

Relatively hyperbolic groups with free abelian second cohomology

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Abstract

Suppose G is a 1-ended finitely presented group that is hyperbolic relative to \mathcal{P} a finite collection of 1-ended finitely presented proper subgroups of G . Our main theorem states that if the boundary $\partial(G, \mathcal{P})$ is locally connected and the second cohomology group $H^2(P, \mathbb{Z}P)$ is free abelian for each $P \in \mathcal{P}$, then $H^2(G, \mathbb{Z}G)$ is free abelian. When G is 1-ended it is conjectured that $\partial(G, \mathcal{P})$ is always locally connected. Under mild conditions on G and the members of \mathcal{P} the 1-ended and local connectivity hypotheses can be eliminated and the same conclusion is obtained. When G and each member of \mathcal{P} is 1-ended and $\partial(G, \mathcal{P})$ is locally connected, we prove that the “Cusped Space” for this pair has semistable fundamental group at ∞ . This provides a starting point in our proof of the main theorem.

1 Introduction

We are interested in an old conjecture (probably due to H. Hopf) speculating that $H^2(G, \mathbb{Z}G)$ is free abelian for any finitely presented group G . T. Farrell [Far74] proved that if G is finitely presented and contains an element of infinite order then $H^2(G, \mathbb{Z}G)$ is either 0, \mathbb{Z} or not finitely generated. A result of B. Bowditch [Bow04] implies $H^2(G, \mathbb{Z}G) = \mathbb{Z}$ if and only if G contains a nontrivial closed surface group as a subgroup of finite index. For R a ring, the R -module $H^2(G, RG)$ is torsion free (see [Geo08] 13.7.1). In Section 13.8 of [Geo08], an example of a finite aspherical 3-pseudomanifold is constructed. Hence, its fundamental group G has type F and geometric dimension 3, but

$H^3(G, \mathbb{Z}G)$ is isomorphic to \mathbb{Z}_2 . This was the first exhibited example of a group of type F_n for which $H^n(G, \mathbb{Z}G)$ is not free abelian. The connection between $H^k(G, \mathbb{Z}G)$ and $H_{k-1}(\varepsilon G)$ (the $(k-1)$ -homology of the end of G) was explored in [GM85], also see Section 13.7 of Geoghegan's book [Geo08]. Our main theorem is:

Theorem 1.1 *Suppose a finitely presented 1-ended group G is hyperbolic relative to $\mathcal{P} = \{P_1, \dots, P_n\}$ a set of 1-ended finitely presented subgroups (with $G \neq P_i$ for all i). If the boundary $\partial(G, \mathcal{P})$ is locally connected and for each i , $H^2(P_i, \mathbb{Z}P_i)$ is free abelian, then $H^2(G; \mathbb{Z}G)$ is free abelian.*

When G is 1-ended and hyperbolic relative to \mathcal{P} then it may always be the case that $\partial(G, \mathcal{P})$ is locally connected. Note that there is no hypothesis on the number of ends of the P_i and no local connectedness hypotheses on $\partial(G, \mathcal{P})$ in the next result.

Corollary 1.2 *Suppose a 1-ended finitely presented group G is hyperbolic relative to $\mathcal{P} = \{P_1, \dots, P_n\}$ a set of finitely presented subgroups (with $G \neq P_i$ for all i). If for each i , P_i contains no infinite torsion subgroup and $H^2(P_i, \mathbb{Z}P_i)$ is free abelian, then $H^2(G; \mathbb{Z}G)$ is free abelian.*

Section 3 explores what it means for a space and group to have semistable fundamental group/first homology at ∞ . If a finitely presented group G has semistable fundamental group at ∞ then G also has semistable first homology at ∞ . The group $H^2(G, \mathbb{Z}G)$ is free abelian if and only if G has semistable first homology at ∞ (see Corollary 3.4). At present, it is unknown if all finitely presented groups have semistable fundamental group at ∞ , but there are many classes of groups (including the class of word hyperbolic groups) which are known to only contain groups that have semistable fundamental group at ∞ . At this point we know of only one result other than our main theorem that concludes all members G of class of groups have free abelian second cohomology with $\mathbb{Z}G$ coefficients without passing through a fundamental group at ∞ result. I. Biswas and M. Mj [BM17] prove that if G is a holomorphically convex group (in particular, if G is a linear projective group) then $H^2(G, \mathbb{Z}G)$ is free abelian.

Combining work of M. Bestvina and G. Mess [BM91], B. Bowditch [Bow99] and G. Swarup [Swa96] one can conclude that if G is a word hyperbolic group then G has semistable fundamental group at ∞ (and so $H^2(G, \mathbb{Z}G)$ is free

abelian). Interestingly, our proof of Theorem 1.1 does not translate into a proof that G has semistable fundamental group at ∞ when the parabolic subgroups $P_i \in \mathcal{P}$ have semistable fundamental group at ∞ . Basically the problem is that fundamental group is a pointed functor (while first homology is not). Still, the main theorem of [MS] is:

Theorem 1.3 ([MS], Theorem 1.2) *Suppose G is a 1-ended finitely generated group that is hyperbolic relative to a collection of 1-ended finitely generated proper subgroups $\mathcal{P} = \{P_1, \dots, P_n\}$. If $\partial(G, \mathcal{P})$ has no cut point, then G has semistable fundamental group at ∞ .*

Note that there is no semistability hypotheses on the parabolic subgroups P_i in this last result. On the other hand, any time G splits over a proper subgroup of a parabolic subgroup, $\partial(G, \mathcal{P})$ contains a cut point. Theorem 2.1 implies that in most cases, when G is 1-ended $\partial(G, \mathcal{P})$ is locally connected even when it contains cut points.

Let X be the cusped space for (G, \mathcal{P}) (defined in §4), and $Y \subset X$ be the Cayley 2-complex for G (with finite presentation containing a subpresentation for each of the parabolics). The pair (G, \mathcal{P}) is relatively hyperbolic if and only if X is δ -hyperbolic. A 1-ended space A has *semistable fundamental group at ∞* if and only if any two proper rays in A are properly homotopic.

Our first goal is to prove the following theorem, which is an important component in our proof of Theorem 1.1. The semistability of the fundamental group at ∞ of the cusped space for (G, \mathcal{P}) does not seem to be of much help in showing that G has semistable fundamental group at ∞ .

Theorem 1.4 *Suppose a finitely presented group G is hyperbolic relative to $\mathcal{P} = \{P_1, \dots, P_n\}$ a set of 1-ended finitely presented subgroups (with $G \neq P_i$ for all i). If the boundary $\partial(G, \mathcal{P})$ is locally connected then the cusped space for (G, \mathcal{P}) has semistable fundamental group at ∞ .*

When G is word hyperbolic it acts simplicially on a contractible locally finite and finite dimensional δ -hyperbolic simplicial complex K usually called the Rips complex. The group G acts freely and transitively on the vertices of K , and the quotient space G/K is compact so that $\partial G = \partial K$. Combining [Bow99] and [Swa96] we see that when G is 1-ended then ∂G is locally connected. Applying [BM91] we see ∂K is a Z -set in K and so; the word hyperbolic group G has semistable fundamental group at ∞ if and only if

$\partial K (= \partial G)$ has the shape of a locally connected continuum. Unfortunately, the analogous approach for relatively hyperbolic groups comes up a bit short for our purposes. The set $\partial(G, \mathcal{P})$ need not be a Z -set for a cusped space, but when the pair (G, \mathcal{P}) is F_∞ , Corollary 3.17 of [MW] states that for $n > 0$, there is an $(n + 1)$ -dimensional cusped space X_{n+1} such that $\partial(G, \mathcal{P})$ is a “ Z_n -set” in $X_{n+1} \cup \partial(G, \mathcal{P})$. In the case $n = 1$ and when $\partial(G, \mathcal{P})$ has the shape of a locally connected continuum, the Z_n -set conclusion implies the corresponding cusped space X_2 has semi-stable fundamental group at ∞ . Manning and Wang remark that the F_∞ hypothesis of their Corollary 3.17 can easily be relaxed to an F_{n+2} hypothesis (on G and each member of \mathcal{P}) and one still obtains that $\partial(G, \mathcal{P})$ is a Z_n set for X_{n+1} . In particular, if G and each member of \mathcal{P} is F_3 , then X_2 has semistable fundamental group at ∞ . It seems this approach will not produce a semistability result in the more general setting of finitely presented groups. Hence the need for Theorem 1.4 is apparent.

In Section 2 we list three established results from the literature that along with Theorem 1.1 and Duwoody’s accessibility result directly imply Corollary 1.2. Section 3, is devoted to basic definitions and background for the semistability of the first homology and fundamental group at ∞ of a finitely presented group G , including the connection to $H^2(G, \mathbb{Z}G)$. Section 4 covers basic definitions and background for relatively hyperbolic groups and cusped spaces. Section 5 contains our proof of Theorem 1.4. In section 6 we prove two cellular decomposition results for $[0, \infty) \times [0, \infty)$, and section 7 concludes with a proof of the Main Theorem 1.1.

