Relatively Hyperbolic Groups with Semistable Fundamental Group at Infinity

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Abstract

Suppose G is a 1-ended finitely generated group that is hyperbolic relative to $\mathbf{P}$ a finite collection of 1-ended finitely generated proper subgroups of G. Our main theorem states that if the boundary $\partial(G, \mathbf{P})$ has no cut point, then G has semistable fundamental group at ∞. Under mild conditions on G and the members of $\mathbf{P}$ the 1-ended hypotheses and the no cut point condition can be eliminated to obtain the same semistability conclusion. We give an example that shows our main result is somewhat optimal. Finally, we improve a "double dagger" result of F. Dahmani and D. Groves.

1 Introduction

In this paper, we are interested in the asymptotic behavior of relatively hyperbolic groups. We consider a property of finitely presented groups that has been well studied for over 40 years called semistable fundamental group at ∞. It is unknown at this time, whether or not all finitely presented groups have semistable fundamental group at ∞. The finitely presented group G satisfies a weaker condition called semistable first homology at ∞ if and only if $H^2(G : \mathbb{Z}G)$ is free abelian (see [GM85]) and it is also a long standing problem if this is always the case. Our main interest is in showing certain relatively hyperbilic groups have semistable fundamental group at ∞. The

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work of B. Bowditch [Bow99b] and G. Swarup [Swa96] show that if G is a 1-ended word hyperbolic group then ∂G, the boundary of G, has no (global) cut point. By work of M. Bestvina and G. Mess [BM91], this is equivalent to ∂G being locally connected. It was pointed out by R. Geoghegan that G has semistable fundamental group at ∞ if and only if ∂G has the shape of a locally connected continuum (see [DS78] for a proof of this fact). In particular, all 1-ended word hyperbolic groups have semistable fundamental group at ∞.

Relatively hyperbolic groups are a much studied generalization of hyperbilic groups. In section 4, we define what it means for a finitely generated 1-ended group to be hyperbolic relative to a finite collection of finitely generated subgroups. If a group G is hyperbolic relative to a collection of subgroups $\mathbf{P}$ the pair $(G, \mathbf{P})$ has a well-defined boundary. While all 1-ended hyperbolic groups have locally connected boundary without cut points, the boundary of the relatively hyperbolic pair $(G, \mathbf{P})$ may be locally connected and contain cut points. At this time, there is no known example of a finitely generated relatively hyperbolic group with a connected boundary that is not locally connected. If the boundary $\partial(G, \mathbf{P})$ is connected and locally connected then it is the continuous image of the interval $[0, 1]$ and so it is path connected and locally path connected, but these facts seems to be of no use in our approach to showing G has semistable fundamental group at ∞. The absence of cut points is critical.

Our main result is as follows. Note that there is no semistability hypothesis on the peripheral subgroups P_i.

Theorem 1.1 *Suppose G is a 1-ended finitely generated group that is hyperbolic relative to a collection of 1-ended finitely generated proper subgroups $\mathbf{P} = \{P_1, \ldots, P_n\}$. If $\partial(G, \mathbf{P})$ has no cut point, then G has semistable fundamental group at ∞.*

Example 1 of §2 indicates the importance of the no cut point hypothesis in our theorem. We explain that if one could eliminate that hypothesis, then the resulting theorem would "nearly" imply that all finitely presented groups have semistable fundamental group at ∞.

All of our work is done in a "cusped" space for $(G, \mathbf{P})$. This is a Gromov hyperbolic, locally finite 2-complex on which our relatively hyperbolic group acts by isometries, but not co-compactly (see §5). It follows from ([Bow12], §6 and §9) that the Bowditch boundary for $(G, \mathbf{P})$ agrees with the Gromov boundary of a corresponding cusped space. One of the main tools of our

proof is a "double dagger" result that appears as Lemma 4.2 of [DG08]. This result is analogous to the original double dagger lemma of [BM91]. The key hypothesis in both of these lemmas is that the boundary of the hyperbolic space under consideration has no cut point. The conclusion of Lemma 4.2 of [DG08] gives a certain "replacement" path, but this path is only guaranteed to have image in the cusped space X. We are able to improve this result (see Theorem 8.1) to provide a replacement path in X_m, a fixed subspace of X on which G acts cocompactly by isometries. The base space Y in X is a universal cover of a finite complex with fundamental group G and our goal in Theorem 1.1 is to show that Y (and hence G) has semistable fundamental group at ∞. It is straightforward to show that Y has semistable fundamental group at ∞ if and only if the neighborhood X_m of Y in X has semistable fundamental group at ∞ (see Theorem 7.4). This is where the specialized double dagger replacement paths play a fundamental role in our proof of the main theorem.

The remainder of the paper is organized as follows: In §2 we describe a process for generalizing our main theorem from 1-ended groups G and 1-ended peripheral subgroups to infinite ended groups. We also describe how it may be possible to extend our theorem to the situation when $\partial(G, \mathbf{P})$ has cut points. These generalizations depend on classical results in the literature. We also point out limitations to our techniques. The proof of our main theorem is independent of §2 and §3. We list a few results about the asymptotic behavior and relative hyperbolicity of CAT(0) groups in §3. In §4 we give several equivalent definitions of semistability that are used in our proofs. In §5 we define cusped spaces and extract three results from [GM08]. All of our technical work takes place in cusped spaces. The main lemmas 6.7 and 6.8 are proved in §6. Also an elementary, but critical observation, Lemma 6.5, about "dead ends" in a general finitely generated group is proved. Section 7 contains the proof of the main theorem and a proof that the subspaces X_m of the cusped space X have semistable fundamental group at ∞ (under the hypotheses of the main theorem). Section §8 contains an argument that uses the main theorem and improves the double dagger result of Dahmani-Groves. Instead of producing "far out" paths in the cusped space X, we produce "far out" paths in X_m. Finally in section §9 a result that seems far more general than one of our key lemmas (Lemma 6.7) is proved.

2   Related Results and Generalizations

We will list several results in the literature which, in conjunction with our theorem, imply that in many cases we can drop the 1-ended hypotheses and the no cut point hypothesis and still obtain the same semistability conclusion. There seems to be no umbrella statement covering all of our applications of Theorem 1.1. The results in this section can be combined multiple times and in various ways to show many relatively hyperbolic groups have semistable fundamental group at ∞. In order to avoid accessibility issues for infinite ended groups, we begin with a finitely presented group G hyperbolic relative to a finite collection $\mathbf{P}$ of finitely presented subgroups. The following steps describe how one might expect to show this group has semistable fundamental group at ∞. We follow this list of steps with statements of theorems that make each step viable. Finally, we conclude the section with a few exotic situations that might make it impossible to carry out one of the steps.

Step 1. Replace the members of $\mathbf{P}$ by a collection $\hat{\mathbf{P}}$ of 1-ended subgroups of the members of $\mathbf{P}$ so that G is hyperbolic relative to $\hat{\mathbf{P}}$.

Step 2. Replace G by a finite collection of 1-ended subgroups G_i of G, such that each G_i is hyperbolic relative to $\mathbf{P_i}$ a collection of (1-ended) members of $\hat{\mathbf{P}}$.

Step 3. If $\partial(G_i, \mathbf{P_i})$ has a cut point, then split G_i and replace it with vertex groups in an "accessible" way so that the splittings eventually cease (i.e. there are no boundary cut points and all groups involved are 1-ended).

Step 4. Apply Theorem 1.1 to show that the resulting G_i have semistable fundamental group at ∞.

Step 5. Use a semistability combination result to show the original group G has semistable fundamental group at ∞.

If a group is G is finitely presented and hyperbolic relative to a finite collection of finitely generated subgroups P_i, then the P_i are finitely presented as well (see [DG13] or for a more general result [DGO17], Theorem 2.11). The following result can be derived from work of C. Drutu and M. Sapir [DtS05] (Corollary 1.14) or D. Osin [Osi06].

Theorem 2.1 *Suppose G is finitely generated and hyperbolic relative to the finitely generated groups $P_1, \ldots, P_n$. If $\mathcal{P}_i$ is a finite graph of groups decomposition of P_i such that each edge group of $\mathcal{P}_i$ is finite, then G is also hyperbolic relative to $\{P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n\} \cup V(\mathcal{P}_i)$ where $V(\mathcal{P}_i)$ is the set of vertex groups of $\mathcal{P}_i$.*

If a peripheral subgroup P_i is either finite or 2-ended, it may be removed from the collection of peripheral subgroups and G remains hyperbolic relative to the remaining subgroups. Recall that a finitely generated group is *accessible* if it has a finite graph of groups decomposition with each edge group finite and each vertex group either finite or 1-ended. By M. Dunwoody's accessibility theorem [Dun85], all (almost) finitely presented groups are accessible. Combining Dunwoody's accessibility theorem with Theorem 2.1 we can carry out *Step 1*.

The following results about splittings are all due to B. Bowditch.

Proposition 2.2 *([Bow12], Propositions 10.1-3). Let $(G, \mathbf{P})$ be a relatively hyperbolic group. Its boundary $\partial(G, \mathbf{P})$ is disconnected if and only if G splits nontrivially over a finite group relative to $\mathbf{P}$. In this case, every vertex group is hyperbolic relative to the parabolic subgroups it contains. In particular, if the parabolic subgroups of G are all 1-ended, then G is 1-ended if and only if ∂G is connected.*

So Dunwoody accessibility theorem and Proposition 2.2 can be combined to carry out *Step 2*. A splitting is *relative to* $\mathbf{P}$ if each element of $\mathbf{P}$ is conjugate into a vertex group of the spllitting.

