set S and a subset T of S, we say an edge path in $\Gamma(G,S)$ is a T*-path* if each edge of the path is labeled by an element of $T^{\pm}$. If the path is infinite and proper we call it a T*-ray*.

4 Proof of Semistability in the Main Theorem

We prove a more general one-ended result than that stated in our main result.

Proposition 4.1 *Suppose A is a finitely generated, infinite subgroup of infinite index in a finitely generated group G, and $gAg^{-1} \cap A$ is infinite for all $g \in G$. Then G is one-ended.*

Proof Suppose S is a finite set of generators for G containing a generating set S_A for A. Let Γ be the Cayley graph of G with respect to S and let Γ_A be the Cayley graph of A with respect to S_A. We consider Γ_A to be a subset of Γ containing $*$, the identity vertex. Let C be a finite subcomplex of Γ. List elements $g_1, \ldots, g_n$ of G such that $g_iA \neq g_jA$ for $i \neq j$ and $g\Gamma_A \cap C \neq \emptyset$ if and only if $gA = g_iA$ for some $i \in \{1, \ldots, n\}$. Choose g_0 such that $g_0\Gamma_A \cap C = \emptyset$. Let D be a finite subcomplex of Γ containing C and all bounded components of $g_i\Gamma_A - C$ for all $i \in \{1, \ldots, n\}$.

It suffices to show that any vertex of $\Gamma - D$ can be joined by a path in $\Gamma - C$ to g_0. Suppose v is a vertex of $\Gamma - D$.

First we consider the case $v\Gamma_A \cap C = \emptyset$. Choose y (in the infinite set) $(v\Gamma_A v^{-1}) \cap (g_0\Gamma_A g_0^{-1})$ such that $d(y, C) > max\{|v|, |g_0|\}$. Then there are paths from y to $v\Gamma_A$ and from y to $g_0\Gamma_A$ avoiding C. Hence there is a path from v to g_0 avoiding C and the first case is finished.

Next suppose $v\Gamma_A \cap C \neq \emptyset$. Then v is in an unbounded component K of $v\Gamma_A - C$. Let $N = max\{|g_0|, \ldots, |g_n|\}$. Choose k a vertex of K such that $d(k, C) > N$. Then there are paths from v to k and from k to kg_i for each $i \in \{0, \ldots, n\}$, all avoiding C. At least one of kg_iA does not intersect C, so by the first case we can connect v to g_0 avoiding C. □

For the remainder of the proof of the main theorem, $\mathcal{Q} = \{q_1, q_2, \ldots, q_n\}$ is a finite generating set for Q and $S = \{q_1, q_2, \ldots, q_n, k_1, k_2, \ldots, k_t\}$ is a generating set for G where $k_i \notin Q$. Let $\mathcal{K} = \{k_1, \ldots, k_t\}$. Our hypothesis states that for each $g \in G$, the Hausdorff distance between Q and gQ is finite in $\Gamma(G, S)$.

Consider the left (Scherier) coset graph $\Lambda(S, Q, G)$ with vertex set, the set of all cosets gQ in G. A directed edge labeled s will have initial vertex g_1Q and terminal vertex g_2Q if there is an edge labeled s in $\Gamma(G, S)$ beginning in g_1Q and ending in g_2Q. By proposition 2.2, $\Lambda(S, Q, G)$ is locally finite. There is a quotient map $\rho : \Gamma(G, S) \to \Lambda(S, Q, G)$ respecting the left action of G on these graphs, such that each edge labeled by an element of Q is mapped to a point.

Lemma 4.2 *Suppose S is a finite generating set for the group G and Q is a finitely generated commensurated subgroup of G (with generating set a subset of S). There is an integer F such that if gQ and hQ are distinct cosets (vertices) of $\Lambda(S, Q, G)$ connected by an edge labeled $s \in S^{\pm 1}$, then for each $v \in gQ \subset \Gamma(S, G)$ there is a Q-path α at v in $\Gamma(S, G)$ of length $< F$ such that the path (α, s) ends in hQ.*

In particular: Suppose $\alpha \equiv (e_1, e_2, \ldots)$ is an edge path (possibly infinite) at $v \in \Lambda(S, Q, G)$ (with i^{th} edge labeled e_i) and v' is a vertex of $\Gamma(G, S)$ such that $\rho(v') = v$ (equivalently $v'Q = v$), then there is an edge path $\alpha' \equiv (\alpha'_0, e_1, \alpha'_1, e_1, \ldots)$ at v' with α'_i a Q-edge path of length $< F$ such that the edge path (determined by) $\rho\alpha'$ is α. I.e. there is (Q, F)- "approximate" path lifting for ρ.

Proof Suppose $v \in gQ$ and the edge labeled s at v ends in hQ. By translation, we assume $v = 1 \in G$, $g = 1$ and $h = s$. As Q is commensurated in G, $sQs^{-1} \cap Q$ has finite index in Q. Hence there is an integer F_s, such that for any vertex $w \in Q$, there is a Q-edge path in $\Gamma(S,G)$ of length $< F_s$ from w to a vertex w' of $Q \cap sQs^{-1}$. As $w' \in sQs^{-1}$, $w's \in sQ$. I.e. the edge labeled s at w' ends in sQ. Let $F = max\{F_s\}_{s \in S^{\pm 1}}$. □

Remark 2. For α and α' as in Lemma 4.2, we call α' a (Q,F)*-approximate lift of* α. Note that Lemma 4.2 does not imply that if v and w are vertices of the same coset uQ, then there are approximate lifts of a path α at $\rho(v) \in \Lambda(S,Q,G)$ to v and w that are G translates of one another in $\Gamma(G,S)$.

The next lemma basically has the same proof as Lemma 3 [19].

Lemma 4.3 *For each vertex v of $\Lambda(S,Q,G)$, there is an edge path ray s_v at v, such that for any finite subgraph C of $\Lambda(S,Q,G)$ only finitely many s_v intersect C. Furthermore, if $w \in v \equiv wQ$ let s_w be a (Q,F)-approximate lift of $s_{\rho(w)}$ to $w \in \Gamma(G,S)$, then*

i) for any finite subgraph D of $\Gamma(G,S)$ there are only finitely many vertices $w \in \Gamma(G,S)$ such that s_w intersects D non-trivially, and

ii) for any $w \in G$, only finitely many vertices z of s_w are such that zQ intersects D non-trivially.

Proof If $\mathcal{G}$ is a locally finite, infinite graph, then for each vertex v of $\mathcal{G}$ there is an edge path ray s_v at v, such that for any finite subgraph C of $\mathcal{G}$, only finitely many v are such that s_v intersects C. (The idea is this: Choose a base vertex x. For any integer $n > 0$, $\mathcal{G} - St^n(x)$ has only finitely many components. For the finitely many vertices v in $St(x)$ or a bounded component of $\mathcal{G} - St(x)$ choose s_v to be an arbitrary edge path ray at v. If v is a vertex of $St^2(x)$ or of a bounded component of $\mathcal{G} - St^2(x)$, and s_v is not defined, then v belongs to an unbounded component of $\mathcal{G} - St(x)$. Choose s_v to be an edge path ray at v in $\mathcal{G} - St(x)$. Continue in this fashion.) Now pick such edge path rays for the vertices of $\Lambda(S,Q,G)$.