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2 The Proof of Corollary 1.2

Several results mesh well with our main theorem and combine to imply Corollary 1.2.

Theorem 2.1 ([Bow01], Theorem 1.5) *Suppose (G, \mathcal{P}) is relatively hyperbolic, G is 1-ended and each $P \in \mathcal{P}$ is finitely presented, does not contain an infinite torsion group, and is either 1 or 2-ended, then $\partial(G, \mathcal{P})$ is locally connected.*

Theorem 2.2 (see [DtS05], Corollary 1.14; or [Osi06]) *Suppose the group G is finitely generated and hyperbolic relative to the finitely generated groups P_1, \dots, P_n . If \mathcal{P}_i is a finite graph of groups decomposition of P_i such that each edge group of \mathcal{P}_i is finite, then G is also hyperbolic relative to the subgroups $\{P_1, \dots, P_{i-1}, P_i, \dots, P_n\} \cup V(\mathcal{P}_i)$ where $V(\mathcal{P}_i)$ is the set of vertex groups of \mathcal{P}_i .*

If a parabolic subgroup P_i is either finite or 2-ended, it may be removed from the collection of parabolic subgroups and G remains hyperbolic relative to the remaining subgroups. Recall that a finitely generated group is *accessible* if it has a finite graph of groups decomposition with each edge group finite and each vertex group either finite or 1-ended. By M. Dunwoody's accessibility theorem [Dun85], all (almost) finitely presented groups are accessible. We show there is a finite collection \mathcal{P}' of 1-ended subgroups of G (each distinct from G) such that $\partial(G, \mathcal{P}')$ is locally connected and $H^2(P, \mathbb{Z}P)$ is free abelian for each $P \in \mathcal{P}'$. Then apply Theorem 1.1.

For each i let \mathcal{G}_i be a Dunwoody decomposition of P_i . Let \mathcal{P}_1 be the set of vertex groups of the \mathcal{G}_i for all $i \in \{1, \dots, n\}$. By Theorem 2.2, G is hyperbolic relative to \mathcal{P}_1 . By the following result, if $P \in \mathcal{P}_1$ is 1-ended, then $H^2(P; \mathbb{Z}P)$ is free abelian.

Theorem 2.3 ([Mih87], Theorem 4) *Suppose G is a finitely presented group and \mathcal{G} is a finite graph of groups decomposition of G such that each vertex group is either finite or 1-ended and each edge group is finite. Then $H^2(G, \mathbb{Z}G)$ is free abelian if and only if for each 1-ended vertex group V of \mathcal{G} , $H^2(V, \mathbb{Z}V)$ is free abelian.*

Let \mathcal{P}' be obtained from \mathcal{P}_1 by removing all finite groups. Again, G is hyperbolic relative to \mathcal{P}' . By Theorem 2.1, $\partial(G, \mathcal{P}')$ is locally connected, and by Theorem 1.1, $H^2(G, \mathbb{Z}G)$ is free abelian.

3 Semistability at ∞

The best reference for the notion of semistable fundamental group (homology) at ∞ is [Geo08] and we use this book as a general reference throughout this section. While semistability makes sense for multiple ended spaces, we are only interested in 1-ended spaces in this article. Suppose K is a 1-ended locally finite and connected CW complex. A *ray* in K is a continuous map

$r : [0, \infty) \rightarrow K$. A continuous map $f : X \rightarrow Y$ is *proper* if for each compact set C in Y , $f^{-1}(C)$ is compact in X . The space K has *semistable fundamental group at ∞* if any two proper rays in K are properly homotopic. We say K has *semistable first homology at ∞* if any two proper rays r and s in K are properly homologous.

Remark 1. In a CW-complex, any proper ray is properly homotopic to a proper edge path ray, so for semistability one only need show proper edge path rays are properly homotopic/homologous.

We are only interested in simply connected complexes. In this case we need only consider edge path rays with the same initial vertex $*$, and homotopies relative to $*$. By properly homologous, we mean there is a proper map $m : M \rightarrow K$ where M is a connected 2-manifold with boundary homeomorphic to $\mathbb{R}^1 = (-\infty, \infty)$, and such that for $t \in [0, \infty)$, $m(t) = r(t)$ and for $t \in (-\infty, 0]$, $m(t) = s(-t)$. Suppose C_0, C_1, \dots is a collection of compact subsets of a locally finite complex K such that C_i is a subset of the interior of C_{i+1} and $\cup_{i=0}^{\infty} C_i = K$. If $r : [0, \infty) \rightarrow K$ is proper, then $\pi_1(\varepsilon K, r)$ is (up to pro-isomorphism) the inverse system of groups:

$$\pi_1(K - C_0, r) \leftarrow \pi_1(K - C_1, r) \leftarrow \dots$$

When r and s are properly homotopic, $\pi_1(\varepsilon K, r)$ is pro-isomorphic to $\pi_1(\varepsilon K, s)$. When K is 1-ended and has semistable fundamental group at ∞ , $\pi_1(\varepsilon K, r)$ is independent of r (up to pro-isomorphism) and is called *the fundamental group of the end of K* . The inverse limit of $\pi_1(\varepsilon K, r)$ is denoted $\pi_1^\infty(K, r)$ and is called *the fundamental group at ∞ of K* . The group $H_1(\varepsilon K)$ is (up to pro-isomorphism) the inverse system:

$$H_1(K - C_0) \leftarrow H_1(K - C_1) \leftarrow \dots$$

These inverse systems are pro-isomorphic to inverse systems of groups with epimorphic bonding maps if and only if K has semistable fundamental group at ∞ (respectively, semistable first homology at ∞). An inverse system of groups that is pro-isomorphic to one with epimorphic bonding maps is called *semistable* or *Mettag-Leffler*.

Remark 2. If K has semistable fundamental group at ∞ , then abelianizing, it immediately follows that K has semistable first homology at ∞ .

Semistability is an invariant of proper homotopy type and quasi-isometry type. There are a number of equivalent forms of semistability. The equivalence of the conditions in the next theorem is discussed in [CM14].

Theorem 3.1 (see Theorem 3.2, [CM14]) *Suppose K is a connected 1-ended locally finite CW-complex. Then the following are equivalent:*

1. K has semistable fundamental group at ∞ .
2. Suppose $r : [0, \infty) \rightarrow K$ is a proper base ray. Then for any compact set C , there is a compact set D such that for any third compact set E and loop α based on r and with image in $K - D$, α is homotopic $\text{rel}\{r\}$ to a loop in $K - E$, by a homotopy with image in $K - C$.
3. For any compact set C there is a compact set D such that if r and s are proper rays based at v and with image in $K - D$, then r and s are properly homotopic $\text{rel}\{v\}$, by a proper homotopy in $K - C$.

Removing the base rays and replacing π_1 by H_1 in Theorem 3.1 gives the corresponding homology result. As a consequence of these homology results we have the following result which we will use in §7:

Theorem 3.2 *Suppose K has semistable first homology at ∞ . If C is a compact subset of K then there is a compact set D of K such that if α is a proper map of \mathbb{R}^1 or the circle S^1 into $K - D$ then there is a manifold M with boundary a line/circle, and a proper map $H : M \rightarrow K - C$ such that H restricted to ∂M agrees with α .*

If G is a finitely presented group and X is a finite connected complex with $\pi_1(X) = G$ then G has *semistable fundamental group at ∞* (respectively, *semistable first homology at ∞*) if the universal cover of X has semistable fundamental group (respectively, first homology) at ∞ . This definition only depends on G and it is unknown if all finitely presented groups have semistable fundamental group at ∞ . By Remark 2, if G has semistable fundamental group at ∞ , then G has semistable first homology at ∞ .

The following is a partial statement of a result of M. Mihalik and R. Geoghegan. Here \tilde{X} is the universal cover of X .

Theorem 3.3 ([GM85], Theorem 1.1) *Let G be a group of type $F(n)$ and let X be a $K(G, 1)$ CW-complex having finite n -skeleton.*

(i) *For $k \leq n$, $H^k(G, \mathbb{Z}G)$ mod torsion is free abelian if and only if $H_{k-1}(\varepsilon\tilde{X}^n)$ is semistable.*

(ii) *For $k \leq n$, $H^k(G, \mathbb{Z}G)$ is torsion free if and only if $H_{k-2}(\varepsilon\tilde{X}^n)$ is pro-torsion free.*

When $n = 2$ observe that $H_0(\varepsilon\tilde{X}^n)$ is trivially an inverse system with epimorphic bonding maps (semistable). We immediately deduce:

Corollary 3.4 *If G is a finitely presented group, then $H^2(G, \mathbb{Z}G)$ is free abelian if and only if G has semistable first homology at ∞ .*

A space X is *simply connected at ∞* if for each compact set C in X there is a compact set D in X such that loops in $X - D$ are homotopically trivial in $X - C$. If X is simply connected at ∞ then X has semistable fundamental group at ∞ and in fact, $\pi_1(\varepsilon X, r)$ is pro-trivial for any proper ray r . Lemma 5.4 shows that certain “horoballs” are simply connected at ∞ , an important fact in our proof of Theorem 1.4

4 Relatively Hyperbolic Groups and Cusped Spaces

We are only interested in locally finite 2-complexes in this paper and we only need define what it means for the 1-skeleton of such a complex to be hyperbolic. There are a number of equivalent definitions of hyperbolicity for a geodesic metric space and it is convenient for us to use the following *thin triangles* definition (see [Definition 1.5, [ABC⁺91]]).