Definition 1. ([Bow01]). Let $(G, \mathbf{P})$ be a relatively hyperbolic group. A *peripheral splitting* of G is a representation of G as a finite bipartite graph of groups where $\mathbf{P}$ consists precisely of the (conjugacy classes of) vertex groups of one color. A peripheral splitting is a refinement of another if there is a color preserving folding of the first into the second.

The following organization of results of Bowditch appears in [DG08].

Theorem 2.3 *(B. Bowditch) Let $(G, \mathbf{P})$ be a relatively hyperbolic group. Assume that ∂G is connected.*

(1) ([Bow99b], Theorem 0.2) If every maximal parabolic subgroup of G is (1 or 2)-ended, finitely presented, and without an infinite torsion subgroup, then, every global cut point of ∂G is a parabolic fixed point.

(2) ([Bow99a], Theorem 1.2) If there is a global cut point that is a parabolic point, then G admits a proper peripheral splitting.

(3) ([Bow01], Theorem 1.2) If G admits a proper peripheral splitting, then ∂G admits a global cut point.

It is established in ([Bow01], Theorem 1.3) that if ∂G is connected, then any non-peripheral vertex group of a peripheral splitting also has connected boundary and is hyperbolic relative to its adjacent edge groups.

Bowditch also proves an accessibility result.

Theorem 2.4 *([Bow01], Theorem 1.4) Suppose that G is relatively hyperbolic with connected boundary. Then G admits a (possibly trivial) peripheral splitting which is maximal in the sense that it is not a refinement of any other peripheral splitting.*

Combining Theorems 2.3 and Theorem 2.4 might allow us to carry out *Step 3*. There are potential issues in carrying out this step that will be discussed later in this section. At this point we are in a position to carry out *Step 4*. Apply Theorem 1.1 to the various (vertex) subgroups of G and conclude that each has semistable fundamental group at ∞.

The following result of M. Mihalik and S. Tschantz meshes well with Steps 1-4 and allows us to work our way back from the semistable groups of Step 4 and show (*Step 5*) the original group G has semistable fundamental group at ∞. Notice that in this next result, there is no restriction on the number of ends of G, the vertex groups or the edge groups.

Theorem 2.5 *[MT92] Suppose $\mathcal{G}$ is a finite graph of groups decomposition of the finitely presented group G where each edge group is finitely generated and each vertex group is finitely presented with semistable fundamental group at ∞. Then G has semistable fundamental group at ∞.*

There are obvious limitations to our results. If G is a 1-ended finitely generated group that is hyperbolic relative to an inaccessible group P then our results do not apply. We list a few other instances where our approach may break down and explain why our result seems optimal.

Example 1. Suppose B is a hyperbolic surface group with infinite cyclic subgroup C and A is an arbitrary finitely presented 1-ended group with infinite cyclic subgroup C'. The group $G = A *_{C=C'} B$ is a finitely presented 1-ended group. By ([Dah03], Theorem 0.1) G is hyperbolic relative to A. The boundary $\partial(G, A)$ has a parabolic cut point (since G splits over C). It would be unreasonable to expect the theory of relatively hyperbolic groups to imply that G is semistable at ∞. Such a result would imply that (the arbitrary 1-ended group) A is "nearly" semistable at ∞. In fact, if A contained an

infinite cyclic group whose two ends were "properly homotopic in A" (any copy of $\mathbb{Z} \times \mathbb{Z}$ in A would suffice), then in fact, the semistablity of G would imply the semistability of A. In our 5-step process, this example breaks down at Step 3. Bowditch's results would leave us with the vertex group A which is hyperbolic with respect to itself. Obviously, that is not enough to prove A has semistable fundamental group at ∞.

The boundaries of the non-peripheral vertex groups of a maximal splitting guaranteed by Theorem 2.4 do not contain parabolic cut points and it is conjectured that they contain no global cut points of any sort. If that is the case, and G is a finitely presented group that is hyperbolic relative to a finite collection $\mathbf{P}$ of finitely presented subgroups then the only thing that could prevent us from showing that G is semistable at ∞ is the possibility of running into an inaccessible peripheral subgroup or a non-semistable peripheral vertex group of the maximal splitting (perhaps the group A in Example 1).

If G and the members of $\mathbf{P}$ are all accessible, then Theorems 2.1 and 2.2 reduce to the case where G and the members of $\mathbf{P}$ are 1-ended. If the edge groups of non-peripheral vertex groups of a maximal peripheral splitting of G are finitely presented, 1-ended and without infinite torsion subgroups, then the boundaries of the non-peripheral vertex groups have no global cut points and Theorem 1.1 implies each of these vertex groups has semistable fundamental group at ∞. If the peripheral vertex groups also have semistable fundamental group at ∞ then Theorem 2.5 implies G has semistable fundamental group at ∞. In this sense, and in light of Example 1, Theorem 1.1 seems optimal for this theory.

3   Complimentary CAT(0) Results

The CAT(0) groups with isolated flats, are intuitively the CAT(0) groups that are closest to hyperbolic groups without actually being hyperbolic. As an application to their main theorem C. Hruska and K. Ruane prove:

Theorem 3.1 *([HR], Theorem 1.2) Any CAT(0) group with isolated flats has semistable fundamental group at ∞.*

CAT(0) groups with isolated flats are hyperbolic relative to their maximal abelian subgroups and this fact plays a critical role in their argument.

While it is known that a boundary of a 1-ended CAT(0) group cannot have a global cut point (P. Papasoglu and E. Swenson [PS09] and E. Swenson [Swe99]) it is unknown if all CAT(0) groups are semistable at ∞ (see [GS]). Let F_2 be the free group of rank 2. The CAT(0) group $F_2 \times F_2$ is not hyperbolic with respect to any collection of subgroups (see [DtS05]).

4   Semistability Background

The best reference for the notion of semistable fundamental group at ∞ is [Geo08] and we use this book as a general reference throughout this section. While semistability makes sense for multiple ended spaces, we are only interested in 1-ended spaces in this article. Suppose K is a 1-ended locally finite connected CW complex. A *ray* in K is a continuous map $r : [0, \infty) \to K$. A continuous map $f : X \to Y$ is *proper* if for each compact set C in Y, $f^{-1}(C)$ is compact in X. The space K has *semistable fundamental group at ∞* if any two proper rays in K are properly homotopic. Suppose $C_0, C_1, \ldots$ is a collection of compact subsets of a locally finite complex K such that C_i is a subset of the interior of C_{i+1} and $\cup_{i=0}^{\infty} C_i = K$, and $r : [0, \infty) \to K$ is proper, then $\pi_1^\infty(K, r)$ is the inverse limit of the inverse system of groups:

$$\pi_1(K - C_0, r) \leftarrow \pi_1(K - C_1, r) \leftarrow \cdots$$

This inverse system is pro-isomorphic to an inverse system of groups with epimorphic bonding maps if and only if K has semistable fundamental group at ∞ (see Theorem 2.1 of [Mih83] or Theorem 16.1.2 of [Geo08]). It is an elementary exercise to see that semistable fundamental group at ∞ is an invariant of proper homotopy type and S. Brick [Bri93] proved that semistability is a quasi-isometry invariant. When K is 1-ended with semistable fundamental group at ∞, $\pi_1^\infty(K, r)$ is independent of proper base ray r (in direct analogy with the fundamental group of a path connected space being independent of base point). There are a number of equivalent forms of semistability. Theorem 2.1 of [Mih83], and Lemma 9 of [Mih86a], provide several equivalent notions of semistability. The space considered in [Mih83] is simply connected, but simple connectivity is not important in that argument. A slight modification of proofs give the following result. Condition 2. is the semistability criterion used in the proof of our main theorem.

Theorem 4.1 *(see Theorem 3.2, [CM14]) Suppose K is a connected 1-ended locally finite CW-complex. Then the following are equivalent:*

1. K has semistable fundamental group at ∞.

2. Suppose $r : [0, \infty) \to K$ is a proper base ray. Then for any compact set C, there is a compact set D such that for any third compact set E and loop α based on r and with image in $K - D$, α is homotopic $rel\{r\}$ to a loop in $K - E$, by a homotopy with image in $K - C$.

3. For any compact set C there is a compact set D such that if r and s are proper rays based at v and with image in $K - D$, then r and s are properly homotopic $rel\{v\}$, by a proper homotopy in $K - C$.

If G is a finitely presented group and X is a finite connected complex with $\pi_1(X) = G$ then G has *semistable fundamental group at* ∞ if the universal cover of X has semistable fundamental group at ∞. This definition only depends on G (see the proof of Theorem 3 of [LR75] or the opening paragraph of section 16.5 of [Geo08]) and it is unknown if all finitely presented groups have semistable fundamental group at ∞.

In order to prove certain finitely presented groups have semistable fundamental group at ∞, the notion of semistable fundamental group at ∞ for a finitely generated group was introduced in [Mih86b]. Our main interest here is also finitely presented groups, but we might begin with a finitely presented group G and step 3 of our process in §2 could produce a finitely generated vertex group A. In order to show that G has semistable fundamental group at ∞ we would need to show the finitely generated group A has semistable fundamental group at ∞. In a personal communication [HH18], M. Haulmark and C. Hruska have constructed examples of finitely presented groups that decompose according to Bowditch's accessibility result (Theorem 2.4) with vertex groups that are finitely generated and not finitely presented. In particular, for any (finitely generated but not finitely presented) recursively presented group Q and finitely presented group P containing a subgroup isomorphic to Q, they show there is a finitely generated relatively hyperbolic group A which contains Q as a maximal parabolic subgroup. Since Q is not finitely presented neither is A ([DG13]). Then they show $G = A *_Q P$ is finitely presented and hyperbolic relative to P.