As $\rho(s_w) = s_{\rho(w)}$, s_w intersects D if and only if $s_{\rho(s)}$ intersects $\rho(D)$. Hence, we may finish the proof of i), by showing at most finitely many vertices v of a coset gQ are such that s_v intersects D. Otherwise, there are infinitely many distinct vertices $v_1, v_2, \ldots$ in $gQ \subset \Gamma(G,S)$ such that each edge path ray s_{v_i} passes through the vertex d of D. In $\Lambda(S,Q,G)$, write the edge path ray $s_{gQ} \equiv (e_1, e_2, \ldots)$. By Lemma 4.2, we may write $s_{v_i} = (\alpha_{i,1}, e_1, \alpha_{i,2}, e_2, \ldots)$ in $\Gamma(G,S)$, where $\alpha_{i,j}$ is a Q-edge path of length $< F$. Let $n(i)$ be such that some vertex of $\alpha_{i,n(i)}$ is d. Since the v_i are distinct and the length of each $\alpha_{i,j}$ is $< F$, the sequence of integers $\{n(1), n(2), \ldots\}$ is unbounded. But then the initial vertex of $e_{n(i)}$ (on $s_{gQ} \equiv (e_1, e_2, \ldots)$) is $\rho(d)$. This is impossible since $s_{gQ} \equiv (e_1, e_2, \ldots)$ is proper, and *i*) is proved.

Part ii) follows immediately from the fact that $\rho(s_w) = s_{\rho(w)}$ is a proper map. □

By Lemma 4.2, if two distinct cosets g_1Q and g_2Q of G are connected by an edge in $\Gamma(G,S)$, then they are of Hausdorff distance $\leq F$. Choose M such that if two vertices of Q in $\Gamma(G,S)$ are within $2F+1$ of one another, then their $\mathcal{Q}$-distance is $\leq M$. Let P be the set of all S-relations in G of length $\leq 2F+1+M$. Let $\tilde{\Gamma}$ be $\Gamma_{(G,S)}(P)$.

The next result is Lemma 2 [19].

Lemma 4.4 *At each vertex v of $\Gamma(G,S)$ there exists a $\mathcal{Q}$-ray q_v, such that for any finite subcomplex C in $\Gamma(G,S)$ there are only finitely many vertices v such that q_v meets C.* □

For each S-relation r of G, consider the $\mathcal{K}$-word $r_{\mathcal{K}}$ obtained by eliminating from r, the $\mathcal{Q}$-letters (and their inverses). If v is a vertex of $\Gamma(G,S)$ and α the edge path loop corresponding to r at v, then $\rho(\alpha)$ (in $\Lambda(S,Q,G)$) has labeling $r_{\mathcal{K}}$. Let $\tilde{\Lambda}(S,Q,G)$ be the 2-complex obtained from $\Lambda(S,Q,G)$ by attaching a 2-cell to each loop ρr (with label $r_{\mathcal{K}}$) where r is a loop of $\Gamma(G,S)$ of length $\leq 2F+M+1$ (only one 2-cell for a given such loop in $\Lambda(S,Q,G)$). Then $\tilde{\Lambda}(S,Q,G)$ is locally finite and there is a natural map $\tilde{\rho}:\tilde{\Gamma}(G,S)\to\tilde{\Lambda}(S,Q,G)$ extending ρ and respecting the action of G.

Lemma 4.5 *If $k\in\mathcal{K}^{\pm}$ labels an edge of $\tilde{\Gamma}$ from v to w and $r=(e_1,e_2,e_3,\ldots)$ is a $\mathcal{Q}$-ray at v, then r is properly homotopic rel$\{v\}$ to $k*(f_1,f_2,\ldots)$, for $(f_1,f_2,\ldots)$ a $\mathcal{Q}$-ray at w, by a homotopy H with image a subset of $St^{2F+M+1}(Im(r),\tilde{\Gamma})$, and the image of $\tilde{\rho}\circ H$ is a subset of the finite complex $St(\tilde{\rho}(k))$.*

Proof Let v_i be the terminal vertex of e_i. Let $v_0=v$, $w_0=w$, α_0 be the empty path. For each $i\geq 1$, Lemma 4.2 implies there is a $\mathcal{Q}$-edge path α_i of length $<F$ at v_i so that (α_i,k) ends at $w_i\in kQ$. Note that in $\tilde{\Gamma}$ the distance from w_i to w_{i+1} is $\leq 2F+1$. For $i\geq 1$, let f_i be a $\mathcal{Q}$-edge path in $\tilde{\Gamma}$ of length $\leq M$ from w_{i-1} to w_i. The loop $(\alpha_i,k,f_{i+1},k^{-1}\alpha_{i+1}^{-1},e_{i+1}^{-1})$ has length $\leq 2F+1+M$ and so bounds a 2-cell of $\tilde{\Gamma}$. Hence $(e_1,e_2,\ldots)$ is properly homotopic to $k*(f_1,f_2,\ldots)$ by a homotopy H with image in $St^{2F+1+M}(Im(r)),\tilde{\Gamma})$. As each α_i and each f_i is a $\mathcal{Q}$-word, $\tilde{\rho}\circ H$ has image in $St(\tilde{\rho}(k))$. □

Recall, for each vertex $v\in\tilde{\Gamma}$, s_v is a (Q,F)-approximate lift of $s_{\rho(v)}$ (see Lemma 4.3).

Lemma 4.6 *Suppose D is a finite subcomplex of $\tilde{\Gamma}$. Then there exists a finite complex $E_1(D)\subseteq\tilde{\Gamma}$ such that if $b=(e_1,e_2,e_3,\ldots)$ is a $\mathcal{Q}$-ray at v with image in $\tilde{\Gamma}-E_1(D)$, then b is properly homotopic rel$\{v\}$ to s_v by a homotopy in $\tilde{\Gamma}-D$.*

Proof Let $L=2F+M+1$ (the constant of Lemma 4.5). There are only finitely many vertices $w\in\tilde{\Lambda}$ such that the edge path rays s_w of Lemma 4.3 intersect $St(\tilde{\rho}(D))$, non-trivially. Call these vertices $y_1,y_2,\ldots,y_l$. Since each s_{y_i} is proper, there are integers J_i such that each edge of the ray s_{y_i} following the J_i^{th}-edge is in $\tilde{\Lambda}-St(\tilde{\rho}(D))$. Let J be the maximum J_i for $i=1,2,\ldots,l$. By

Lemma 4.3, if w is any vertex of $\tilde{\Gamma}$ and e is the j^{th}-edge of s_w for $j > FJ$, then $\tilde{\rho}(e) = d$ (or a vertex of d) for d the k^{th}-edge of $s_{\rho(w)}$ for some $k > J$. By the definition of J, d does not intersect $St(\tilde{\rho}(D))$ and so $\tilde{\rho}(e)$ does not intersect $St(\tilde{\rho}(D))$. In particular,

($*$) If w is any vertex of $\tilde{\Gamma}$, and e is the j^{th}-edge of s_w for $j > FJ$, then $\tilde{\rho}(e) \subset \tilde{\Lambda} - St(\tilde{\rho}D)$.