Definition 1. Suppose Γ is a locally finite 1-complex with edge path metric d . Suppose $T = \Delta(x_1, x_2, x_3)$ is a geodesic triangle (with vertices x_1, x_2 and x_3) in Γ . By inscribing maximal circles in Euclidean comparison triangles (in \mathbb{R}^2), there is a point c_i on the side of T opposite x_i such that $d(x_1, c_3) = d(x_1, c_2)$ and similarly for x_2 and x_3 (see Figure 1). The points c_1, c_2 and c_3 are called the *internal points* of T . Let $t \in [x_i, c_j]$ and $s \in [x_i, c_k]$ (where i, j and k are distinct) be such that $d(x_i, t) = d(x_i, s)$. If there is a number $\delta \geq 0$ such that $d(t, s) \leq \delta$ for every geodesic triangle T in Γ and all such t and s , then Γ is (Gromov) δ -hyperbolic. In particular $d(c_i, c_j) \leq \delta$ for all i, j . As a

notational matter, we let $[a, b]$ be an arbitrary geodesic between the points $a, b \in \Gamma$.

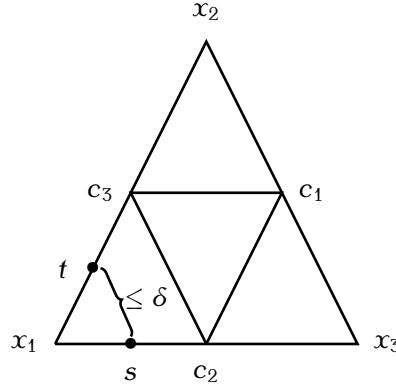


Figure 1

D. Groves and J. Manning [GM08] consider a locally finite Gromov hyperbolic space X constructed from a finitely generated group G and a collection \mathcal{P} of finitely generated subgroups. The following definitions are directly from [GM08]

Definition 2. Let Γ be any 1-complex. The *combinatorial horoball* based on Γ , denoted $\mathcal{H}(\Gamma)$, is the 2-complex formed as follows:

A) $\mathcal{H}^{(0)} = \Gamma(0) \times (\{0\} \cup \mathbb{N})$

B) $\mathcal{H}^{(1)}$ contains the following three types of edges. The first two types are called horizontal, and the last type is called vertical.

(B1) If e is an edge of Γ joining v to w then there is a corresponding edge \bar{e} connecting $(v, 0)$ to $(w, 0)$.

(B2) If $k > 0$ and $0 < d_\Gamma(v, w) \leq 2^k$, then there is a single edge connecting (v, k) to (w, k) .

(B3) If $k \geq 0$ and $v \in \Gamma^{(0)}$, there is an edge joining (v, k) to $(v, k + 1)$.

C) $\mathcal{H}^{(2)}$ contains three kinds of 2-cells:

(C1) If $\gamma \subset \mathcal{H}^{(1)}$ is a circuit composed of three horizontal edges, then there is a 2-cell (a horizontal triangle) attached along γ .

(C2) If $\gamma \subset \mathcal{H}^{(1)}$ is a circuit composed of two horizontal edges and two vertical edges, then there is a 2-cell (a vertical square) attached along γ .

(C3) If $\gamma \subset \mathcal{H}^{(1)}$ is a circuit composed of three horizontal edges and two vertical ones, then there is a 2-cell (a vertical pentagon) attached along γ , unless γ is the boundary of the union of a vertical square and a horizontal

triangle.

Definition 3. Let Γ be a graph and $\mathcal{H}(\Gamma)$ the associated combinatorial horoball. Define a *depth function*

$$\mathcal{D} : \mathcal{H}(\Gamma) \rightarrow [0, \infty)$$

which satisfies:

- (1) $\mathcal{D}(x) = 0$ if $x \in \Gamma$,
- (2) $\mathcal{D}(x) = k$ if x is a vertex (v, k) , and
- (3) \mathcal{D} restricts to an affine function on each 1-cell and on each 2-cell.

Definition 4. Let Γ be a graph and $\mathcal{H} = \mathcal{H}(\Gamma)$ the associated combinatorial horoball. For $n \geq 0$, let $\mathcal{H}_n \subset \mathcal{H}$ be the full sub-graph with vertex set $\Gamma^{(0)} \times \{0, \dots, N\}$, so that $\mathcal{H}_n = \mathcal{D}^{-1}[0, n]$. Let $\mathcal{H}^n = \mathcal{D}^{-1}[n, \infty)$ and $\mathcal{H}(n) = \mathcal{D}^{-1}(n)$.

Lemma 4.1 ([GM08], Lemma 3.10) *Let $\mathcal{H}(\Gamma)$ be a combinatorial horoball. Suppose that $x, y \in \mathcal{H}(\Gamma)$ are distinct vertices. Then there is a geodesic $\gamma(x, y) = \gamma(y, x)$ between x and y which consists of at most two vertical segments and a single horizontal segment of length at most 3.*

Moreover, any other geodesic between x and y is Hausdorff distance at most 4 from this geodesic.

Definition 5. Let G be a finitely generated group, let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a family of finitely generated subgroups of G , and let S be a generating set for G containing generators for each of the P_i . For each $i \in \{1, \dots, n\}$, let T_i be a left transversal for P_i (i.e. a collection of representatives for left cosets of P_i in G which contains exactly one element of each left coset).

For each i , and each $t \in T_i$, let $\Gamma_{i,t}$ be the full subgraph of the Cayley graph $\Gamma(G, S)$ which contains tP_i . Each $\Gamma_{i,t}$ is isomorphic to the Cayley graph of P_i with respect to the generators $P_i \cap S$. Then define

$$X = \Gamma(G, S) \cup (\cup \{\mathcal{H}(\Gamma_{i,t})^{(1)} \mid 1 \leq i \leq n, t \in T_i\}),$$

where the graphs $\Gamma_{i,t} \subset \Gamma(G, S)$ and $\Gamma_{i,t} \subset \mathcal{H}(\Gamma_{i,t})$ are identified in the obvious way. We call the space X a *cusped space* for G , \mathcal{P} and S .

The next result shows cusped spaces are fundamentally important spaces. We prove our results in these spaces.

Theorem 4.2 ([GM08], Theorem 3.25) *Suppose that G is a finitely generated group and $\mathcal{P} = \{P_1, \dots, P_n\}$ is a finite collection of finitely generated subgroups of G . Let S be a finite generating set for G containing generating sets for the P_i . The cusped space X for G , \mathcal{P} and S is hyperbolic if and only if G is hyperbolic with respect to \mathcal{P} .*

We make some minor adjustments. Assume G is finitely presented and hyperbolic with respect to the finitely presented subgroups $\mathcal{P} = \{P_1, \dots, P_n\}$. Take a finite presentation \mathcal{A} for G that contains finite presentations for each of the P_i as a subpresentation. Let S be the finite generating set for this presentation. Let Y be the Cayley 2-complex for \mathcal{A} . So Y is simply connected with 1-skeleton $\Gamma(G, S)$, and the quotient space G/Y has fundamental group G . Let X be a cusped space for G , \mathcal{P} and S . Replace the Cayley graph $\Gamma(G, S)$ in X with Y in the obvious way. For $g \in G$ and $i \in \{1, \dots, n\}$ we call gP_i a *parabolic coset*. The depth functions on the horoballs over the parabolic cosets extend to X . So that

$$\mathcal{D} : X \rightarrow [0, \infty)$$

where $\mathcal{D}^{-1}(0) = Y$ and for each horoball H (over a parabolic coset) we have $H \cap \mathcal{D}^{-1}(m) = H(m)$, $H \cap \mathcal{D}^{-1}[0, m] = H_m$ and $H \cap \mathcal{D}^{-1}[m, \infty) = H^m$. We call each H^m an *m -horoball*.

Lemma 4.3 ([GM08], Lemma 3.26) *If the cusped space X is δ -hyperbolic, then the m -horoballs of X are convex for all $m \geq \delta$.*

Given two points x and y in a horoball H , there is a shortest path in H from x to y of the form (α, τ, β) where α and β are vertical and τ is horizontal of length ≤ 3 . Note that if α is non-trivial and ascending and β is non-trivial and descending, then τ has length either 2 or 3.