In this type of situation, the finitely generated version of our main theorem might come into play and advance the process of §2 to step 4. It could also be the case that Q is not accessible and our techniques would break down. Now we say what it means for a finitely generated group to have semistable fundamental group at ∞. Suppose S is a finite generating set for

a finitely generated group G. Let $\Gamma(G, S)$ be the Cayley graph of (G, S). If there are a finite set of relations $\mathcal{R}$ of G such that the space resulting from attaching 2-cells to Γ (one at each vertex for each $R \in \mathcal{R}$) produces a space that has semistable fundamental group at ∞, then G is said to have semistable fundamental group at ∞. We first prove our main theorem in the finitely presented case. An elementary adjustment provides the finitely generated version.

5   Cusped Spaces

D. Groves and J. Manning [GM08] investigate a locally finite space X derived from a finitely generated group G and a collection $\mathbf{P}$ of finitely generated subgroups. The following definitions are directly from [GM08]

Definition 2. Let Γ be any 1-complex. The combinatorial horoball based on Γ, denoted $\mathcal{H}(\Gamma)$, is the 2-complex formed as follows:

A) $\mathcal{H}^{(0)} = \Gamma(0) \times (\{0\} \cup \mathbb{N})$

B) $\mathcal{H}^{(1)}$ contains the following three types of edges. The first two types are called horizontal, and the last type is called vertical.

(B1) If e is an edge of Γ joining v to w then there is a corresponding edge $\bar{e}$ connecting $(v, 0)$ to $(w, 0)$.

(B2) If $k > 0$ and $0 < d_\Gamma(v, w) \leq 2^k$, then there is a single edge connecting (v, k) to (w, k).

(B3) If $k \geq 0$ and $v \in \Gamma^{(0)}$, there is an edge joining (v, k) to $(v, k+1)$.

C) $\mathcal{H}^{(2)}$ contains three kinds of 2-cells:

(C1) If $\gamma \subset \mathcal{H}^{(1)}$ is a circuit composed of three horizontal edges, then there is a 2-cell (a horizontal triangle) attached along γ.

(C2) If $\gamma \subset \mathcal{H}^{(1)}$ is a circuit composed of two horizontal edges and two vertical edges, then there is a 2-cell (a vertical square) attached along γ.

(C3) If $\gamma \subset \mathcal{H}^{(1)}$ is a circuit composed of three horizontal edges and two vertical ones, then there is a 2-cell (a vertical pentagon) attached along γ, unless γ is the boundary of the union of a vertical square and a horizontal triangle.

Definition 3. Let Γ be a graph and $\mathcal{H}(\Gamma)$ the associated combinatorial horoball. Define a depth function

$$\mathcal{D} : \mathcal{H}(\Gamma) \to [0, \infty)$$

which satisfies:

(1) $\mathcal{D}(x) = 0$ if $x \in \Gamma$,

(2) $\mathcal{D}(x) = k$ if x is a vertex (v, k), and

(3) $\mathcal{D}$ restricts to an affine function on each 1-cell and on each 2-cell.

Definition 4. Let Γ be a graph and $\mathcal{H} = \mathcal{H}(\Gamma)$ the associated combinatorial horoball. For $n \geq 0$, let $\mathcal{H}_n \subset \mathcal{H}$ be the full sub-graph with vertex set $\Gamma^{(0)} \times \{0, \ldots, N\}$, so that $\mathcal{H}_n = \mathcal{D}^{-1}[0, n]$. Let $\mathcal{H}^n = \mathcal{D}^{-1}[n, \infty)$ and $\mathcal{H}(n) = \mathcal{D}^{-1}(n)$.

Lemma 5.1 *([GM08], Lemma 3.10) Let $\mathcal{H}(\Gamma)$ be a combinatorial horoball. Suppose that $x, y \in \mathcal{H}(\Gamma)$ are distinct vertices. Then there is a geodesic $\gamma(x, y) = \gamma(y, x)$ between x and y which consists of at most two vertical segments and a single horizontal segment of length at most 3.*

Moreover, any other geodesic between x and y is Hausdorff distance at most 4 from this geodesic.

Definition 5. Let G be a finitely generated group, let $\mathbf{P} = \{P_1, \ldots, P_n\}$ be a (finite) family of finitely generated subgroups of G, and let S be a generating set for G containing generators for each of the P_i. For each $i \in \{1, \ldots, n\}$, let T_i be a left transversal for P_i (i.e. a collection of representatives for left cosets of P_i in G which contains exactly one element of each left coset).

For each i, and each $t \in T_i$, let $\Gamma_{i,t}$ be the full subgraph of the Cayley graph $\Gamma(G, S)$ which contains tP_i. Each $\Gamma_{i,t}$ is isomorphic to the Cayley graph of P_i with respect to the generators $P_i \cap S$. Then define

$$X(G, \mathbf{P}, S) = \Gamma(G, S) \cup (\cup \{\mathcal{H}(\Gamma_{i,t})^{(1)} | 1 \leq i \leq n, t \in T_i\}),$$

where the graphs $\Gamma_{i,t} \subset \Gamma(G, S)$ and $\Gamma_{i,t} \subset \mathcal{H}(\Gamma_{i,t})$ are identified in the obvious way.

The space $X(G, \mathbf{P}, S)$ is called the *cusped* space for G, $\mathbf{P}$ and S. If G has a finite presentation $\mathcal{A} = \langle S; R \rangle$ we add 2-cells to $\Gamma(G, S)$ to form the Cayley 2-complex of this presentation. The resulting expansion of $X(G, \mathbf{P}, S)$ is called the cusped space for G, $\mathbf{P}$ and $\mathcal{A}$ and is denoted $X(G, \mathbf{P}, \mathcal{A})$. The next result shows cusped spaces are fundamentally important spaces. We prove our results in cusped spaces.

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Theorem 5.2 *([GM08], Theorem 3.25) Suppose that G is a finitely generated group and $\mathbf{P} = \{P_1, \ldots, P_n\}$ is a finite collection of finitely generated subgroups of G. Let S be a finite generating set for G containing generating sets for the P_i. A cusped space $X(G, \mathbf{P}, S)$ is hyperbolic if and only if G is hyperbolic with respect to $\mathbf{P}$.*

Assume G is finitely presented and hyperbolic with respect to the subgroups $\mathbf{P} = \{P_1, \ldots, P_n\}$ and S is a finite generating set for G containing generating sets for the P_i. For a finite presentation $\mathcal{A}$ of G with respect to S, let Y be the Cayley 2-complex for $\mathcal{A}$. So Y is simply connected with 1-skeleton $\Gamma(G, S)$, and the quotient space G/Y has fundamental group G. The cusped space $X(G, \mathbf{P}, S)$ is quasi-isometric to the cusped space $X(G, \mathbf{P}, \mathcal{A})$ and so one is hyperbolic if and only if the other is hyperbolic, and these two spaces have the same boundary. For $g \in G$ and $i \in \{1, \ldots, n\}$ we call gP_i a *peripheral coset* in a cusped space. The depth functions on the horoballs over the peripheral cosets extend from $X(G, \mathbf{P}, S)$ to $X(G, \mathbf{P}, \mathcal{A})$. So that

$$\mathcal{D} : X(G, \mathbf{P}, \mathcal{A}) \to [0, \infty)$$

where $\mathcal{D}^{-1}(0) = Y$ and for each horoball H (over a peripheral coset) we have $H \cap \mathcal{D}^{-1}(m) = H(m)$, $H \cap \mathcal{D}^{-1}[0, m] = H_m$ and $H \cap \mathcal{D}^{-1}[m, \infty) = H^m$. We call each H^m an *m-horoball*.

Lemma 5.3 *([GM08], Lemma 3.26) If a cusped space X is δ-hyperbolic, then the m-horoballs of X are convex for all $m \geq \delta$.*

Given two points x and y in a horoball H, there is a shortest path in H from x to y of the from (α, τ, β) where α and β are vertical and τ is horizontal of length ≤ 3. Note that if α is non-trivial and ascending and β is non-trivial and decending, then τ has length either 2 or 3.

6   The Main Lemmas

In this section we prove the two lemmas (6.7 and 6.8) that provide the combinatorial base of our semistability proof.

Again assume that G is a 1-ended finitely presented group, hyperbolic relative to $\mathbf{P}$, a finite collection of proper 1-ended subgroups of G. Let $\mathcal{A} = \langle S; R \rangle$ be a finite presentation for G where S contains generators for

each member of **P**. Let Y be the Cayley 2-complex for $\mathcal{A}$ and $*$ the identity vertex of $Y \subset X \equiv X(G, \mathbf{P}, \mathcal{A})$ and d the edge path distance in X. If H is a horoball, and $x, y \in H(m)$ then let $d^m(x, y)$ be the length of the shortest edge path from x to y in $H(m)$. For $z \in H(m)$, let $\hat{B}_n(z)$ be the set of all vertices v of $H(m)$ such that $d^m(v, z) \le n$.