Let $E_1(D)$ be a compact subcomplex of $\tilde{\Gamma}$ such that $St^{FJL}(D) \subseteq E_1(D)$ and such that $E_1(D)$ contains the finite set of vertices v in $\tilde{\Gamma}$ such that s_v intersects $St^{FJL}(D)$. Assume b and v satisfy the hypothesis of the lemma. The edge path ray s_v (in $\tilde{\Gamma} - St^{FJL}(D)$) has the form $(\alpha_0, c_1, \alpha_1, c_2, \ldots)$ where α_i is a $\mathcal{Q}$-path of length $< F$ and c_i is a $\mathcal{K}$-edge. Here s_v is a (Q,F)-approximate lift of $s_{\rho(v)} = (c'_1, c'_2, \ldots)$ (where c'_i has the same label as c_i).

Let v_i, w_i be the initial and terminal vertices of c_i, respectively. Let b_0 be the $\mathcal{Q}$-edge path ray (α_0^{-1}, b). By Lemma 4.5, b_0 is properly homotopic $rel\{v_1\}$ to $c_1 * b_1$, where b_1 is a $\mathcal{Q}$-ray at w_1, by a proper homotopy H_1 with image in $St^L(Im(b_0))$. In particular, b_1 has image in $\tilde{\Gamma} - St^{(FJ-1)L}(D)$. Again by Lemma 4.5, (α_1^{-1}, b_1) is properly homotopic $rel\{v_2\}$ to $c_2 * b_2$, where b_2 is a $\mathcal{Q}$-edge path ray, by a proper homotopy H_2 with image in $St^L(Im(b_1)) \subset \tilde{\Gamma} - St^{(FJ-2)L}(D)$. Iterating the above process, the $\mathcal{Q}$-ray (α_j^{-1}, b_j) is properly homotopic $rel\{v_{j+1}\}$ to $c_{j+1} * b_{j+1}$, where b_{j+1} is a $\mathcal{Q}$-ray, by a proper homotopy H_{j+1} with image in $St^L(Im(b_j))$. Let H be the homotopy of b to s_v obtained by patching together these H_i. For $i \leq FJ$, H_i has image in $\tilde{\Gamma} - D$. By Lemma 4.5, $\tilde{\rho} \circ H_j$ has image in $St(\tilde{\rho}(c_j))$. By ($*$), if $j > FJ$, then $\tilde{\rho}(c_j)$ misses $St(\tilde{\rho}(D))$. So $St(\tilde{\rho}(c_j))$ misses $\tilde{\rho}(D)$. For all positive integers j, H_j misses D and H misses D.

It remanins to show that H is a proper. Let $C \subseteq \tilde{\Gamma}$ be a finite subcomplex. Since $\tilde{\rho}(s_v)$ is proper in $\tilde{\Lambda}$, there exists an integer R such that if $j > R$, then $\tilde{\rho}(c_j)$ misses $St(\tilde{\rho}(C))$. As $\tilde{\rho} \circ H_j$ has image in $St(\tilde{\rho}(c_j))$, H_j misses C when $j > R$. Since only finitely many of the proper homotopies H_j have image that intersect an arbitrary finite subcomplex C, H is proper. □

Lemma 4.7 *Suppose $D \subseteq \tilde{\Gamma}$ is compact. There exists a compact set $E_2(D) \subseteq \tilde{\Gamma}$ such that if e is an edge in $\tilde{\Gamma} - E_2(D)$ from v to w, then the $\mathcal{Q}$-ray q_v is properly homotopic to $e * q_w$ $rel\{v\}$, by a proper homotopy in $\tilde{\Gamma} - D$.*

Proof Again let $L = 2F + M + 1$ (the constant of Lemma 4.5). Let $E_2(D)$ be a compact subcomplex of $\tilde{\Gamma}$ containing $St^L(E_1(D))$ and the finite set of vertices x such that q_x intersects $St^L(E_1(D))$. If $e \in \mathcal{K}^{\pm 1}$, then by Lemma 4.5, q_v is properly homotopic to $e * \beta$ $rel\{v\}$, where β is a $\mathcal{Q}$-ray at w and this homotopy has image in $St^L(Im(q_v))$. In particular, β avoids $E_1(D)$. By Lemma 4.6, β and q_w are properly homotopic $rel\{w\}$ to s_w by proper homotopies in $\tilde{\Gamma} - D$. Combining these homotopies gives the result.

If $e \in \mathcal{Q}^{\pm 1}$, then Lemma 4.6 implies q_v and $e * q_w$ are both properly homotopic $rel(v)$, to s_v by a proper homotopy in $\tilde{\Gamma} - D$. Combining homotopies gives the desired homotopy. □

Lemma 4.8 *Suppose $s = (s_1, s_2, s_3, \ldots)$ is an edge path ray at a vertex v in $\tilde{\Gamma}$, then s is properly homotopic to q_v $rel\{v\}$.*

Proof Choose a sequence of compact subcomplexes $\{C_i\}_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} C_i = \tilde{\Gamma}$, C_i is contained in the interior of C_{i+1}, and such that C_{i+1} contains $E_2(C_i)$. Let v_i be the endpoint of s_i. Define $H : [0,\infty) \times [0,\infty) \to \tilde{\Gamma}$ as follows: If R is the largest integer such that the edge s_i misses C_R, then by definition of C_R, $q_{v_{i-1}}$ is properly homotopic $rel\{v_{i-1}\}$ to $s_i * q_{v_i}$ by a proper homotopy H_i, missing C_{R-1}. Define H on $[i-1,i] \times [0,\infty)$ to be H_i.

In order to check that H is proper, it suffices to show that for any compact set $C \subseteq \tilde{\Gamma}$ only finitely many H_j intersect C. This follows from the fact that $C \subseteq C_i$ for some index i. Since s is proper, there is an integer $W(i)$ such that for all $j \geq W_i$, s_j lies in $\tilde{\Gamma} - C_{i+1}$. So, H_j avoids C and therefore H is proper. □

This completes the semistability part of our main theorem.

If H is a group and $\phi : H \to H$ is a monomorphism the group with presentation $\langle t, H : t^{-1}ht \text{ for all } h \in H\rangle$ is called the *ascending* HNN extension of H by ϕ and is denoted $H*_\phi$. The main theorem of Mihalik [18], states that if H is a finitely presented group and $\phi : H \to H$ a monomorphism, then the ascending HNN extension $H*_\phi$ is one-ended and semistable at ∞. Consider a general finite presentation of the form $\langle t, h_1, \ldots, h_n : r_1, \ldots, r_n, t^{-1}h_1t = w_1, \ldots, t^{-1}h_nt = w_n\rangle$ where r_i and w_i are words in $\{h_1^{\pm 1}, \ldots, h_n^{\pm 1}\}$ for all i. The group G of this presentation is the ascending HNN extension $H*_\phi$ where H is generated by $\{h_1, \ldots, h_n\}$ and ϕ is the monomorphism $\phi : H \to H$, where $\phi(h_i) = w_i$ for all i. While G is finitely presented it would seem rare that the finitely generated group H would be finitely presented. It has long been suggested that ascending HNN extensions of this form may be a good place to search for non-semistable at ∞, finitely presented groups. Conner and Mihalik [3], show that if H is finitely generated and the image of the monomorphism $\phi : H \to H$ has finite index in H, then H is commensurated in $H*_\phi$. As a direct consequence of this result and our main theorem we have:

Corollary 4.9 *Suppose H is a finitely generated group and $\phi : H \to H$ a monomorphism such that $\phi(H)$ has finite index in H, then the ascending HNN extension $H*_\phi$ is semistable at ∞.*

5 A Theorem of Lew

Our goal in this section is to give an alternate proof of a theorem of V. M. Lew [15].