Let $*$ be the identity vertex of $Y \subset X$ and d the edge path distance in X (so d is measured in X^1 , the 1-skeleton of X). For v a vertex of X and $K \in [0, \infty)$, let $B^1(v, K) = \{y \in X^1 : d(y, v) \leq K\}$ and $B(v, K) = B^1(v, K)$ union all 2-cells of X with boundary a subset of $B^1(v, K)$. We call $B(v, K)$ the ball of radius K about v in X .

Lemma 4.4 *Suppose X is a cusped space for (G, \mathcal{P}) and $Y \subset X$ is the (simply connected) Cayley 2-complex of a finite presentation for G . Given an integer K , there is an integer $N(K)$ such that if γ is an edge path loop in X of length $\leq K$, then γ is homotopically trivial in $B(v, N(K))$ for any vertex v of γ .*

Proof: If γ has image in a horoball \mathcal{H} , then by using vertical squares (with two horizontal edges and two vertical edges), γ can be slid up in \mathcal{H} to a loop τ_0 of length $\leq K$ and in a single level of \mathcal{H} (the highest level attained by γ). For v a vertex of γ , and w a vertex of τ_0 , there is a vertical path of length $< K$ from w to a vertex of γ and so $d(v, w) < 2K$. Then γ is homotopic to τ_0 by a homotopy (that only uses vertical squares) with image in $B(v, 2K)$ for any vertex v of γ . Using vertical pentagons (with two vertical edges, two horizontal edges of τ_0 and one horizontal edge in a level above τ_0), and at most one vertical square, τ_0 is homotopic to a loop τ_1 with image one level above the level of τ_0 , and $|\tau_1| \leq \frac{|\tau_0|}{2} + 1$. If v is a vertex of γ and w a vertex of τ_1 , then $d(v, w) < 2K + 1$ and γ is homotopic to τ_1 in $B(v, 2K + 1)$. If k is the smallest integer such that $K < 2^k$, then τ_0 need only be slid up at most $k - 1$ times to the loop τ_{k-1} where $|\tau_{k-1}| = 2$ (a trivial loop of the form (e, e^{-1})). If v is a vertex of γ and w a vertex of τ_{k-1} , then $d(v, w) < 2K + k$, and γ is homotopically trivial in $B(v, 2K + k)$.

If γ does not have image in a horoball, write $\gamma = (\alpha_1, \beta_1, \dots, \alpha_n, \beta_n)$ where α_i is a maximal subpath of γ in Y and β_i has image in a horoball. By cyclically permuting the edges of γ we may assume that α_1 is non-trivial (but β_n might be trivial). Since β_i begins and ends in Y , and has length $< K$ there is an edge path β'_i in Y with the same end points as β_i and $|\beta'_i| < K2^K$ (using vertical squares and pentagons, push β_i down to level 0 and note that each horizontal edge of β_i is pushed to an edge path of length $< 2^K$). The loops (β_i^{-1}, β'_i) have length $< (2^K + 1)K$ and by the first part there is a constant $N_1(K)$ satisfying the conclusion of the lemma for such loops. In particular α is homotopic to a loop α' in Y by a homotopy in $B(v, N_1(K) + K)$ for any vertex v of α , and $|\alpha'| \leq K^2(2^K + 1)$. In Y there are only finitely many edge path loops of a given length up to translation by G . Hence there is an integer $N_2(K)$ such that any loop in Y of length $\leq K^2(2^K + 1)$ is homotopically trivial in $B(v, N_2(K))$ for any vertex v of that loop. Then α is homotopically trivial in $B(v, N_2(K) + N_1(K) + K)$ for any vertex v of α . Choose $N(K)$ to be the larger of $N_2(K) + N_1(K) + K$ and $2K + k$. \square

5 The Proof of Theorem 1.4

For the remainder of the paper, we will assume that G is a finitely presented group hyperbolic relative to $\mathcal{P} = \{P_1, \dots, P_n\}$ a set of 1-ended finitely pre-

sented parabolic subgroups (with $G \neq P_i$ for all i). We assume $\partial(G, \mathcal{P})$ is locally connected. The space X is the cusped space for the group (G, \mathcal{P}) , and $Y \subset X$ is the Cayley 2-complex for G (corresponding to a presentation for G that contains presentations for the parabolics as subpresentations). We have that X is δ -hyperbolic and $\partial X = \partial(G, \mathcal{P})$. A point $p \in \partial(X)$ is an equivalence class of geodesic edge path rays. Geodesic rays r and s belong to the same equivalence class $p \in \partial X$ if there is a number K such that $d(r(t), s(t)) \leq K$ for all $t \in [0, \infty)$. We write $r \in [s] = p \in \partial X$. The space ∂X is a compact metric space.

Remark 3. The only general fact that we use about the boundary ∂X of a δ hyperbolic space X is: Two points $x, y \in \partial X$ are “close” precisely when any two representatives $r_x \in x$ and $r_y \in y$ (with $r(0) = s(0) = *$) fellow travel for a long time. More precisely, given an integer $K > 0$ there is $\epsilon(K) > 0$ such that if x and y are within ϵ in ∂X , then $d(r_x(t), r_y(t)) \leq K$ for all $t \in [0, K]$ all $r_x \in x$ and all $r_y \in y$.

Lemma 5.1 ([MS], Lemma 6.3) *For any vertex v of Y there is a geodesic edge path ray r_v in X such that $r_v(0) = *$, and for some $t_v \in [0, \infty)$ we have $d(r_v(t_v), v) \leq \delta$ and $r_v|_{[t_v, \infty)}$ has image in $\mathcal{D}^{-1}([0, 21\delta])$.*

Remark 4. For $t > 0$, Lemma 4.4 implies there is an integer $N_1(t)$ such that any edge path loop in X of length $\leq 7t + 3$ is homotopically trivial in the ball $B(v, N_1(t))$ for every vertex v of the loop. The constant $N_1(\delta)$ is used in the proofs of Lemma 5.2 and Theorem 1.4.

The next lemma implies that any two geodesic edge path rays in X are properly homotopic (Corollary 5.3), and is an important tool used to prove Theorem 1.4.

Lemma 5.2 *Suppose $\tau : [0, 1] \rightarrow \partial X$ is a path. Let τ_t be a geodesic edge path ray at $*$ representing $\tau(t)$ with $\tau_0 = r$ and $\tau_1 = s$. Suppose $K > 0$ is an integer such that for each $t \in [0, 1]$, $d(\tau_t(K), r(K)) \leq \delta$ (so that if τ is a small diameter path, K can be chosen large). Let β be an edge path from $r(K)$ to $s(K)$ of length $\leq \delta$. Then $r|_{[K, \infty)}$ is properly homotopic rel $\{r(K)\}$ to $(\beta, s|_{[K, \infty)})$ by a homotopy in $X - B(*, K - N_1(\delta))$.*

Proof: Choose points $q(0) = 0 < q(1) < \dots < q(n) = 1$ such that for each i , the diameter of $\tau([q(i), q(i+1)])$ is small enough to ensure that for each

Combining these homotopies we have $(\gamma_0, \dots, \gamma_{n-1})$ is homotopic to β by a homotopy avoiding $B(*, K - N_1(\delta))$. Combining this homotopy with H'_1 we have a null homotopy H_1 of the loop determined by $r([K, 2K])$, $s([K, 2K])$, $(\beta_0, \dots, \beta_{n-1})$ and β , such that H_1 avoids $B(*, K - N_1(\delta))$.

Note that each β_i has image avoiding $B(*, 2K - \frac{\delta}{2})$. For each i , letting β_i , $\tau_{q(i)}$ and $\tau_{q(i+1)}$ play the role of β , r and s respectively and $\tau|_{[q(i), q(i+1)]}$ the role of τ , a completely analogous argument shows that the edge path $((r([2K, 3K]))^{-1}, \beta_0, \dots, \beta_{n-1}, s([2K, 3K]))$ is homotopic to a path with image avoiding $B(*, 3K - \frac{\delta}{2})$ by a homotopy H_2 with image avoiding the ball $B(*, 2K - N_1(\delta))$. Patching together the H_i gives the desired proper homotopy. \square

Corollary 5.3 *Suppose r and s are geodesic edge path rays in X and β is an edge path from $r(0)$ to $s(0)$ then r is properly homotopic to $(\beta, s) \text{ rel}\{r(0)\}$.*

Proof: Let r_1 and s_1 be geodesic edge path rays at $*$ such that $r \in [r_1]$ and $s \in [s_1]$. Let γ_r and γ_s be arbitrary edge paths from $r(0)$ to $*$ and $s(0)$ to $*$ respectively. Certainly r is properly homotopic to $(\gamma_r, r_1) \text{ rel}\{r(0)\}$ and s is properly homotopic to $(\gamma_s, s_1) \text{ rel}\{s(0)\}$. By Lemma 5.2, r_1 is properly homotopic to $s_1 \text{ rel}\{*\}$. (Apply Lemma 5.2 with $K = 0$ and τ any path in ∂X from $[r_1]$ to $[s_1]$. Also note that $B(*, -N_1(\delta)) = \emptyset$.) Since X is simply connected, the loop $(\beta, \gamma_s, \gamma_r^{-1})$ is homotopically trivial. Simply combine the four homotopies. \square