Lemma 6.1 *Suppose t_1 and t_2 are vertices of depth $\bar{d} \ge \delta$ in a horoball H of X. Then for each $i \in \{1, 2\}$, there is a geodesic γ_i from $*$ to t_i such that γ_i has the form $(\eta_i, \alpha_i, \tau_i, \beta_i)$, where the end point x_i of η_i is the first point of γ_i in $H(\bar{d})$, α_i and β_i are vertical and of the same length in $H^{\bar{d}}$ and τ_i is horizontal of length ≤ 3. Furthermore $d(x_1, x_2) \le 2\delta + 1$.*

Proof: The proof of the first part follows directly from Lemma 5.3. For the last part, simply consider the geodesic triangle formed by the geodesics η_1, η_2 and the geodesic between x_1 and x_2. The two vertical segments of this last geodesic cannot be longer than $\delta - 1$ since the triangle is δ-thin. $\quad\square$

We call a path of the form $(\eta_i, \alpha_i, \tau_i, \beta_i)$ a $\mathcal{D}(t_i)$-*standard geodesic* from $*$ to t_i. The next lemma basically says if $\mathcal{D}(t) = \bar{d} \ge \delta$, H is a horoball containing t and z is a closest point of $H(\bar{d})$ to $*$, then the (X) distance $d(t, *)$ is approximately the distance from $*$ to z plus the distance (in $H^{\bar{d}}$) from z to t.

Lemma 6.2 *Suppose t is a vertex of a horoball H of X such that $\mathcal{D}(t) = \bar{d} \ge \delta$. Let z be a point of $H(\bar{d})$ closest to $*$. Then*

$$d(*, t) \le d(*, z) + d(z, t) \le 2\delta + 1 + d(*, t)$$

Proof: The first inequality follows from the triangle inequality. Let $(\eta, \alpha, \tau, \beta)$ a $\bar{d}$-standard geodesic from $*$ to t. Let p be the end point of η. By Lemma 6.1, $d(z, p) \le 2\delta + 1$ and so by the triangle equality

$$d(z, t) \le d(z, p) + d(p, t) \le 2\delta + 1 + d(p, t)$$

Also, $d(*, z) \le d(*, p)$. Adding inequalities,

$$d(*, z) + d(z, t) \le d(*, p) + 2\delta + 1 + d(p, t) = 2\delta + 1 + d(*, t).$$

$\square$

The next two results follow directly from the definition of a cusped space.

Lemma 6.3 *Suppose z and t are vertices of a horoball H of X, $\mathcal{D}(z) = \mathcal{D}(t) = m(\geq \delta)$ and $d^m(z,t) \leq 2^n$. Then $d(z,t) \leq 2n$, and so $\hat{B}_{2n}(z) \subset B_{2n}(z)$.*

Lemma 6.4 *Suppose z and t are vertices of a horoball H, $\mathcal{D}(z) = \mathcal{D}(t) = m(\geq \delta)$ and $d(z,t) \leq 2n+3$ (respectively $2n+2$). Then $d^m(z,t) \leq 3(2^n)$ (respectively 2^{n+1}), and so $B_{2n+3}(z) \cap H(m) \subset \hat{B}_{3(2^n)}(z)$ (respectively $B_{2n+2}(z) \cap H(m) \subset \hat{B}_{2^{n+1}}(z)$). In particular, for any integer $k > 0$, $B_k(z) \cap H(m) \subset \hat{B}_{2^{[\frac{k}{2}]+1}}(z)$.*

Lemma 6.5 *Suppose G is a finitely generated group with finite generating set S. Let Γ be the Cayley graph of G with respect to S. Then for any integer $r > 0$ and any point $x \in \Gamma - B_{2r}(*)$ there is a geodesic ray at x which avoids $B_r(*)$.*

Proof: Let L be a geodesic line in Γ. By translation, we may assume x is a vertex of L. Now consider the two geodesic rays q^+ and q^- in opposite directions on L, both with initial point x. If both q^+ and q^- intersect $B_r(*)$ at say x^+ and x^- respectively, then since $x \notin B_{2r}(*)$ both of x^+ and x^- are of distance at least $r + 1$ from x. This means the distance from x^+ to x^- is at least $2r + 2$. But, since both belong to $B_r(*)$ they are at most distance $2r$ from one another - the desired contradiction. $\square$

Suppose $x, y \in X$ are points of a horoball H and $\mathcal{D}(x) = \mathcal{D}(y) = k$. Any path (without backtracking) in the k-horoball H^k between x and y can be written as
$$\psi = (\tau_0, \alpha_1, \tau_1, \alpha_2, \tau_2, \ldots, \alpha_n \tau_n)$$
where for each i, α_i is non-trivial and vertical and τ_i is horizontal. With the possible exception of τ_0 and τ_n, τ_i is non-trivial. If τ_1 is the edge path $(e_1, \ldots, e_m)$, let $\alpha_{1,j}$ be the vertical path from the end point of e_j to $H(k) \subset H^k$. Also let $\alpha_1^{-1} = \alpha_{1,0}$. Next let $\gamma_{1,j}$ be a shortest edge path in $H(k)$ from the end point of $\alpha_{1,j-1}$ to the end point of $\alpha_{1,j}$ for all $j \in \{1, \ldots, m\}$. Let $\gamma_1 = (\gamma_{1,1}, \ldots, \gamma_{1,m})$ and notice that for each vertex v of γ_1 the vertical edge path of length $|\alpha_1|$ is within 1 (horizontal) unit of a vertex of one of the e_i. For each τ_i similarly construct γ_i. The path $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a *projection* of ψ into $H(k)$.

Lemma 6.6 *Suppose H is a horoball in X, $x, y \in H(k)$ and*

$$\psi = (\alpha_1, \tau_1, \alpha_2, \tau_2, \ldots, \alpha_n \tau_n, \alpha_{n+1})$$

is a path in H^k between x and y, where for each i, α_i is non-trivial and vertical and τ_i is nontrivial and horizontal. If γ is a projection of ψ into $H(k)$ then each vertical line at a vertex of γ passes within 1 horizontal unit of a vertex of one of the τ_i.

The next lemma combines with Lemma 6.2 to show that if ψ is an edge path in $X - B_r(*)$, has image in $H^{\bar{d}}$ ($\bar{d} \geq \delta$) for some horoball H, begins and ends in $H(\bar{d})$ and the image of ψ is "far" from a vertex of $H(\bar{d})$ closest to $*$ then ψ projects to a path in $H(\bar{d})$ that avoids $B_{r-(2\delta+1)}(*)$. The subsequent Lemma 6.8, shows that if ψ has a vertex "close" to a vertex of $H(\bar{d})$ closest to $*$, then ψ can be replaced by a path in $H(\bar{d})$ that avoids $B_{r-(2\delta+5)}(*)$. These are the two main lemmas of the paper. In the proof of our main theorem, we only need these lemmas with $\bar{d} = \delta$. In the proof of Theorem 8.1, we will need the general version of these lemmas. Lemma 6.7 has a variation (Lemma 9.1) that is in some sense stronger and of separate interest. We only need the version that immediately follows in order to prove our main theorem, and so we delay the introduction of Lemma 9.1 to section §9.

Lemma 6.7 *Suppose H is a horoball, $\bar{d}$ is an integer $\geq \delta$, $x \neq y$ are vertices of $H(\bar{d})$, z is a closest vertex of $H(\bar{d})$ to $*$, ψ is a path in $H^{\bar{d}} - B_r(*)$ between x and y that only intersects $H(\bar{d})$ at x and y and $L = |\psi|$.*

If γ is a projection of ψ to $H(\bar{d})$ and each vertex v of ψ is such that $d(v, z) > L + 2$, then each vertex of γ is at distance greater than $L + 2$ from z. Furthermore, the image of γ avoids $B_{r-(2\delta+1)}(*)$.

Proof: By Lemma 5.3, $H^{\bar{d}}$ is convex. Let p be a vertex of the projection γ and $(\alpha_p, \tau_p, \beta_p)$ a geodesic (in $H^{\bar{d}}$) from z to p where α_p and β_p are vertical of the same length and $|\tau_p| \leq 3$. Let y be the end point of τ_p. By Lemma 6.6, there is a vertical segment that begins at p and ends at most 1 (horizontal) unit from a vertex v of ψ (and $\mathcal{D}(v) \geq \bar{d} + 1$). If y is on that vertical line segment (Figure 1(1)), then there is a path ρ from v to z that begins with a horizontal edge from v to a vertex w on the vertical segment, followed by a vertical segment from w to y, followed by τ_p^{-1}, followed by α_p^{-1}.

Since ψ begins and ends in $H(\bar{d})$ has image in $H^{\bar{d}}$ and has length L, at most $\frac{L-1}{2}$ vertical edges are required to go from a vertex of ψ down to $H(\bar{d})$.