Theorem 5.1 **(V. M. Lew)** *Suppose H is an infinite, finitely generated, subnormal subgroup of the finitely generated group G and H has infinite index in G. Then G is one-ended and semistable at ∞.*

Proof Suppose $k > 0$ and $H = N_0 \lhd N_1 \lhd N_2 \lhd \ldots \lhd N_k = G$ is a subnormal series. For $k \in \{1,2\}$ and G finitely presented, semistability was proved by Mihalik [17] and [19]. Those proofs easily generalize to the finitely generated case. The result that G is one-ended can be concluded from results of Cohen [2], or Stallings [26]. A geometric proof of this fact was given by Lee and Raymond [14]. We may assume that the index $[G : N_{k-1}] = \infty$, as G is semistable at ∞ if and only if any subgroup of finite index is semistable at ∞.

Let $\mathcal{H} = \{h_1, h_2, \dots, h_n\}$ be a finite generating set for H. Now, G has generating set $S \equiv \{h_1, h_2, \dots, h_n, a_1, a_2, \dots, a_m, k_1, k_2, \dots, k_t\}$ where, under the projection map $\rho : G \to G/N_{k-1}$, $\rho(k_1), \dots, \rho(k_t)$ generate G/N_{k-1} and the set $\{h_1, \dots, h_n, a_1, \dots, a_m\}$ is a subset of N_{k-1}. Let $\mathcal{K} = \{k_1, \dots, k_t\}$. We also assume that conjugates of the h_i's by the k_j's are among $a_1, \dots, a_m$ with the corresponding defining relations, say $k_i h_j {k_i}^{-1} \equiv a_{ij}$, and $k_i^{-1} h_j k_i \equiv b_{ij}$ for $i = 1, 2, \dots, t$ and $j = 1, 2, \dots, n$ so that $a_{ij}, b_{ij} \in \{a_1, a_2, \dots, a_m\}$. Define Q to be this set of conjugation relations.

$$Q = \{k_i h_j k_i^{-1} a_{ij}^{-1}, k_i^{-1} h_j k_i b_{ij}^{-1} : i = 1, \dots, t \text{ and } j = 1, \dots, n\}$$

Let A be the subgroup of N_{k-1} generated by $\mathcal{A} = \{h_1, \dots, h_n, a_1, \dots, a_m\}$. Let $A_i = N_i \cap A$ for $i \in \{1, \dots, n-2\}$. Then the subnormal sequence

$$H = A_0 \lhd A_1 \lhd \cdots \lhd A_{k-2} \lhd A$$

has length $k-1$. The proof splits naturally into the two cases of whether or not H has finite index in A. In the case H has finite index in A, we give a straightforward argument showing that H is commensurated in G and by our main theorem G is semistable at ∞. Note that if $k = 1$, this is the only case (since $A \subset N_0 = H$). So when the proof of the first case is concluded, we are in position to apply an induction argument (with base case in hand) to the remaining case.

Suppose H has finite index in A. Each point of $\Gamma(A, \mathcal{A})$ is within a bounded distance of aH for any $a \in A$. In particular the Hausdorff distance between H and aH is bounded.

If $k \in \mathcal{K}^{\pm 1}$ and $z \in kH$, then $z = kh$ for some $h \in H$. Note that $khk^{-1} \in A$ (it is a product of the $a_{ij}^{\pm 1}$ or $b_{ij}^{\pm 1}$). Since H has finite index in A, this point is close to H. As each point of kH is close to H, left multiplying by k^{-1} shows that each point of H is close to $k^{-1}H$ for all $k \in \mathcal{K}^{\pm 1}$. We have H is commensurated in G. The conditions of our main theorem are satisfied and so in the case H has finite index in A, G is semistable at ∞.

Now suppose H has infinite index in A. The subnormal sequence $H = N_0 \lhd N_1 \lhd \cdots \lhd N_{k-1} \lhd G$ has length k. Case 1 (or Mihalik's theorem [17]) shows that if $k = 1$, then G is semistable at ∞. Inductively, we assume that if G' is finitely generated and has a subnormal sequence of $H' = N'_0 \lhd N'_1 \lhd \cdots \lhd N'_{k-2} \lhd G'$ of length $k-1$ such that H' is finitely generated and has infinite index in G', then G' is semistable at ∞.

In our case, H has infinite index in A, and the $k-1$ length subnormal series $H = A_0 \lhd A_1 \lhd \cdots \lhd A_{k-2} \lhd A$ implies that A is semistable at ∞. Hence we may choose a finite set of $\mathcal{A}$-relations P so that $\Gamma_{(A,\mathcal{A})}(P)$ is semistable. By using Lemma 3.6 or remark 1, we may assume that if r and s are $\mathcal{A}$-rays at v in $\Gamma_{(A,\mathcal{A})}(P)$, then r and s are properly homotopic $rel\{v\}$ in $\Gamma_{(A,\mathcal{A})}(P)$. In this case, let $\tilde{\Gamma}$ be $\Gamma_{(G,S)}(P \cup Q)$ (where Q is the set of conjugation relations defined at the beginning of this proof).

If $v \in G$ (so v is a vertex of $\tilde{\Gamma}$) and C_v is a compact subcomplex of $v\Gamma_{(A,\mathcal{A})}(P) \subset \tilde{\Gamma}$ there is a compact subcomplex D_v of $v\Gamma_{(A,\mathcal{A})}(P)$ such that if r and s are edge path rays at $w \in v\Gamma_{(A,\mathcal{A})}(P) - D_v$, then, r and s are properly homotopic $rel\{v\}$ by a proper homotopy whose image does not intersect C_v. Hence, if C is a compact subcomplex of $\tilde{\Gamma}$ and we let $C_v = C \cap v\Gamma_{(A,\mathcal{A})}(P)$ (for the finite set of vertices v such $C \cap v\Gamma_{(A,\mathcal{A})}(P) \neq \emptyset$) and let $D = \cup D_v$, then any two $\mathcal{A}$-rays r and s at $w \in v\Gamma_{(A,\mathcal{A})}(P) - D$ are properly homotopic $rel\{w\}$ in $\tilde{\Gamma} - C$.

We use $\mathcal{H}$-rays r_v, as defined in Lemma 4.4.

Choose a sequence of compact subcomplexes $\{C_i\}_{i=1}^{\infty}$ of $\tilde{\Gamma}$ satisfying the following conditions:

(1) $\bigcup_{i=1}^{\infty} C_i = \tilde{\Gamma}$

(2) $St(C_i)$ is contained in the interior of C_{i+1}, and the finite set of vertices v such that r_v intersects C_i, is a subset of C_{i+1}.