Lemma 5.4 *For any group P with finite generating set S , the horoball for (P, S) is simply connected at ∞ .*

Proof: Let \mathcal{H} be the horoball for (P, S) and C a compact subcomplex of \mathcal{H} . Let D be the full subcomplex containing C and all vertices “between C and level-0” (so if v is a vertex of C and l_v is the vertical line thru v , then D contains all vertices of l_v from level 0, to the level of v). So if v is a vertex of $\mathcal{H} - D$, then the vertical line at v avoids D . Let α be an edge path loop in $\mathcal{H} - D$. Then using vertical squares, α can be “slid” directly up to an edge path α_1 in a single level above the top level of D , by a homotopy avoiding D . Write α_1 as the edge path loop (e_1, e_2, \dots, e_n) . Using vertical pentagons (and perhaps one vertical square), each of the pairs of edges $(e_1, e_2), (e_3, e_4), \dots$ can be slid up to edges (d_1, \dots, d_m) respectively, where $m \leq \lfloor \frac{n}{2} \rfloor + 1$. Continuing this process, one ends up with an edge path loop of length 3, which is homotopically trivial in that level. Combining homotopies, α is homotopically trivial in $\mathcal{H} - C$ (actually in $\mathcal{H} - D$). \square

Lemma 5.5 *Suppose r is a proper edge path ray at $* \in Y$, with image in X such that no tail of r has image in a horoball, then r is properly homotopic to an edge path ray in Y .*

Proof: Suppose \mathcal{H} is a horoball. If r has only finitely many edges in \mathcal{H} , then let $\alpha_1, \dots, \alpha_n$ be the maximal subpaths of r that begin and end in Y and have image in \mathcal{H} . Let β_i be an edge path in $Y \cap \mathcal{H}$ with the same initial and end point as α_i . Since \mathcal{H} is simply connected, α_i and β_i are homotopic in \mathcal{H} , relative to their common end points. Let s be the proper edge path ray obtained from r by replacing the α_i by the β_i for every horoball \mathcal{H} such that r meets \mathcal{H} in only finitely many edges. There is an obvious homotopy K from r to s and K is proper since a compact set in X can only intersect finitely many horoballs.

Let Γ_i be the Cayley graph of P_i with respect to the generating set used here. Suppose \mathcal{H} is a horoball and r has infinitely many edges in \mathcal{H} . Let $\alpha_1, \alpha_2, \dots$ be the maximal subpaths of r that begin and end in Y and have image in \mathcal{H} . Assume that the α_i are ordered as they appear as subpaths of r . Note that the α_i are also ordered subpaths of s . Let $C_0 \subset C_1 \subset \dots$ be a collection of compact subsets of X such that C_i is a subset of the interior of C_{i+1} and $\cup_{i=1}^{\infty} C_i = X$. Since each P_i is 1-ended, we may assume that if \mathcal{H} is the horoball corresponding to the coset gP_i (so that $\mathcal{H} \cap Y = g\Gamma_i$) and $\mathcal{H} \cap C_j \neq \emptyset$ then C_{j+1} contains all bounded components of $g\Gamma_i - C_j$. By Lemma 5.4, \mathcal{H} is simply connected at infinity and so we may assume that if γ is a loop in $\mathcal{H} - C_{j+1}$ then γ is homotopically trivial in $\mathcal{H} - C_j$. For convenience let $C_i = \emptyset$ for $i \leq 0$. For each $k \geq 1$, let $j(k) \geq 0$ be the largest integer such that α_k has image in $\mathcal{H} - C_{j(k)}$. Then there is an edge path β_k in $g\Gamma_i - C_{j(k)-1}$ with the same initial and end point as α_k . Since \mathcal{H} is simply connected at infinity, α_k and β_k are homotopic relative to their endpoints by a homotopy in $\mathcal{H} - C_{j(k)-2}$.

Again, any compact set C intersects only finitely many horoballs. Given any horoball \mathcal{H} , only finitely many of the homotopies of the α_k to the β_k intersect C . Combining homotopies, s (and hence r) is properly homotopic to a proper edge path ray in Y . \square

Proof: (of Theorem 1.4) By Corollary 5.3 it is enough to show that each proper edge path ray based at $*$ in X is properly homotopic $\text{rel}\{*\}$, to a geodesic edge path ray at $*$. Suppose r is a proper edge path ray at $*$ in X with tail in the horoball \mathcal{H} . Let z be a closest point of $\mathcal{H}(\delta)$ (the points

of \mathcal{H} in level δ) to $*$ and α a geodesic edge path from $*$ to z . Let s be the vertical geodesic edge path ray in \mathcal{H} beginning at z . By Lemma 4.3, (α, s) is a geodesic edge path ray. Let v be the first vertex of r such that each vertex following v belongs to \mathcal{H} . Let q be the vertical geodesic edge path ray at v . By pushing horizontal edges up along vertical squares, q and the tail of r at v are properly homotopic rel $\{v\}$. Let β be the initial segment of r from $*$ to v . By Corollary 5.3, the edge path rays (α, s) and (β, q) are properly homotopic rel $\{*\}$. But then r is properly homotopic to the geodesic edge path ray (α, s) rel $\{*\}$.

By Lemma 5.5, we now only need show that a general proper edge path ray at $*$ and with image in Y is properly homotopic rel $\{*\}$ to a geodesic edge path ray at $*$ (Corollary 5.3). Suppose r is a proper edge path ray in Y that is based at $*$. List the consecutive edges of r as e_0, e_1, \dots and consecutive vertices as $* = v_0, v_1, \dots$. Let r_0 be a geodesic edge path ray beginning at v_0 and for each $i \geq 1$, let r_i be a geodesic edge path ray in X such that $r_i(0) = *$ and for some $L_i \in [0, \infty)$, $d(r_i(L_i), v_i) \leq \delta$ (see Lemma 5.1). Let α_i be an edge path of length $\leq \delta$ from $r_i(L_i)$ to v_i (see Figure 3).

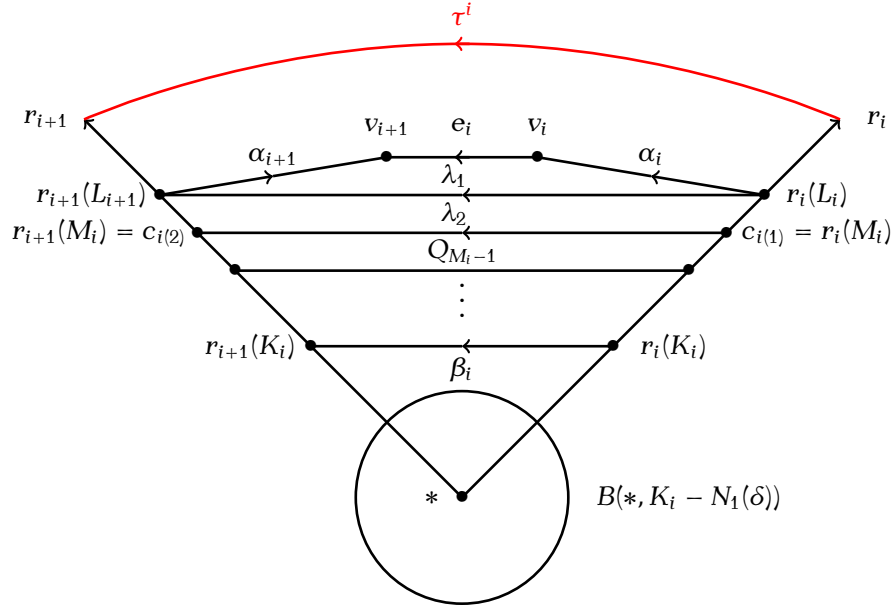


Figure 3

Since r is proper, $\lim_{i \rightarrow \infty} \{L_i\} = \infty$. Let λ_1 be a geodesic from $r_i(L_i)$ to $r_{i+1}(L_{i+1})$. Then $|\lambda_1| \leq 2\delta + 1$. Let T_i be the geodesic triangle with sides

$r_i([0, L_i])$, λ_1 and $r_{i+1}([0, L_{i+1}])$. Let $c_{i(1)}$ and $c_{i(2)}$ be the internal points of T_i on r_i and r_{i+1} respectively. By the definition of internal points:

$$d(c_{i(1)}, r_i(L_i)) + d(c_{i(2)}, r_{i+1}(L_{i+1})) = |\lambda_1| \leq 2\delta + 1$$

Also, $d(c_{i(1)}, c_{i(2)}) \leq \delta$. The definition of internal points implies that if $c_{i(1)} = r_i(M_i)$, then $c_{i(2)} = r_{i+1}(M_i)$. So $L_i - M_i = d(c_{i(1)}, r_i(L_i)) \leq 2\delta + 1$ (and $L_{i+1} - M_i \leq 2\delta + 1$). In particular $\lim_{i \rightarrow \infty} \{M_i\} = \infty$. Let λ_2 be a geodesic (of length $\leq \delta$) from $c_{i(1)} = r_i(M_i)$ to $c_{i(2)} = r_{i+1}(M_i)$. We have:

$$|(\lambda_2, r_{i+1}|_{[M_i, L_{i+1}]}, (\alpha_{i+1}, e_i^{-1}, \alpha_i^{-1}), (r_i|_{[M_i, L_i]})^{-1})| \leq 7\delta + 3$$

So there is a null homotopy H_i for this loop in $X - B(*, M_i - N_1(\delta))$ (see Figures 3 and 4(a)).