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In particular, the sum of the lengths of the two vertical segments of ρ is less than or equal to $\frac{L-1}{2}$ (since v is on ψ), and so

$$L + 2 < d(v, z) \le \frac{L-1}{2} + 4 = \frac{L}{2} + \frac{7}{2} \text{ implying}$$

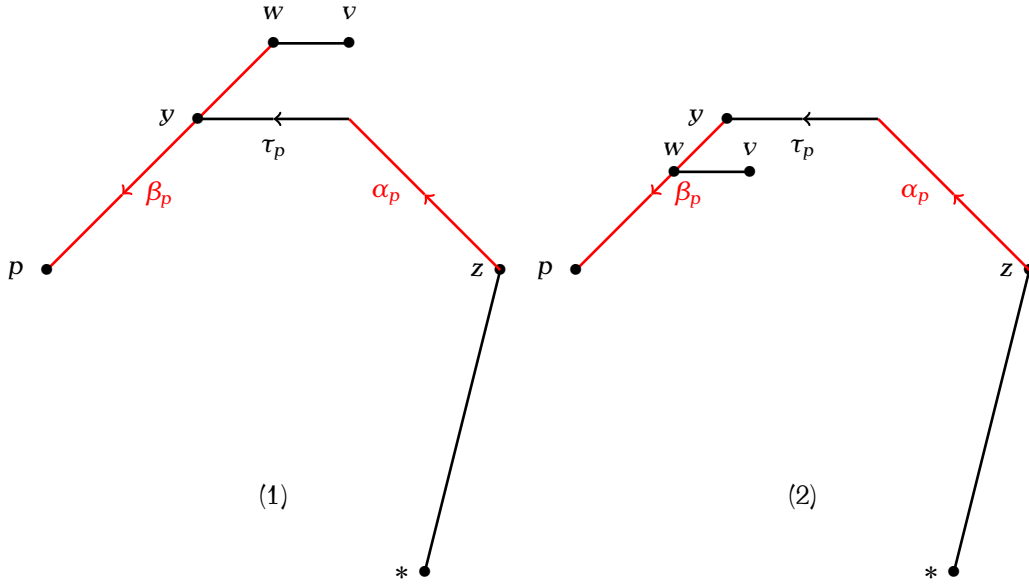$$2L + 4 < L + 7 \text{ and } L < 3,$$

which is impossible since $x \ne y$.

Figure 1: Two possible positions of y relative to v.

Instead, y is on the vertical line at p and the vertical line segment from p to y contains a vertex w (other than y) within 1 horizontal unit of a vertex v of ψ (Figure 1(2)). Now

$$d(z, p) = d(p, w) + d(w, y) + |\tau_p| + |\alpha_p| \text{ and}$$

$$L + 2 < d(v, z) \le 1 + d(w, y) + |\tau_p| + |\alpha_p|$$

Since $d(p, w) \ge 1$, $d(z, p) > L + 2$. Completing the first part of the lemma.

Since the depth of v (and hence the depth of w) is at least $\bar{d} + 1$,

$$d(p, z) \ge 1 + d(w, z) \ge d(v, z)$$

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Combining this inequality with Lemma 6.2 and the triangle inequality, we have for each vertex p of γ:

$$d(p, *) + 2\delta + 1 \geq d(p, z) + d(z, *) \geq d(v, z) + d(z, *) \geq d(v, *) > r$$

So $d(p, *) > r - 2\delta - 1$. □

Remark 1. Assume all of the hypotheses of Lemma 6.7, except $d(v, z) > L + 2$ for all vertices v of γ. Consider the integer $k = r - d(z, *)$, so that $r = d(z, *) + k$. Note that if $v \in B_{L+2}(z)$ then $d(v, *) \leq d(v, z) + d(z, *) \leq L + 2 + d(z, *) = L + 2 + r - k$. So if $k \geq L + 2$ then $d(v, *) \leq r$ and $B_{L+2}(z) \subset B_r(*)$. Since ψ has image in $X - B_r(*)$, each vertex v of ψ is such that $d(v, z) > L + 2$. Then the hypotheses (and hence the conclusion) of Lemma 6.7 is satisfied.

If we consider Lemma 6.7 and Lemma 6.8 simultaneously, we see that any path ψ connecting points x and y (as described in these lemmas) must satisfy the conclusion of one of these lemmas. By Remark 1, the situation where $k = r - d(z, *) \geq L + 2$ (and hence every vertex of ψ is of distance $> L + 2$ from the special vertex z), is covered by Lemma 6.7. The case when some vertex of ψ is within $L + 2$ of z, (and hence by Remark 1, $k < L + 2$) is covered by Lemma 6.8.

Lemma 6.8 *Suppose L is a fixed integer, $\bar{d}$ an integer $\geq \delta$, $x, y \in H(\bar{d}) - B_r(*)$ for some horoball H, ψ is a path in $H^{\bar{d}} - B_r(*)$ between x and y, the image of ψ intersects $H(\bar{d})$ only at x and y, z is a closest point of $H(\bar{d})$ to $*$, the length of ψ is L, and some vertex v of ψ is such that $d(v, z) \leq L + 2$. Then there is an integer $F_{6.8}(L)$ (independent of r) and a path in $H(\bar{d})$ of length $\leq F_{6.8}$ from x to y that avoids $B_{r-(2\delta+5)}(*)$.*

Proof: First observe that by Remark 1, $k = r - d(z, *) < L + 2$. Since $|\psi| = L$ and $d(v, z) \leq L + 2$, both $d(x, z)$ and $d(y, z)$ are $\leq 2L + 2$. By Lemma 6.4, $x, y \in \hat{B}_{2L+2}(z)(\subset H(\bar{d}))$. If $k = r - d(z, *) < 0$ then (since z is a closest point of $H(\bar{d})$ to $*$) no point of $H(\bar{d})$ belongs to $B_r(*)$. Hence any path in $H(\bar{d})$ between x and y will avoid $B_r(*)$. In particular, a projection of ψ to $H(\bar{d})$ will have length $\leq 2^{\frac{L-1}{2}}$ and avoid $B_r(*)$. Let $F_{-1}(L) = 2^{\frac{L-1}{2}}$ (independent of r).

Now we need only consider the cases where $k \in \{0, 1, \ldots, L + 1\}$. For simplicity, assume k is a positive even number. If $d(t, z) \leq k = r - d(*, z)$ then

$r \geq d(*, z) + d(z, t) \geq d(*, t)$, and so $t \in B_r(*)$. In particular, $B_k(z) \subset B_r(*)$. By Lemma 6.3

$$\hat{B}_{2^{\frac{k}{2}}}(z) \subset B_k(z) \subset B_r(*)$$

so that $x, y \in \hat{B}_{2^{L+2}}(z) - \hat{B}_{2^{\frac{k}{2}}}(z)$. For every horoball H in X, $H(\bar{d})$ is a Cayley graph for some peripheral subgroup and hence is 1-ended. Lemma 6.5 provides a path $\gamma(x, y)$ from x to y, with image in $H(\bar{d}) - \hat{B}_{2^{\frac{k}{2}-1}}(z)$.

Next, we show that $B_{r-(2\delta+5)}(*) \cap H(\bar{d}) \subset \hat{B}_{2^{\frac{k}{2}-1}}(z)$ (in order to ensure that $\gamma(x, y)$ avoids $B_{r-(2\delta+5)}(*)$). By Lemma 6.4, $B_m(z) \cap H(\bar{d}) \subset \hat{B}_{2^{\lceil\frac{m}{2}\rceil+1}}(z)$ and so $B_{k-4}(z) \cap H(\bar{d}) \subset \hat{B}_{2^{\frac{k}{2}-1}}(z)$. It suffices to show that $B_{r-(2\delta+5)}(*) \cap H(\bar{d}) \subset B_{k-4}(z) \cap H(\bar{d})$.

Say $t \in B_r(*) \cap H(\bar{d})$ and let p be the first vertex of $H(\bar{d})$ on a geodesic from $*$ to t. By Lemma 6.1, $d(z, p) \leq 2\delta + 1$. Then $r \geq d(t, *) = d(*, p) + d(p, t)$ so that $k = r - d(*, z) \geq r - d(*, p) \geq d(p, t)$, and

$$d(z, t) \leq d(z, p) + d(p, t) \leq 2\delta + 1 + k$$

In particular, $t \in B_{2\delta+1+k}(z) \cap H(\bar{d})$ and

$$B_r(*) \cap H(\bar{d}) \subset B_{2\delta+1+k}(z) \cap H(\bar{d}) = B_{2\delta+1+r-d(*,z)}(z) \cap H(\bar{d})$$

This last formula is true for all r, and replacing r by $r - 2\delta - 5$ gives

(1) $\quad B_{r-2\delta-5}(*) \cap H(\bar{d}) \subset B_{r-4-d(*,z)}(z) \cap H(\bar{d}) = B_{k-4}(z) \cap H(\bar{d}) \subset \hat{B}_{2^{\frac{k}{2}-1}}(z).$

Now, for $x, y \in \hat{B}_{2^{L+2}}(z) - \hat{B}_{2^{\frac{k}{2}}}(z)$, $\gamma(x, y)$ is a path from x to y with image in $H(\bar{d}) - \hat{B}_{2^{\frac{k}{2}-1}}(z)$ and so $\gamma(x, y)$ avoids $B_{r-2\delta-5}(*)$. Let $F_k(L)$ be the length of a longest path $\gamma(x, y)$ for $x, y \in \hat{B}_{2^{L+2}}(z) - \hat{B}_{2^{\frac{k}{2}}}(z)$. We will show that for any G-translate $H'(\bar{d})$ of $H(\bar{d})$ with vertex $z' \in H'(\bar{d})$ closest to $*$ and such that $k = k' = r' - d(z', *)$, the element $g \in G$ such that $gz = z'$ can be used to translate the paths $\gamma(x, y)$ to paths connecting any pair of points in $\hat{B}_{2^{L+2}}(z') - \hat{B}_{2^{\frac{k}{2}}}(z')$ so that $g\gamma(x, y)$ avoids $B_{r'-(2\delta+5)}(*)$.