(3) If r and s are $\mathcal{A}$-rays both based at a vertex v with images missing C_i, then r and s are properly homotopic $rel\{v\}$ by a proper homotopy missing C_{i-1}.

For convenience define $C_i = \emptyset$ for $i < 1$ and observe that conditions (1), (2), and (3) remain valid for all C_i. The next lemma concludes the proof of the second case and the theorem.

Lemma 5.2 *If v is a vertex of $\tilde{\Gamma}$, and $s = (s_1, s_2, \dots)$ is an S-ray at v, then s is properly homotopic to r_v, $rel\{v\}$.*

Proof Assume that s has consecutive vertices $v = v_0, v_1, \dots$. By construction, if $v_j \in C_i - C_{i-1}$, then r_{v_j} avoids C_{i-1}. Assume j is the largest integer such that C_j avoids s_i. We will show $r_{v_{i-1}}$ is properly homotopic to $s_i * r_{v_i}$ $rel\{v_{i-1}\}$ by a proper homotopy H_i with image avoiding C_{j-2}.

If $s_i \in \mathcal{A}^{\pm 1}$, this is clear by condition (3) with H_i avoiding C_{j-1}. If $s_i \in \mathcal{K}^{\pm 1}$, then $s_i * r_{v_i}$ is properly homtopic $rel\{v_{i-1}\}$ to an $\mathcal{A}$-ray, $t_{v_{i-1}}$ (using only 2-cells arising from Q) and this homotopy has image in $St(Im(s_i * r_{v_i})) \subset \tilde{\Gamma} - C_{j-1}$. Since $t_{v_{i-1}}$ and $r_{v_{i-1}}$ are $\mathcal{A}$-rays with images avoiding C_{j-1}, condition (3) on the sets C_i gives a proper homotopy between them $rel\{v_{i-1}\}$, whose image avoids C_{j-2}. Patch these two proper homotopies together to obtain H_i.

Let H be the homotopy $rel\{v\}$ of s to r_v, obtained by patching together the homotopies H_i. We need to check that H is proper. Let $C \subset \tilde{\Gamma}$ be compact. Choose an index j such that $C \subseteq C_j$. Since s is a proper edge path to infinity, choose an index N such that all edges after the N^{th} edge of s avoid C_{j+2}. Then for all $i > N$, H_i avoids C_j, so H is proper. □

This concludes the proof of the theorem. □

6 Simple Connectivity at ∞

Recall, a connected locally finite CW-complex X is simply connected at ∞ if for each compact set C in X there is a compact set D in X such that loops in $X-D$ are homotopically trivial in $X-C$. A group G is simply connected at ∞ if given some, equivalently any (see Lee and Raymond [14], Theorem 3), finite complex X with $\pi_1(X)=G$, then the universal cover of X is simply connected at ∞.

If G is a group and H a subgroup of G there are various notions for the number of ends of the pair (G,H). Chapter 14 of Geoghegan [7], gives a good account of these notions. In particular, the idea of the number of filtered ends of the pair (G,H) is developed and compared to the standard number of ends of a pair. In any case, the number of filtered ends of the pair (G,H) is greater than or equal to the number of standard ends of the pair. Proposition 14.5.9 [7], shows that if H is a normal subgroup of G, then the number of ends of G/H, the standard number of ends of (G,H) and the number of filtered ends of (G,H) are all the same. Conner and Mihalik [3], show that if G is a group with finite generating set S and Q is a finitely generated commensurated subgroup of G, then the number of filtered ends of (G,Q) equals the number of ends of $\Lambda(S,Q,G)$.

Theorem 6.1 *Suppose G is a finitely presented group with finite generating set S, and Q is a finitely presented, infinite commensurated subgroup of infinite index in G. If Q or $\Lambda(S,Q,G)$ is one-ended, then G is simply connected at ∞.*

Proof Suppose $\mathcal{P}=\langle q_1,\ldots,q_a,k_1,\ldots,k_b : R\rangle$ is a finite presentation of the group G such that the q_i generate the infinite commensurated subgroup Q, no k_i is an element of Q, and R contains relations R' such that $\langle q_1,\ldots,q_a : R'\rangle$ is a finite presentation of Q. Assume that Q has infinite index in G. Let X be the Cayley 2-complex of $\mathcal{P}$, $\tilde X$ the universal cover for X and $\tilde X(Q,v)\subset\tilde X$ the copy of the universal cover of the Cayley 2-complex for $\langle q_1,\ldots,q_a : R'\rangle$ containing v. Let $\mathcal{K}=\{k_1,\ldots,k_b\}$ and $\mathcal{Q}=\{q_1,\ldots q_a\}$.

Let N_1 be an integer such that if cosets gQ and hQ of G are connected by an edge in $\tilde X$, then the Hausdorff distance between gQ and hQ in $\tilde X$ is $\leq N_1$. For each relator $r\in R$, let r' be the word obtained from r by removing $\mathcal{Q}$ letters. For each such (non-trivial) r' and edge loop in $\Lambda(S,Q,G)$ with edge label r', attach a 2-cell and call the resulting locally finite 2-complex $\hat\Lambda(S,Q,G)$. Note that $\Gamma(S,G)$ is the 1-skeleton of $\tilde X$. Extend the map $\rho:\Gamma(S,G)\to\Lambda(S,Q,G)$ (see proposition 2.2), to $\rho:\tilde X\to\hat\Lambda(S,Q,G)$. Let C be a finite subcomplex of $\tilde X$. Let $d_1\geq 1$ be an integer such that for each vertex v of $\rho(C)$, there is a $\mathcal{K}$-edge path in $\hat\Lambda(S,Q,G)$ of length $\leq d_1$ from v to a vertex of $\hat\Lambda(S,Q,G)-\rho(C)$. In particular, for each vertex v of $\tilde X$, there is an edge path at v of length $\leq N_1d_1$ and with end point w such that $\tilde X(Q,w)\cap C=\emptyset$. For each $k\in\{k_1,\ldots,k_b\}$ assume that Q and kQ are within Hausdorff distance N_1. Choose N_2 so that if q_1 and q_2 are two Q-vertices of $\tilde X$ with the edge path distance in $\tilde X$ between q_1 and q_2 less than or equal to $2N_1+1$, then the edge path distance between q_1 and q_2 in $\tilde X(Q,q_1)$ is $\leq N_2$. In particular, there is a $\mathcal{Q}$-edge path between q_1 and q_2 of length $\leq N_2$. Choose N_3 such that if α is an edge path loop at $* \in \tilde{X}$ of length $\leq 2N_1 + N_2 + 1$, then α is homotopically trivial in $St^{N_3}(*)$.