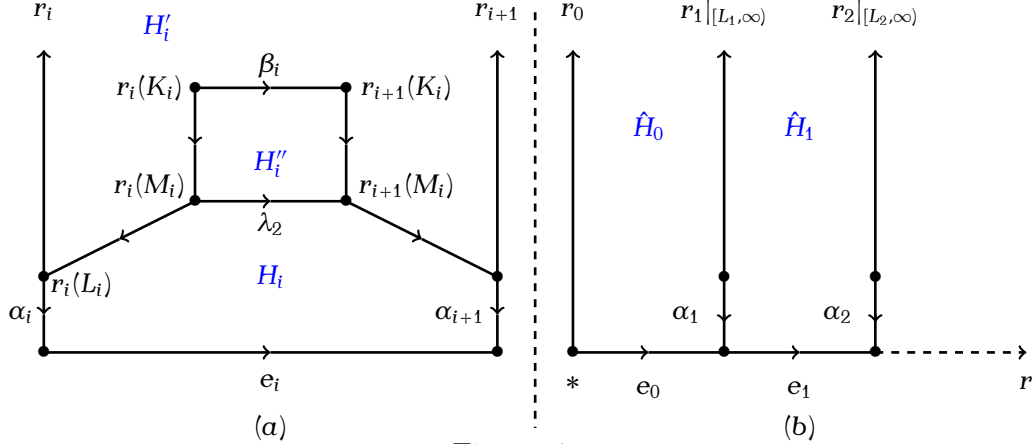


Figure 4

For large i , r_i and r_{i+1} follow travel for a long time. By the local connectivity of ∂X , there is a path τ^i in ∂X of diameter ϵ_i connecting $[r_i]$ and $[r_{i+1}]$ where $\lim_{i \rightarrow \infty} \epsilon_i = 0$. Following the notation of Lemma 5.2, let $\tau_t^i \in [\tau^i(t)]$. Then we may choose $K_i \leq M_i$ such that for all $t \in [0, 1]$, $d(\tau_t^i(K_i), r_i(K_i)) \leq \delta$, and $\lim_{i \rightarrow \infty} \{K_i\} = \infty$. Let β_i be a geodesic edge path (of length $\leq \delta$) from $r_i(K_i)$ to $r_{i+1}(K_i)$. By Lemma 5.2, $r_i|_{[K_i, \infty)}$ is properly homotopic $rel\{r_i(K_i)\}$ to $(\beta_i, r_{i+1}|_{[K_i, \infty)})$ by a homotopy H'_i with image in $X - B(*, K_i - N_1(\delta))$.

Just as with Lemma 5.2 the geodesic quadrilateral R_i with sides λ_2 , β_i , $r_i|_{[K_i, M_i]}$ and $r_{i+1}|_{[K_i, M_i]}$ can be subdivided by geodesic quadrilaterals Q_j , where two opposite sides of Q_i are corresponding edges of r_i and r_{i+1}

$(r_i([k, k+1])$ and $r_{i+1}([k, k+1])$ for some integer k) and the other two sides are geodesics of length $\leq \delta$ (see Figures 2 and 3). Since each Q_i has boundary path of length $\leq 2\delta+2$, it is homotopically trivial in $X - B(*, K_i - N_1(\delta))$, and so R_i is null homotopic by a homotopy H_i'' with image in $X - B(*, K_i - N_1(\delta))$. Combining the homotopies H_i, H_i' and H_i'' , (as in Figure 4(a)) $r_i|_{[L_i, \infty)}$ is properly homotopic to $(\alpha_i, e_i, \alpha_{i+1}^{-1}, r_{i+1}|_{[L_{i+1}, \infty)})$ by a homotopy \hat{H}_i with image in $X - B(*, K_i - N_1(\delta))$.

Since r_0 begins at $*$, we may assume that α_0 is trivial. Patching together the homotopies \hat{H}_i (for $i \in \{0, 1, \dots\}$) we have a homotopy \hat{H} of r_0 to r (see Figure 4 (b)). For any compact set C , only finitely many of the \hat{H}_i have image that intersect C and so \hat{H} is proper. \square

6 Proper Cellular Homotopies

An important combinatorial tool in the proof of our main theorem is the existence of special proper cellular homotopies between proper edge path rays in the cusped space. Throughout this section X will be the cusped space for the relatively hyperbolic pair (G, \mathcal{P}) where G is 1-ended and finitely presented and each subgroup $P \in \mathcal{P}$ of G is proper and 1-ended. Again, $Y \subset X$ is the Cayley 2-complex of G with respect to some finite presentation of G . A finite presentation for each $P \in \mathcal{P}$ is a subpresentation of the one for G . As usual, a proper homotopy between proper rays $r, s : [0, \infty) \rightarrow Z$ is a proper map $H : [0, \infty) \times [0, \infty) \rightarrow Z$ such that $H|_{[0, \infty) \times \{0\}} = r$ and $H|_{\{0\} \times [0, \infty)} = s$.

We need two decomposition theorems in this section. The first will be applied with $Z = X$ and r and s edge path rays in $Y \subset X$.

Theorem 6.1 *Let Z be a locally finite 2-dimensional CW-complex such that each 2-cell is attached to the 1-skeleton of Z by an edge path attaching map where the edge path has length $\leq N$. Suppose r and s are proper edge path rays at $* \in Z$ and $H : [0, \infty) \times [0, \infty) \rightarrow Z$ is a proper homotopy of r to s . Then there is a cellular decomposition \mathcal{B} of $[0, \infty) \times [0, \infty)$ and proper map K from the 1-skeleton of the decomposition \mathcal{B} of $[0, \infty) \times [0, \infty)$ to Z such that:*

- 1) $K|_{[0, \infty) \times \{0\}}$ (respectively $K|_{\{0\} \times [0, \infty)}$) is properly homotopic to r (respectively s) by a homotopy with image in the image of r (respectively s),
- 2) K restricted to an edge either has image a vertex, or K linearly maps the edge onto an edge of Z , and

3) If α is an edge path loop bounding a 2-cell of $[0, \infty) \times [0, \infty)$ in the compliment of the 1-skeleton of \mathcal{B} , then $|\alpha| \leq 2N$.

Proof: Our proof of this result is a simple refinement of the Proper Cellular Approximation Theorem (see Theorem 10.2.3 of [Geo08]). Begin with the decomposition of $[0, \infty) \times [0, \infty)$ with vertex set the integer grid in the first quadrant of the plane and corresponding squares (with side length 1). Next we define an open cover \mathcal{O} of Z . All open cells of Z belong to \mathcal{O} . If v is a vertex of Z , and N_v is the ball of radius $\frac{1}{2}$ about v in the 1-skeleton of Z , let U_v be the $\frac{1}{4}$ -neighborhood of N_v in Z . The U_v complete the list of open sets in \mathcal{O} . The only important facts about these U_v are:

- i) The U_v cover the 1-skeleton of Z
- ii) The open set U_v intersects U_w if and only if there is an edge connecting v and w , and
- iii) The only element of \mathcal{O} containing v is U_v .

Suppose that Z contains a bigon (when Z contains a Cayley 2-complex there may be generators of order 2 to contend with) bounded by edges e and d and vertices v and w . Then $U_v \cap U_w$ is the disjoint union of the two balls of radius $\frac{1}{4}$ with centers the midpoints of d and e . For any other type of edge e , with end points v and w , $U_v \cap U_w$ is the ball of radius $\frac{1}{4}$ with center the midpoint of e .

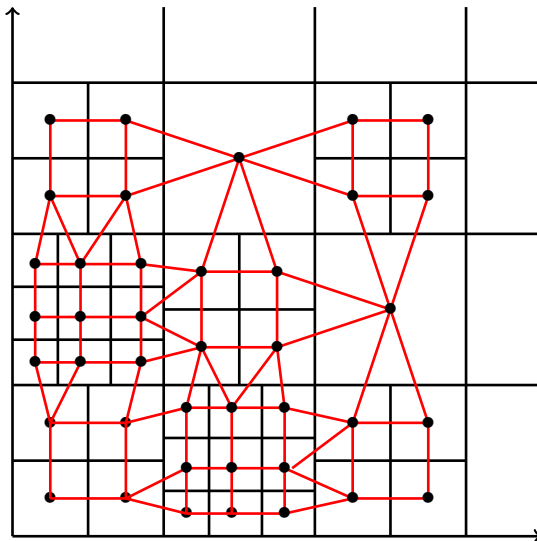


Figure 5

For each unit square A of our grid decomposition of $[0, \infty) \times [0, \infty)$, Lebesgue numbers, give a decomposition of A (by squares with all the same side length) such that each of the squares in the decomposition is mapped by H , into a member of \mathcal{O} . Consider the resulting decomposition \mathcal{A} of $[0, \infty) \times [0, \infty)$. (See Figure 5.)