Say $H'(\bar{d})$ is a G-translate of $H(\bar{d})$, z' is a closest point of $H'(\bar{d})$ to $*$, $k = k' = r' - d(z', *)$ and $gz = z'$ (so $gH(\bar{d}) = H'(\bar{d})$). Note that g maps $\hat{B}_s(z)$ to $\hat{B}_s(z')$ for any integer s. So if $x', y' \in \hat{B}_{2^{L+2}}(z') - \hat{B}_{2^{\frac{k'}{2}}}(z')$ then there is a path connecting x' and y' of length $\leq F_k$ in $H'(\bar{d}) - \hat{B}_{2^{\frac{k'}{2}-1}}(z')$. By the same argument that establishes equation (1), $B_{r'-(2\delta+5)}(*) \cap H'(\bar{d}) \subset \hat{B}_{2^{\frac{k'}{2}-1}}(z')$, so the paths connecting x' and y' avoid $B_{r'-(2\delta+5)}(*) \cap H'(\bar{d})$. Lastly, let $F_{6.8}(L) = \max\{F_{-1}, F_0, \ldots, F_{L+1}\}$. $\square$

7   Proof of the Main Theorem

The proof of Theorem 1.1 is completed in this section. The two papers [GM08]-Groves Manning and [DG08]-Dahmani-Groves along with Bowditch's many results are used. We list a few results that we will need for our proof.

Theorem 3.33 of [GM08] insures that there are geodesic lines in X that are axes for elements of G. Our next result is a simple application of Theorem 3.33.

Theorem 7.1 *There is an infinite order element $g \in G$ so that if ρ is a geodesic from $*$ to $g*$, then the line $(\ldots, g^{-1}\rho, \rho, g\rho, g^2\rho, \ldots)$ is a bi-infinite geodesic that has image in $\mathcal{D}^{-1}([0, 19\delta])$.*

Proof: By Theorem 3.33 [GM08] there is a geodesic line ℓ in $\mathcal{D}^{-1}([0, 19\delta])$ and an infinite order element $g_1 \in G$ such that $g_1 \ell = \ell$. Certainly the image of ℓ is not a subset of a horoball and so ℓ must contain a vertex $v = h*$ (for $h \in G$) of Y. The element $g = h^{-1} g_1 h$ stabilizes the geodesic line $h^{-1}\ell$ (containing $*$). If ρ is the geodesic from $*$ to $g*$, then $h^{-1}\ell = (\ldots, g^{-1}\rho, \rho, g\rho, g^2\rho, \ldots)$. $\square$

We need a result of [DG08], but some background first. In ([DG08], §2.3) several constants are defined. There δ is any positive integer such that X is δ-hyperbolic and $C = 3\delta$. Following [DG08], define

$$M = 6(C + 45\delta) + 2\delta + 3 = 6(45\delta) + 20\delta + 3 \text{ and } K = 2M.$$

Two conditions on points in X are defined. First:
Given $\epsilon \geq 0$, and two points $x, y \in X$, say that x and y satisfy $\star_\epsilon$ if

$$\star_\epsilon : |d(*, x) - d(*, y)| \leq \epsilon \text{ and } d(x, y) \leq M.$$

Given an integer N, say $x, y \in X$ satisfying $\star_\epsilon$, satisfy condition $\ddagger(\epsilon, N)(x, y)$ if there is a path of length at most N from x to y in the compliment of the ball $B_{m-48\delta}(*)$ where $m = min\{d(*, x), d(*, y)\}$.

Lemma 7.2 *([DG08], Lemma 4.2) If the boundary ∂X is connected, and has no global cut point, then there is an integer N such that $\ddagger(10\delta, N)(x, y)$ holds for all $x, y \in X_K = \mathcal{D}^{-1}([0, K])$ satisfying $\star_{10\delta}$.*

Note that the path between x and y of this last lemma has image in X (not X_K). Later we improve this result (see Theorem 8.1) to get a path in X_K.

Lemma 7.3 *For any vertex v of Y there is a geodesic ray r_v in X such that $r_v(0) = *$, there is $t_v \in [0, \infty)$ such that $d(r_v(t_v), v) \leq \delta$ and $r_v|_{[t_v, \infty)}$ has image in $\mathcal{D}^{-1}([0, 21\delta])$.*

Proof: Let ℓ be the line of Theorem 7.1. By definition, the vertices of Y are the elements of G, so that $v\ell$ is a geodesic line through v and with image in $\mathcal{D}^{-1}[0, 19\delta]$. Write the consecutive vertices of $v\ell$ as $\{\dots, v_{-1}, v_0, v_1, \dots\}$ where $v_0 = v$. Let ρ_i be a geodesic from $*$ to v_i for $i \geq 1$. Infinitely many of the ρ_i have the same first edge e_1. Of these, infinitely many have the same second edge e_2. Continuing, the geodesic ray $r_v = (e_1, e_2, \dots)$ converges to $v\ell^{+\infty}$. Similarly, the geodesic ray s_v is constructed converging to the other end $v\ell^{-\infty}$ of $v\ell$. By construction, for any integer $n \in [0, \infty)$, $r_v([0, n])$ can be extended to some ρ_i (ending at the vertex v_i of $v\ell$). The same is true for s_v. Let $k > d(v, *) + \delta$ be an integer. Extend $r_v([0, k])$ to a geodesic α ending on $v\ell([0, \infty))$ and extend $s_v([0, k])$ to a geodesic β ending on $v\ell((-\infty, 0])$. Let γ be the geodesic segment of $v\ell$ connecting the endpoints of α and β and note that v is a vertex of γ. Consider the geodesic triangle formed by α, β and γ. Without loss assume that v is within δ of α. Then since α extends $r_v([0, k])$ and $k = d(v, *) + \delta$, v is within δ of $r_v([0, k])$. (In slightly different terms, the ideal geodesic triangle formed by r_v, s_v and $v\ell$ is δ-thin.) In any case, v is within δ of some vertex of r_v or s_v. Assume there is $t_v \in [0, \infty)$ such that $d(r_v(t_v), v) \leq \delta$. Since for any $n \geq t_v$, $r_v|_{[0,n]}$ can be extended to a geodesic ending on $v\ell$, $d(r_v(n), v\ell) \leq 2\delta$. As $v\ell$ has image in $\mathcal{D}^{-1}([0, 19\delta])$, $r_v|_{[t_v, \infty)}$ has image in $\mathcal{D}^{-1}([0, 21\delta])$. $\square$

Proof: *(of the main theorem)* First assume that G is finitely presented, so that Y is simply connected. All of our compact sets will be finite subcomplexes of X and all paths will be edge paths. Given a compact set C in Y, our goal is to find a compact set D in Y with the following semistability property (see Theorem 4.1(2)): For any third compact set $E \subset Y$ and Y-loop β based on (the base ray) $\hat{r}$ and with image in $Y - D$, show that β is homotopic rel$\{\hat{r}\}$ to a loop in $Y - E$ by a homotopy in $Y - C$. We will define a set of integers N_0, N_1, N_2, N_3 and N_4. Our compact set D will be $B_{N_4}(C) \cap Y$, the ball in X of radius N_4 about C intersected with Y.

Next we construct the base ray $\hat{r}$ for our semistability criterion. Let g and ρ be as in Lemma 7.1 and say ρ is the edge path $(e_1, \ldots, e_{L_0})$ so that $|\rho| = L_0$. Say the consecutive vertices of ρ are $* = v_0, \ldots, v_{L_0}$. If $v_i \in Y$, then let $y_i = v_i$. Otherwise, let $y_i \in Y$ be the end point of a vertical path (of length $\leq 19\delta$) from v_i to Y. Let $\hat{\tau}_i$ be a shortest path in Y from y_{i-1} to y_i.

Let $\hat{\rho}$ be the Y-path $\hat{\rho} = (\hat{\tau}_1, \ldots, \hat{\tau}_{L_0})$. Then our base ray $\hat{r} = (\hat{\rho}, g\hat{\rho}, g^2\hat{\rho}, \ldots)$ is a proper edge path ray in Y beginning at $*$. Notice that the X-geodesic ray $r = (\rho, g\rho, g^2\rho, \ldots)$ tracks $\hat{r}$ and for any vertex v of r, there is a vertical path of length $\leq 19\delta$ from v to a point of $\hat{r}$. Furthermore, r and $\hat{r}$ share the vertices $*, g, g^2, \ldots$ (where $g^n = g^n*$). (See Figure 2.)
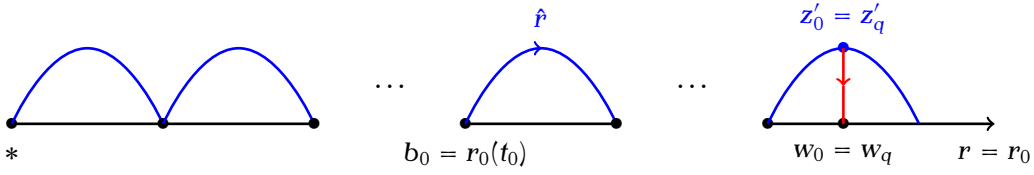

Figure 2: Constructing the base ray.