Lemma 6.2 *Suppose G is a finitely presented group, Q is a finitely presented, infinite commensurated subgroup of infinite index in G, $\mathcal{P}$ is a presentation of G as above, and X is the Cayley 2-complex of $\mathcal{P}$. If α is a $\mathcal{Q}$-loop in $\tilde{X}$, with image in $\tilde{X} - St^{d_1N_1N_3}(C)$, then α is homotopically trivial in $\tilde{X} - C$.*

Proof We may assume v is the initial vertex of α. If $\tilde{X}(Q, v) \cap C = \emptyset$, then α is homotopically trivial in $\tilde{X}(Q, v)$ and we are finished. If $\tilde{X}(Q, v) \cap C \neq \emptyset$, there is an edge path $\beta = (b_1, \ldots, b_k)$ at v with $k \leq N_1 d_1$ and with end point w such that $\tilde{X}(Q, w) \cap C = \emptyset$. Let $v \equiv v_0, \ldots, v_k \equiv w$ be the consecutive vertices of α. For each vertex x of β, there is an edge path of length $\leq N_1$ from x to a vertex of $\tilde{X}(Q, v_1)$ (if b_1 is a $\mathcal{Q}$-edge, this path is trivial) and hence α is homotopic $rel\{0, 1\}$ to a loop $(b_1, \alpha_1, b_1^{-1})$, where α_1 is a $\mathcal{Q}$-loop in $\tilde{X}(Q, v_1)$, by a homotopy in $St^{N_3}(im(\alpha))$. Inductively, α is freely homotopic to a $\mathcal{Q}$-loop α_k at the end point of β, by a homotopy in $St^{kN_3}(im(\alpha)) \subset \tilde{X} - C$. As $\tilde{X}(Q, w) \cap C = \emptyset$ and $im(\alpha_k) \subset \tilde{X}(Q, w)$, α_k (and hence α) is homotopically trivial in $\tilde{X} - C$. □

Case 1: Q is one-ended. There are finitely many vertices $w_1, \ldots, w_n \in \tilde{X}$ such that $\tilde{X}(Q, w_i) \cap St^{(d_1N_1+1)N_3}(C) \neq \emptyset$. As $\tilde{X}(Q, w_i)$ is one-ended, there is a compact subcomplex D of $\tilde{X}$ such that $St^{(d_1N_1+1)N_3}(C) \subset D$ and for all $i \in \{1, \ldots, n\}$ and vertices $x, y \in \tilde{X}(Q, w_i) - D$, x and y can be joined by a $\mathcal{Q}$-edge path in $\tilde{X}(Q, v_i) - St^{(d_1N_1+1)N_3}(C)$. Now, suppose α is an arbitrary loop in $\tilde{X} - D$ with initial vertex v. Choose L a positive integer such that if q_1 and q_2 are vertices of $\tilde{X}(Q, *)$ that are of distance $\leq N_1|\alpha|$ apart in $\tilde{X}$, then they are of distance $\leq L$ in $\tilde{X}(Q, *)$. Choose E such that any edge path loop τ at a vertex x of $\tilde{X}$, of length $\leq N_1|\alpha| + L$, is homotopically trivial in $St^E(x)$. Let β_1 be a $\mathcal{Q}$-path in $\tilde{X}(Q, v) - St^{(d_1N_1+1)N_3}(C)$ from v to a point

$$w \in \tilde{X} - (St^E(C) \cup St^{d_1N_1N_3+L}(C) \cup St^{N_1|\alpha|}(D))$$

Write the edge path α as $(e_1, \ldots, e_m)$ with consecutive vertices $v = v_1, \ldots, v_m$. As $w \in \tilde{X}(Q, v)$ there is an edge path τ_1 of length $\leq N_1$ from $w = w_1$ to $w_2 \in \tilde{X}(Q, v_2)$. Let τ_2 be an edge path of length $\leq N_1$ from w_2 to $w_3 \in \tilde{X}(Q, v_3)$. Inductively, τ_m is an edge path of length $\leq N_1$ from w_m to a vertex $w_{m+1} \in \tilde{X}(Q, v)$. (Note that τ_i may be taken as the trivial path if e_i is a $\mathcal{Q}$-edge.) As the edge path $(\tau_1, \ldots, \tau_m)$ has length $\leq N_1|\alpha|$, there is a $\mathcal{Q}$-path λ, from w_{m+1} to w of length $\leq L$. By the definition of E, the loop $\tau \equiv (\tau_1, \ldots, \tau_m, \lambda)$ at w is homotopically trivial in $\tilde{X} - C$. Hence, it suffices to show that α is freely homotopic to τ in $\tilde{X} - C$. (See Figure 1.)

First note that each vertex of $(\tau_1, \ldots, \tau_m)$ is in $\tilde{X} - D$, since the vertex $w \in \tilde{X} - St^{N_1|\alpha|}(D)$. Next, write β_1 as the edge path $(b_1, \ldots, b_s)$. Let $\phi_0 = e_1$ and let ϕ_i be an edge path of length $\leq N_1$ from the end point of b_i to a point of $\tilde{X}(Q, v_2)$. Let ψ_i be a $\mathcal{Q}$-edge path of length $\leq N_2$ from the end point of ϕ_{i-1} to the end point of ϕ_i. (Choose $\phi_s = \tau_1$.) Then the loop $(\phi_{i-1}, \psi_i, \phi_i^{-1}, b_i^{-1})$ has length $\leq 2N_1 + N_2 + 1$ and is homotopically trivial by a homotopy in the N_3-star of the