Now we define the dual decomposition \mathcal{B}_1 of $[0, \infty) \times [0, \infty)$ for \mathcal{A} (also depicted in Figure 5). For each square $A \in \mathcal{A}$, let b_A be the barycenter of A and a vertex of \mathcal{B}_1 . There is an edge in \mathcal{B}_1 between b_A and b_B if A and B intersect in a non-trivial line segment (but not if the intersection is a single point/corner of both). The resulting graph decomposes $[0, \infty) \times [0, \infty)$ into triangles and (convex) quadrilaterals. A triangle (respectively quadrilateral) contains a vertex of \mathcal{A} which belongs to three (respectively four) squares of \mathcal{A} . The open triangles and quadrilaterals are the 2-cells of \mathcal{B}_1 . We will refine the 1-skeleton of \mathcal{B}_1 to obtain the desired decomposition \mathcal{B} and then define our proper map K , but first we define K on the current vertices of \mathcal{B}_1 . Perip

If $A \in \mathcal{A}$ and $H(A)$ contains a vertex v of Z , then $H(A)$ is a subset of U_v and no other member of \mathcal{O} . In this case we define $K(b_A) = v$. If $H(A)$ does not contain a vertex, but is a subset of some U_v , then pick some such vertex v (there are at most two) and define $K(b_A) = v$. Otherwise, H maps A into an open cell D of Z . For each open cell D in Z select a vertex v_D in the boundary of D . In this last case (when $H(A) \subset D$ but $H(A)$ is not a subset of U_v for any vertex v of Z) define $K(b_A) = v_D$.

Next we decide how to subdivide edges of \mathcal{B}_1 and define K on these edges. Say e is an edge of \mathcal{B}_1 with vertices b_A and b_B so:

$$A \cap B \neq \emptyset$$

If $K(b_A) = K(b_B)$ then e is not subdivided and K is constant on e . From this point forward, assume:

$$K(b_A) \neq K(b_B)$$

If $K(b_A)$ and $K(b_B)$ are the vertices of an edge d of Z then linearly extend K from e onto d (if d belongs to a bigon of Z , then pick one of the two edges of the bigon and map e to that edge).

We may now assume that $K(b_A)$ and $K(b_B)$ are not vertices of a single edge of Z . It is not the case that $H(A) \subset U_v$ and $H_B \subset U_w$ for vertices v and w of Z . Otherwise, either $v = w$ and $K(b_A) = K(b_B)$ - which is not the

case, or $v \neq w$. But if $v \neq w$, then $H(A \cap B) \subset U_v \cap U_w$ so there is an edge between $v = K(b_A)$ and $w = K(b_B)$ in Z and this is also not the case.

We can assume that $H(B) \not\subset U_v$ when v is a vertex of Z . Then $H(B) \subset D$ for some open 2-cell D of Z and $K(b_B) = v_D$. In this case we show $H(A) \subset U_w$ for some vertex w of Z . Otherwise, $H(A) \subset D_A$ and $K(b_A) = v_{D_A}$, where D_A is an open 2-cell of Z . Since $v_{D_A} = K(b_A) \neq K(b_B) = v_D$, $D_A \neq D$. This is impossible since it implies $H(A \cap B) \subset D_A \cap D$ for D_A and D disjoint open 2-cells of Z .

The only remaining possibility is that $H(B) \subset D$ (and $K(b_B) = v_D$ for some open 2-cell D of Z) and $H(A) \subset U_w$ for w a vertex of Z (and $K(b_A) = w$). Since $H(A \cap B) \subset U_w \cap D \neq \emptyset$, the open 2-cell D of Z contains w in its boundary. Let α be shortest edge path in the boundary of D from w to v_D . In particular $|\alpha| \leq \frac{N}{2}$. Subdivide the edge e with as many edges that are in α and define K to map e to α accordingly. We have now defined our decomposition \mathcal{B} of $[0, \infty) \times [0, \infty)$, and our proper map K on the 1-skeleton of this decomposition, satisfying condition 2). If D is a component of the compliment of the 1-skeleton of \mathcal{B} in $[0, \infty) \times [0, \infty)$, then D is bounded by an edge path of length $\leq 4\frac{N}{2} = 2N$ so that condition 3) is satisfied.

Now we examine the map K restricted to $\{0\} \times [0, \infty)$. Consider an edge e of $\{0\} \times [0, \infty)$ in our original unit square decomposition of $[0, \infty) \times [0, \infty)$. The edge e is decomposed into edges (e_1, \dots, e_n) of \mathcal{A} and e_i is the base of a square $S(i)$ of \mathcal{A} . The vertices $b_{S(i)}$ form an edge path that is part of the \mathcal{A} -decomposition of $\{0\} \times [0, \infty)$. Say H maps e to the edge d of Z then $K(b_{S(i)})$ is a vertex of d . Furthermore, $K(b_{S(1)})$ is the initial vertex of d and $K(b_{S(n)})$ is the end vertex of d . Clearly K restricted to $\{0\} \times [0, \infty)$ is properly homotopic to r by a homotopy with image in the image of r and similarly, K restricted to $[0, \infty) \times \{0\}$ is properly homotopic to s by a homotopy with image in the image of s . Condition 1) is satisfied. \square

Theorem 6.2 *Suppose r and s are proper edge path rays at $* \in Y$ and $H : [0, \infty) \times [0, \infty) \rightarrow X$ a proper homotopy of r to s . Let \mathcal{B} be the cellular decomposition of $[0, \infty) \times [0, \infty)$ given by Theorem 6.1 and K the corresponding map on the 1-skeleton of \mathcal{B} . There is an integer M , subdivision \mathcal{C} of \mathcal{B} and proper map L of the 1-skeleton of \mathcal{C} in $[0, \infty) \times [0, \infty)$ such that:*

1) $L|_{[0, \infty) \times \{0\}} = K|_{[0, \infty) \times \{0\}}$ (respectively $L|_{\{0\} \times [0, \infty)} = K|_{\{0\} \times [0, \infty)}$) and so is properly homotopic to r (respectively s) by a homotopy with image in the image of r (respectively s),

2) L restricted to an edge either has image a vertex, or L linearly maps the edge onto an edge of X ,

3) If α is an edge path loop bounding a 2-cell of $[0, \infty) \times [0, \infty)$ in the compliment of the 1-skeleton of \mathcal{C} , then $|\alpha| \leq M$ and either L maps α to an edge path loop in Y or an edge path loop in a horoball.

4) If e is a edge of \mathcal{C} that is mapped to an edge d with one vertex in Y and one vertex in level 1 of a horoball, then any edge path loop bounding a 2-cell of \mathcal{C} and containing e , has length 3, 4 or 5, and is mapped by L to d or homeomorphically to the boundary of one of the cells of the horoball (with boundary edge path length 4 or 5) connecting Y to an edge in level 1 of the horoball.

Proof: If e is an edge of \mathcal{B} that is mapped to an edge of X connecting a vertex of Y to a level 1 vertex of a horoball, then color e red. Suppose α bounds a 2-cell D of \mathcal{B} and α contains a red edge e . If α is as described in 4), there is nothing to do. Without loss assume e is the first edge of α and the initial point of e is mapped to a vertex of Y . Set $\alpha = (e = e_1, \dots, e_n)$. By Theorem 6.1, $n \leq 2N$ where N is the length of a longest edge path bounding a 2-cell of X . Then α has at least one more red edge and say that e_i is the first red edge of α following e_1 . Then there are two cases to deal with. First assume that K maps e and e_i to the same edge d of X . If $i = 2$ then add an edge to the interior of D connecting the initial point of e to the terminal point of e_2 (forming a triangle) and extend K on that edge to be constant. If $i \neq 2$, then add two edges in the interior of D one connecting the initial point of e and the end point of e_i the other connecting the terminal point of e and the initial point of e_i . Again extend K on these edges to be constant on each.

In the second case, K maps e to d and e_i to d_i where $d \neq d_i$. The vertices of (e_2, \dots, e_{i-1}) are mapped by K to a single horoball Q , and the initial vertex v of d and the terminal vertex w of d_i belong to Q . Since the distance between v and w in X is $\leq N$, There is an integer M_1 (only depending on N and X) such that there is an edge path β_1 connecting v and w of length $\leq M_1$ and with image in the 0-level of Q (a sub-Cayley graph of Y given by one of the parabolics). Add an edge path γ_1 to the interior of D of length $|\beta_1|$ connecting the initial vertex of e_1 and the terminal vertex of e_i . Extend K to γ_1 by mapping it to β_1 in the obvious way. For each edge a of γ_1 consider the square in Q containing $K(a)$ and two edges connecting level 0 to level 1. For each edge of γ_1 add a corresponding square to D on the same side of γ_1

(in D) as (e_2, \dots, e_{i-1}) and extend K to these edges. (See Figure 6, where the outer most edge path is α , bounding the cell D , and the edge path loop $(e_2, \dots, e_{i-1}, \tau_1)$ is mapped by K to levels 1 and above of Q with $K(\tau_1)$ in level 1. Also $K(\gamma_1) = \beta_1$ a path in level 0 of Q .)