Suppose β, with consecutive vertices $b_0 = r(t_0), b_1, b_2, \ldots, b_{q-1}, b_q = b_0$, is a loop based at a common vertex b_0 of $\hat{r}$ and r. Let $r = r_0 = r_q$. By Lemma 7.3, for each $i \in \{1, \ldots, q-1\}$, there is a geodesic ray r_i at $*$ such that for some integer $t_i \in [0, \infty)$, $d(r_i(t_i), b_i) \leq \delta$ and $r_i|_{[t_i, \infty)}$ has image in $\mathcal{D}^{-1}([0, 21\delta])$. Define τ_i to be a path from b_i to $r_i(t_i)$ of length $\leq \delta$. (See Figure 3.) Since $b_0 = r_0(t_0)$, τ_0 is the trivial path. Note that $d(r_{i-1}(t_{i-1}), r_i(t_i)) \leq 2\delta + 1$. Let $w_i = r_i(t_i + 3(45\delta) + 9\delta)$ for all i. Then

$$d(w_{i-1}, w_i) \leq 2[3(45\delta) + 9\delta] + 2\delta + 1 = 6(45\delta) + 20\delta + 1 < M.$$

Since $d(r_{i-1}(t_{i-1}), r_i(t_i)) \leq 2\delta + 1$ and $d(r_i(t_i), w_i)$ is the same for all i, $|d(w_{i-1}, *) - d(w_i, *)| \leq 2\delta + 1 < 10\delta$. As $w_i \in \mathcal{D}^{-1}([0, 21\delta])$ and $21\delta < K$, Lemma 7.2 implies w_{i-1} and w_i satisfy $\ddagger(10\delta, N_0)(w_{i-1}, w_i)$ for some positive integer N_0. That means there is a path μ_i in X from w_{i-1} to w_i of length $\leq N_0$ in the compliment of the ball $B_{m-48\delta}(*)$ where $m = min\{d(*, w_{i-1}), d(*, w_i)\}$. As $\mathcal{D}(w_i) \leq 21\delta$ there is a vertical path α_i of length $\leq 20\delta$ from a point $z_i \in \mathcal{D}^{-1}(\delta)$ to w_i. (By the definition of $\hat{r}$, the vertical path from w_0 to Y ends at a point in the image of $\hat{r}$ and has length $\leq 19\delta$.) Hence the vertical path from w_0 to the point $z_0 \in \mathcal{D}^{-1}(\delta)$ is of length $\leq 18\delta$. Let α_0 be the vertical path from z_0 to w_0. *The path $\psi_i = (\alpha_{i-1}, \mu_i, \alpha_i^{-1})$ begins and ends in $\mathcal{D}^{-1}(\delta)$, has image in $X - B_{m-68\delta}(*)$ and length $\leq N_0 + 40\delta$. The path ψ_i*

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can be decomposed into subpaths where each either has image in $\mathcal{D}^{-1}([0,\delta])$, or satisfies the conclusion of either Lemma 6.7 or Lemma 6.8. Let λ be one of these subpaths.
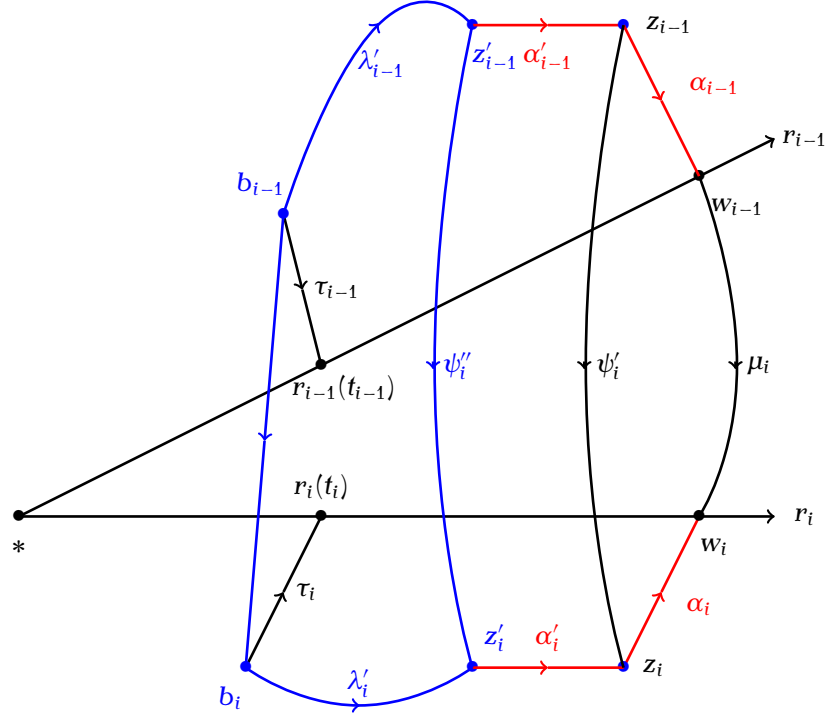

Figure 3: Moving the edge $[b_{i-1}, b_i]$ to ψ_i''.

If λ satisfies the conclusion of Lemma 6.7, then it has a projection λ' (to $\mathcal{D}^{-1}(\delta)$) with image in $X - B_{m-68\delta-(2\delta+1)}(*)$. The length of λ, and in particular the length of λ', is bounded by a constant only depending on $|\psi_i| \leq N_0 + 40\delta$.

If λ satisfies the conclusion of Lemma 6.8, then there is path λ' (in $\mathcal{D}^{-1}(\delta)$) connecting the end points of λ, with image in $X - B_{m-68\delta-(2\delta+5)}(*)$, and again the length of λ' is bounded by a constant only depending on the length of λ, and in particular, by a constant depending only on $|\psi_i| \leq N_0 + 40\delta$. Replacing each of the λ subpaths by the corresponding λ' paths in $\mathcal{D}^{-1}(\delta)$ gives us a path ψ_i' in $\mathcal{D}^{-1}([0,\delta])$ from z_{i-1} to z_i, of length $\leq N_1$ (a constant depending only on the number $N_0 + 40\delta$). *Then $|\psi_i'| \leq N_1$ and the image of ψ' is in*

$X - B_{m-68\delta-(2\delta+5)}(*)$.

Let α_i' be the vertical path of length δ from $z_i' \in Y$ to z_i. Note that z_0' is in the image of $\hat{r}$. Let ψ_i'' be a projection of ψ_i' to Y. Again, the length of ψ'' is $\leq N_2$ (a constant depending only on the number $N_0 + 40\delta$). By Lemma 6.6, each vertex of ψ_i'' is within $\delta + 1$ of a vertex of ψ_i'. Hence, $|\psi_i''| \leq N_2$ and the image of ψ_i'' is in $X - B_{m-68\delta-(2\delta+5)-(\delta+1)}(*) = X - B_{m-71\delta-6}(*)$.

Our goal is to show that the edges $[b_{i-1}, b_i]$ are compatibly homotopic to the ψ_i'', and that each vertex of ψ_i'' is further from $*$ than either of b_{i-1} or b_i. We must also show the homotopies that exchange the edges of β for the ψ_i'' avoid the compact set C (but first we must decide how large the yet to be defined N_4 must be). Once this is accomplished, then repeating the process moves the original path (rel $\{\hat{r}\}$) to a path outside of any preassigned compact set E (by a homotopy avoiding C) to finish the proof.

First we show each vertex v of ψ_i'' is further from $*$ than either b_i or b_{i-1}. We have

$$d(v, *) \geq m - 71\delta - 6 \text{ where}$$

$$m = min\{d(*, w_{i-1}), d(*, w_i)\} =$$

$$min\{d(*, r_{i-1}(t_{i-1})), d(*, r_i(t_i))\} + 3(45\delta) + 9\delta$$

Then,

$$d(v, *) \geq min\{d(*, r_{i-1}(t_{i-1})), d(*, r_i(t_i))\} + 73\delta - 6$$

Since $d(r_i(t_i), b_i) \leq \delta$ for all i:

$$d(v, *) \geq min\{d(b_{i-1}, *), d(b_i, *)\} + 72\delta - 6$$

Since $d(b_{i-1}, b_i) \leq 1$, v is at least $72\delta - 7$ further from $*$ than either b_{i-1} or b_i is from $*$. We have shown:

(2) Each vertex of ψ_i'' is at least $72\delta - 7$ further from $*$ than either b_{i-1} or b_i is from $*$.

Recall that $|\tau_i| \leq \delta$, $d(r(t_i), w_i) = 3(45\delta) + 9\delta$ and $|(\alpha_i', \alpha)| \leq 21\delta$. Consider the X path $(\tau_i, [r_i(t_i), w_i], \alpha_i^{-1}, \alpha_i'^{-1})$ from b_i to z_i' (both vertices in Y) of length $\leq 3(45\delta) + 31\delta = L_1$. There is a positive integer N_3 only depending on L_1 so that for any edge path λ in X of length $\leq L_1$, that begins and ends in Y, there is an edge path λ' of length $\leq N_3$ in Y connecting the end points of λ. In particular, there is a Y-edge path λ_i' from b_i to z_i of length $\leq N_3$. When $i = 0$ or $i = n$ we want $\lambda_0' = \lambda_n'$ to be the subpath of $\hat{r}$ from $b_0 = b_n$ to $z_0 = z_n$. Since $b_0 = r_0(t_0)$ and $d(r_0(t_0), w_0) = 3(45\delta) + 9\delta$, we can assume

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N_3 is larger than the length of the subpath of $\hat{r}$ from $b_0 = r_0(t_0) \in im(\hat{r})$ to $z_0 = z_n$.