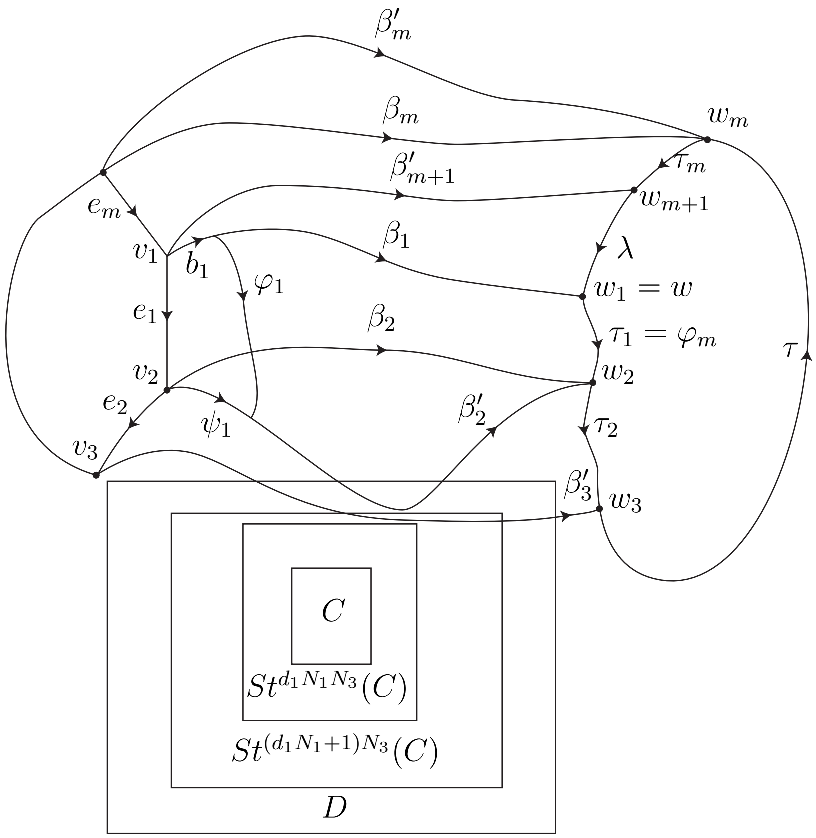

Figure 1

initial point of b_i. String together these homotopies and we have that the edge path $\langle e_1^{-1}, \beta_1, \tau_1\rangle$ is homotopic $rel\{0,1\}$ to the $\mathcal{Q}$-edge path $\beta'_2 \equiv (\psi_1, \ldots, \psi_m)$ by a homotopy with image in $St^{N_3}(im(\beta_1)) \subset \tilde{X} - St^{d_1N_1N_3}(C)$. By the definition of D, there is an $\mathcal{Q}$-edge path β_2 with the same end points as β'_2 and with image in $\tilde{X} - St^{(d_1N_1+1)N_3}(C)$. By Lemma 6.2, β_2 and β'_2 are homotopic $rel\{0,1\}$ by a homotopy in $\tilde{X} - C$. Continue inductively until β_m and β'_{m+1} are defined. Since $w \in \tilde{X} - St^{d_1N_1N_3+L}(C)$, the path λ (of length $\leq L$) has image in $\tilde{X} - St^{d_1N_1N_3}(C)$. By Lemma 6.2, the $\mathcal{Q}$-loop $(\beta'_{m+1}, \lambda, \beta_1^{-1})$ is homotopically trivial in $\tilde{X} - C$.

Case 2: $\Lambda(S,Q,G)$ **is one-ended.** The letters N_1, N_2 and N_3 remain as in case 1 and we recycle letters used for any other constant.

Given C a finite subcomplex of $\tilde{X}$. Consider $\rho(C) \subset \hat{\Lambda}(S,Q,G)$. Choose D a finite subgraph of $\Lambda(S,Q,G)$ such that any two vertices of $\hat{\Lambda}(S,Q,G) - D$ can be connected by a path in $\hat{\Lambda}(S,Q,G) - \rho(St^{N_3}(C))$. For each vertex v of D choose a path $\bar{\alpha}_v$ from v to a vertex of $\hat{\Lambda}(S,Q,G) - D$. If v is a vertex of $\hat{\Lambda}(S,Q,G) - D$, let $\bar{\alpha}_v$ be the trivial path. Let N be the length of the longest path $\bar{\alpha}_v$ for $v \in D$. If v is a vertex of $\tilde{X}$ such that $\rho(v) \in D$ let α_v be an edge path of the form $(\beta_1, \ldots, \beta_m)$ where each β_i has length $\leq N_1$ and $\rho(\beta_i)$ has the same end points as the i^{th} edge of $\bar{\alpha}_{\rho(v)}$ (so $|\alpha_v| \leq N_1 N$). In analogy with previous terminology, we call α_v an N_1-approximate lift of $\bar{\alpha}_{\rho(v)}$. If $\rho(v) \notin D$, let α_v be the trivial path.

Choose an integer M such that if v and w are adjacent vertices of $St(D)$, then there is an edge path $\bar{\alpha}_{v,w}$ in $\hat{\Lambda}(S,Q,G) - \rho(St^{N_3}(C))$ of length $\leq M$ from the end point of $\bar{\alpha}_v$ to the end point of $\bar{\alpha}_w$. Choose an integer B such that if β is a $\tilde{X}$-edge path of length $\leq (2N+M)N_1 + 1$ connecting $*$ (the vertex of $\tilde{X}$ corresponding to the identity element of G) to a vertex $q \in Q$, then there is a $\mathcal{Q}$-edge path of length $\leq B$ connecting $*$ to q. Choose an integer A such that if β is an edge path loop at $*$ of length $\leq (2N+M)N_1 + B + 1$, then β is homotopically trivial in $St^A(*)$.

We next show: If β is an edge path loop in $\tilde{X} - St^A(C)$, then β is freely homotopic to a loop $\hat{\beta}$ by a homotopy in $\tilde{X} - C$ where $\hat{\beta}$ can be chosen so that for each vertex v of $\hat{\beta}$, $\rho(v) \notin \rho(St^{N_3}(C))$. If e is a directed edge of $\tilde{X}$ or $\Lambda(S,Q,G)$, with initial point a and terminal point b, then let $[a,b]$ represent this edge. Suppose β is the edge path $(d_1, d_2, \ldots, d_n)$ with consecutive vertices $b_1, \ldots, b_{n+1}$. If (cyclically) neither $\rho(b_i)$ nor $\rho(b_{i+1})$ is in D, then let $\hat{\beta}_i$ be the single edge d_i. Otherwise, $\rho(b_i)$ and $\rho(b_{i+1})$ belong to $St(D)$. In this case, consider the edge path $\delta_i \equiv (\alpha_{b_i}^{-1}, d_i, \alpha_{b_{i+1}})$ of $\tilde{X}$.

If $\rho(b_i) \neq \rho(b_{i+1})$, the edge path $\bar{\alpha}_{\rho(b_i),\rho(b_{i+1})}$ joins the end points of $\rho(\delta_i)$ and has length $\leq M$. Let α_i be an N_1-approximate lift of $\bar{\alpha}_{\rho(b_i),\rho(b_{i+1})}$ to the initial point of δ_i (otherwise, let α_i be the trivial path at the initial point of δ_i).

Note that the end point of α_i and the end point of δ_i belong to the same left Q-coset. As the length of $(\alpha_i^{-1}, \delta_i)$ is $\leq (2N+M)N_1 + 1$ there is a $\mathcal{Q}$-edge path γ_i of length $\leq B$ from the initial point to the end point of $(\alpha_i^{-1}, \delta_i)$. The loop $(\gamma_i^{-1}, \alpha_i^{-1}, \delta_i)$ has length $\leq 2N + M + B + 1$ and so is homotopically trivial in $\tilde{X} - C$ (by the definition of A). Let $\hat{\beta}_i = (\alpha_i, \gamma_i)$, for $i \in \{1, \ldots, n\}$. Let $\hat{\beta}$ be the loop $(\hat{\beta}_1, \ldots, \hat{\beta}_n)$. Combining homotopies shows that β is freely homotopic to $\hat{\beta}$ by a homotopy in $\tilde{X} - C$. As $\rho(\alpha_i)$ avoids $\rho(St^{N_3}(C))$, $\rho(\hat{\beta})$ avoids $\rho(St^{N_3}(C))$. (See Figure 2.)

We conclude the proof of case 2 by showing $\hat{\beta}$ is homotopically trivial in $\tilde{X} - C$. The proof is analogous to the closing argument of case 1. Let v be the initial vertex of $\hat{\beta}$. Choose L a positive integer such that if q_1 and q_2 are vertices of $\tilde{X}(Q,*)$ that are of distance $\leq N_1|\hat{\beta}|$ apart in $\tilde{X}$, then they are of distance $\leq L$ in $\tilde{X}(Q,*)$. Choose E such that any edge path loop τ at a vertex x of $\tilde{X}$ and of length $\leq N_1|\hat{\beta}| + L$, is homotopically trivial in $St^E(x)$. Let β_1 be a $\mathcal{Q}$-path from v to a point $w \in \tilde{X} - St^E(C)$. Write the edge path $\hat{\beta}$ as $(e_1, \ldots, e_m)$ with consecutive vertices $v \equiv v_1, v_2, \ldots, v_m$. As $w \in \tilde{X}(Q,v)$ there is an edge path τ_1 of length $\leq N_1$ from w to $w_2 \in \tilde{X}(Q,v_2)$. Let τ_2 be an edge path of length $\leq N_1$ from w_2 to $w_3 \in \tilde{X}(Q,v_3)$. Inductively, τ_m is an edge path of length $\leq N_1$ from w_m to a vertex $w_{m+1} \in \tilde{X}(Q,v)$. (Note that τ_i may be taken as the trivial path if e_i is a $\mathcal{Q}$-edge.) As the edge path $(\tau_1, \ldots \tau_m)$ begins and ends in $\tilde{X}(Q,v)$ and has length $\leq N_1|\hat{\beta}|$, there is a $\mathcal{Q}$-path λ, from w_{m+1} to w of length $\leq L$. By the definition of E, the loop $\tau \equiv (\tau_1, \ldots, \tau_m, \lambda)$ at w is homotopically trivial in $\tilde{X} - C$. Hence, it suffices to show that α is freely homotopic to τ in $\tilde{X} - C$.

Each vertex b of β_1 is such that $\rho(v) = \rho(b) \in \hat{\Lambda}(S,Q,G) - \rho(St^{N_3}(C))$ and so the image of β_1 avoids $St^{N_3}(C)$. As in case 1, this implies that the path $(e_1^{-1}, \beta_1, \tau_1)$ is homotopic $rel\{0,1\}$

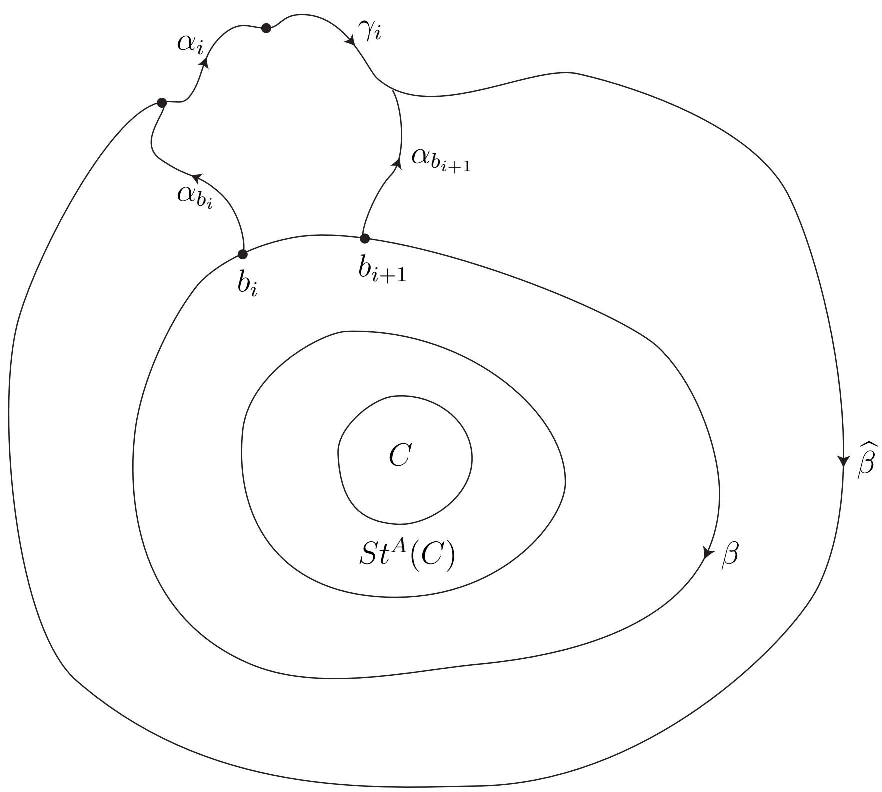

Figure 2

to a $\mathcal{Q}$-edge path β_2 by a homotopy with image in $St^{N_3}(im(\beta_1)) \subset \tilde{X} - C$. Each vertex b of β_2 is such that $\rho(b) = \rho(v_2) \in \hat{\Lambda}(S,Q,G) - \rho(St^{N_3}(C))$ and so the image of β_2 avoids $St^{N_3}(C)$. The path $(e_2^{-1}, \beta_2, \tau_2)$ is homotopic $rel\{0,1\}$ to a $\mathcal{Q}$-edge path β_3 by a homotopy with image in $St^{N_3}(im(\beta_2)) \subset \tilde{X} - C$. Continue inductively until β_{m+1} is defined (as a $\mathcal{Q}$-path from v to w_{m+1}). As $\rho(v) \in \hat{\Lambda}(S,Q,G) - \rho(St^{N_3}(C))$, the $\mathcal{Q}$-loop $(\beta_1, \lambda^{-1}, \beta_{m+1}^{-1})$ has image in $\tilde{X}(Q,v) \subset \tilde{X} - C$, and so is homotopically trivial in $\tilde{X} - C$. (See Figure 3.) Combining homotopies produces a null homotopy of $\hat{\beta}$ with image in $\tilde{X} - C$. □

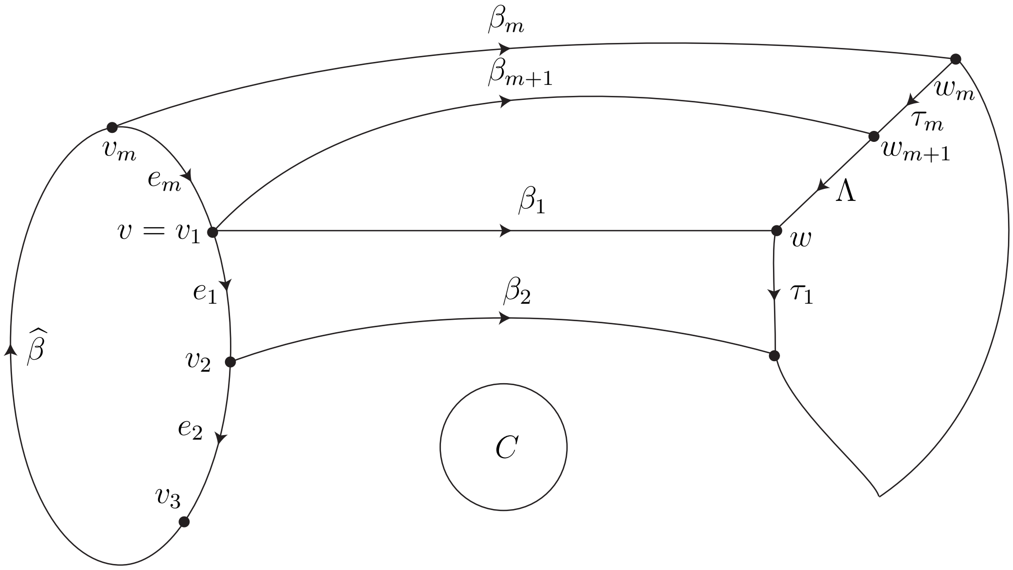

Figure 3

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