The loops $\ell_i = ([b_{i-1}, b_i], \lambda'_i, \psi''^{-1}_i, \lambda'^{-1}_{i-1})$ have image in Y and are of length $\leq 1 + 2N_3 + N_2$. Choose an integer N_4 so that for any loop of length $\leq 1 + 2N_3 + N_2$ in Y the loop is homotopically trivial by a homotopy in $B^Y_{N_4}(v)$ (the ball in Y of radius N_4) for any vertex v of the loop. Define the compact set D to be $B_{N_4}(C) \cap Y$ (the ball in X of radius N_4 about C, intersected with Y). Assume that β is a loop in Y based at a vertex of r common to $\hat{r}$ and with image in $Y - D$. The homotopies killing the ℓ_i in Y (with image in $St_Y^{N_4}(v)$ for any vertex v of ℓ_i) move each edge $[b_{i-1}, b_i]$ to a path ψ''_i by a homotopy in Y whose image avoids the compact set C. These homotopies string together compatibly to move β further out along $\hat{r}$. Continuing, β is eventually moved outside of any compact subset E of Y. We have shown that Y, and hence G has semistable fundamental group at ∞.

Note that in order to show that Y has semistable fundamental group at ∞, the only homotopies we need are those that kill loops of length $\leq 1 + 2N_3 + N_2$ in Y. If G is only finitely generated (as opposed to finitely presented), then let Y be obtained from $\Gamma(G, S)$ the Cayley graph of G with respect to the generating set S, by attaching 2-cells at each vertex for each loop of length $\leq 1 + 2N_3 + N_2$. The same proof (with only minor changes in the last paragraph) shows that Y has semistable fundamental group at ∞. $\square$

Theorem 7.4 *Suppose G is a 1-ended finitely generated group that is hyperbolic relative to the finitely generated proper subgroups $P_1, \ldots, P_n$. Let X be a corresponding cusped space. If Y has semistable fundamental group at ∞ then X_k has semistable fundamental group at ∞ for all $k \geq 0$.*

Proof: Let r and s be proper rays at $*$ in X_k for $k \geq 0$. Let r' and s' be projections of r and s to Y. Clearly, r is properly homotopic to r' and s to s'. Since r' and s' are properly homotopic (by the semistablity of Y), a combination of proper homotopies shows r is properly homotopic to s in X_k. $\square$

8 An Improved ‡ Result

Throughout this section G is a finitely generated group that is hyperbolic relative to a finite collection **P** of proper 1-ended finitely generated subgroups. Assume again that X is a cusped space for $(G, \mathbf{P})$. Say X is δ_0 hyperbolic

and that $\delta \geq \delta_0$ is an integer. Lemmas 6.7 and 6.8 allow us to improve Lemma 7.2 from a partial result about $X_{K(\delta)}$ to one completely about $X_{K(\delta)}$. More precisely, Lemma 7.2 guarantees a path in $X - B_{m-48\delta}(*)$ and we want to replace it by a path in $X_{K(\delta)} - B_{m-(50\delta+5)}(*)$.

First some observations. As Lemma 7.2 gives a separate result for each $\delta \geq \delta_0$, the integers K, M and N will in fact be $K(\delta)$, $M(\delta)$ and $N(\delta)$. Consider $\delta \geq \delta_0$. Say $x, y \in X_{K(\delta)}$ satisfy $\star_{10\delta}$ and ψ is a path in X of length $\leq N(\delta)$ joining x and y in $X - B_{m-48\delta}(*)$ where $m = min\{d(*, x), d(*, y)\}$. Lemmas 6.7 and 6.8 allow us to replace the segments of ψ that leave $X_{K(\delta)}$ by paths in $H(K(\delta))$ (for some horoball H). The number of such segments and the lengths of the replacement paths are bounded in terms of $N(\delta)$ and the image of each such replacement path is in $X_{K(\delta)} - B_{m-(48\delta+2\delta+5)}(*)$.

Recall the $\star_\epsilon$ and $\ddagger$ condition of [DG08], where for any $\delta \geq \delta_0$ we define

$$M(\delta) = 6(45\delta) + 20\delta + 3 \text{ and } K(\delta) = 2M(\delta).$$

Given $\epsilon \geq 0$, and two points $x, y \in X$, say that x and y satisfy $\star_\epsilon$ if

$$\star_\epsilon : |d(*, x) - d(*, y)| \leq \epsilon \text{ and } d(x, y) \leq M.$$

Note that if $\epsilon' \leq \epsilon$ and $x, y \in X$ satisfy $\star_{\epsilon'}$ then x, y satisfy $\star_\epsilon$.

Given an integer N, say $x, y \in X$ satisfying $\star_\epsilon$ satisfy condition $\ddagger(\epsilon, N)(x, y)$ if there is a path of length at most N from x to y in $X - B_{m-48\delta}(*)$ where $m = min\{d(*, x), d(*, y)\}$.

This induces the following definition.

Given an integer N, say $x, y \in X_{K(\delta)}$ satisfying $\star_\epsilon$ satisfy condition $\hat{\ddagger}(\epsilon, N)(x, y)$ if there is a path of length at most N from x to y in $X_{K(\delta)} - B_{m-50\delta-5}(*)$ where $m = min\{d(*, x), d(*, y)\}$.

Lemma 7.2 states that if the boundary ∂X is connected, and has no global cut point, then there is an integer $N(\delta)$ such that $\ddagger(10\delta, N)(x, y)$ holds for all $x, y \in X_{K(\delta)} = \mathcal{D}^{-1}([0, K(\delta)])$ satisfying $\star_{10\delta}$. The path of length $\leq N$ connecting x and y and guaranteed by Lemma 7.2 has image in X. Lemmas 6.7 and 6.8 enhance this result to give a path in X_K (although N will become larger and the image of the path may come $2\delta + 5$ units closer to $*$ than the one given by $\ddagger(10\delta, N(x, y))$.

Combining these observations we have:

Theorem 8.1 *Suppose G is a finitely generated group that is relatively hyperbolic with respect to a finite collection of finitely generated 1-ended proper*

subgroups, a cusped space X is δ_0-hyperbolic and δ is an integer $\geq \delta_0$. If the boundary ∂X is connected, and has no global cut point, then there is an integer $N(\delta)$ such that $\hat{\ddagger}(10\delta, N)(x, y)$ holds for all $x, y \in X_{K(\delta)} (= \mathcal{D}^{-1}([0, K(\delta)]))$ satisfying $\star_{10\delta}$.

9 An Interesting Technical Lemma

The following lemma seems more general than Lemma 6.7, since there is no restriction on the length of the path ψ considered here, only a restriction on the depth of the path. The proof parallels that of Lemma 6.7 and the second half is exactly the same.

Lemma 9.1 *Suppose H is a horoball, $\bar{d}$ an integer $\geq \delta$, $x \neq y$ vertices of $H(\bar{d})$ and ψ is a path in $H^{\bar{d}} - B_r(*)$ between x and y that only intersects $H(\bar{d})$ at x and y. Let L be such that $\mathcal{D}(\psi) = [\bar{d}, L]$ (the depth of ψ).*

Suppose γ is a projection of ψ to $H(\bar{d})$. If z is a closest vertex of $H(\bar{d})$ to $*$ and each vertex v of ψ is such that $d(v, z) > L - \frac{\bar{d}}{2} + 2$, then each vertex of γ is at distance greater than $L - \frac{\bar{d}}{2} + 2$ from z. Furthermore, the image of γ avoids $B_{r-(2\delta+1)}(*)$.

Proof: By Lemma 5.3, $H^{\bar{d}}$ is convex. Let p be a vertex of γ and $(\alpha_p, \tau_p, \beta_p)$ a geodesic (in $H^{\bar{d}}$) from z to p where α_p and β_p are vertical of the same length and $|\tau_p| \leq 3$. Let y be the end point of τ_p. By Lemma 6.6, there is a vertical segment that begins at p and ends at most 1 (horizontal) unit from a vertex v of ψ (and $\mathcal{D}(v) \geq \bar{d} + 1$). If y is on that vertical line segment (Figure 1.1), then there is a path ρ from v to z that begins with a horizontal edge from v to a vertex w on the vertical segment, followed by a vertical segment from w to y, followed by τ_p^{-1}, followed by α_p^{-1}.

The sum of the lengths of the two vertical segments of ρ is less than or equal to $\frac{L-\bar{d}-1}{2}$ (since v is on ψ), and so

$$L - \frac{\bar{d}}{2} + 2 < d(v, z) \leq \frac{L - \bar{d} - 1}{2} + 4 = \frac{L}{2} - \frac{\bar{d}}{2} + \frac{7}{2} \text{ implying}$$

$$2L + 4 < L + 7 \text{ and } L < 3,$$

which is impossible since $x \neq y$.

Instead, y is on the vertical line at p and the vertical line segment from p to y contains a vertex w (other than y) within 1 horizontal unit of a vertex v of ψ (Figure 1.2). Now

$$d(z, p) = d(p, w) + d(w, y) + |\tau_p| + |\alpha_p| \text{ and}$$

$$L - \frac{\bar{d}}{2} + 2 < d(v, z) \leq 1 + d(w, y) + |\tau_p| + |\alpha_p|$$

Since $d(p, w) \geq 1$, $d(z, p) > L - \frac{\bar{d}}{2} + 2$. Completing the first part of the lemma.

Since the depth of v (and hence the depth of w) is at least $\bar{d} + 1$,

$$d(p, z) \geq 1 + d(w, z) \geq d(v, z)$$

Combining this inequality with Lemma 6.2 and the triangle inequality, we have for each vertex p of γ:

$$d(p, *) + 2\delta + 1 \geq d(p, z) + d(z, *) \geq d(v, z) + d(z, *) \geq d(v, *) > r$$

So $d(p, *) > r - 2\delta - 1$. $\quad\square$

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