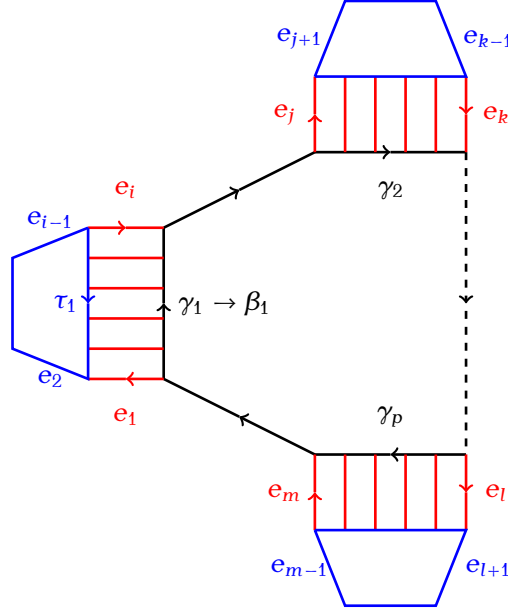


Figure 6

The cells of this refinement of \mathcal{B} are bounded by edge paths of length $< M_1 + 2N$. Find the next red edge e_j of (e_{i+1}, \dots, e_n) and add squares accordingly (Figure 6). Since α has fewer than N red edges, no cell of the final decomposition of $[0, \infty) \times [0, \infty)$ is bounded by more than $N(M_1 + 2)$ edges. Also observe that edge path loops bounding 2-cells of this decomposition of $[0, \infty) \times [0, \infty)$ that are mapped into a horoball (e.g. $(e_2, \dots, e_{i-1}, \tau_1)$) have length $< M_1 + 2N$. \square

7 The Proof of Theorem 1.1

Let H be a proper homotopy in X relative to a base point $*$ from the proper Y -edge path ray r to the proper Y -edge path ray s . So $H : [0, \infty) \times [0, \infty) \rightarrow X$ is a proper map such that $H(t, 0) = r(t) \in Y$, $H(0, t) = s(t) \in Y$ and $H(0, 0) = *$ for all t . Let L be the proper map of the 1-skeleton of the

cellular decomposition \mathcal{C} of $[0, \infty) \times [0, \infty)$ to X given by Theorem 6.2. Then $r_1 = L|_{[0, \infty) \times \{0\}}$ is properly homotopic to r in Y and $s_1 = L|_{\{0\} \times [0, \infty)}$ is properly homotopic to s in Y . Also, let M be the constant guaranteed by Theorem 6.2. First we extend L to $[0, \infty) \times [0, \infty)$. Note that Y admits only finitely many edge path loops of length $\leq M$, up to translation by the action of G . Consider an edge path loop α of $[0, \infty) \times [0, \infty)$ bounding a 2-cell in the compliment of the 1-skeleton of \mathcal{C} (and hence α has length $\leq M$). Then $L(\alpha)$ is an edge path loop in Y of length $\leq M$. There is an integer $T_1(M)$ such that $H(\alpha)$ is homotopically trivial in the T_1 neighborhood (in Y) of any vertex of $L(\alpha)$. This means that L can be extended to the 2-cell bounded by α (for all such α) and remain proper. Similarly there exists T_2 such that if $L(\alpha)$ has image in a horoball of X it is homotopically trivial in that horoball by a homotopy with image in the T_2 neighborhood of any vertex of $L(\alpha)$, so L extends properly to the cells bounded by such α . Finally, extend L to the cells bounded by loops α containing a “red” edge (an edge that L maps to an edge that connects a vertex of Y to a level-1 vertex of a horoball). In this case $H(\alpha)$ has image either a single vertex, a single edge or a loop bounding a cell with two vertical edges and either two or three horizontal edges. Now $L : [0, \infty) \times [0, \infty) \rightarrow X$ is proper.

Repeating notation from §6, each red edge belongs to a *connector* 2-cell that contains exactly 2 red edges (and either two or three horizontal edges) and each red edge belongs to exactly two connector 2-cells in $[0, \infty) \times [0, \infty)$.

The connector cells of $[0, \infty) \times [0, \infty)$ can be partitioned into sets so that each set either:

1) Determines a loop of connector cells which we call a *wheel*. The red edges of the wheel are called the *spokes* of the wheel. Each wheel is bounded by two loops such that L maps one loop to Y and the other loop to level 1 of a horoball, or

2) A real line of connector cells which we call a ladder. The red edges of the ladder are called *rungs* and each ladder is bounded by two lines such that one is mapped by L to Y and the other is mapped to level 1.

If we connect the midpoints of two red edges of a connector cell by a geodesic and concatenate these geodesics in a wheel, we obtained an embedded circle in $[0, \infty) \times [0, \infty)$ called the *center circle* of the wheel. Similarly the midpoint geodesics for a ladder form an embedded line called the *center line* of the ladder. The loops of 1) and the lines of 2) may intersect themselves, but this last observation implies they cannot “cross” themselves (see Figure 7). We call the loops/lines of a wheel/ladder which are mapped to Y

by L , the (*level*) 0 -boundary path of that wheel/ladder. The level 0 loop of the wheel of Figure 7 is the union of a square and triangle).

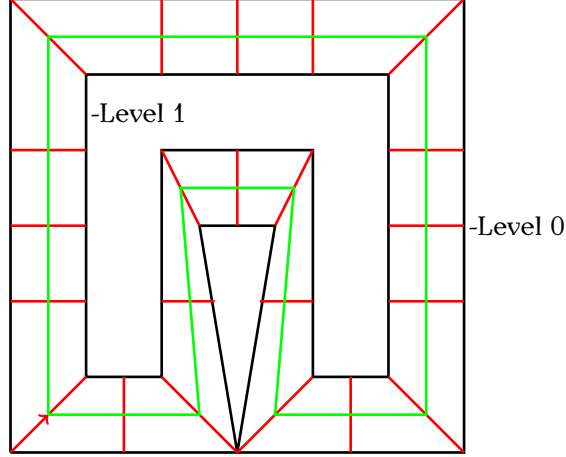


Figure 7

Cut out a center circle/line of a wheel/ladder in $[0, \infty) \times [0, \infty)$ if its 0-boundary path can be connected to $(\{0\} \times [0, \infty)) \cup ([0, \infty) \times \{0\})$ by an edge path containing no red edge and call the resulting space W . Let W_1 be the closure (in X) of the component of W containing $(\{0\} \times [0, \infty)) \cup ([0, \infty) \times \{0\})$. First observe that W_1 is a 2-manifold with boundary and L “nearly” maps W_1 to Y . Since W_1 contains no red edge, only the half connector cells of W_1 are mapped to $X - Y$. Also observe that there is a strong deformation retraction R of W_1 to a subcomplex of $[0, \infty) \times [0, \infty)$ that is mapped by L to Y (simply deform each half cell of a connector cell to its base). Each boundary component of W_1 is a center circle/line that is mapped by R to the 0-boundary path of that wheel/ladder, which in turn is mapped by L to the 0 level of a horoball of X (and hence to a parabolic Cayley graph in Y). At this point we alter L restricted to W_1 to have image in Y . Define $L_1 : W_1 \rightarrow Y$ by first retracting the half cells of W_1 to the 0 level paths of those connector cells, and then composing with L . Then $L_1 : W_1 \rightarrow Y$ is a proper map.

Using the fact that the parabolic subgroups of \mathcal{P} have semistable first homology at ∞ , we next attach 2-manifolds to the center circles/lines of W_1 and in a uniformly proper way, map these 2-manifolds into Y such that on the center circles/lines, these maps agree with L_1 . In this way, we obtain a

2-manifold manifold M_1 with boundary $(\{0\} \times [0, \infty)) \cup ([0, \infty) \times \{0\})$ and a proper map of M_1 into Y that agrees with L on $(\{0\} \times [0, \infty)) \cup ([0, \infty) \times \{0\})$. In particular, r_1 and s_1 (and hence r and s) are properly homologous in Y , as desired.

Suppose C is a compact subset of Y . If $P \in \mathcal{P}$ and Γ is a copy of the Cayley 2-complex of P in Y , then only finitely many G translates of Γ intersect C . Hence there are only finitely many Cayley 2-complexes $\Gamma_1, \dots, \Gamma_m$ such that Γ_i is a copy of a Cayley 2-complex in Y for some $P \in \mathcal{P}$ and $C \cap \Gamma_i \neq \emptyset$. Each $P \in \mathcal{P}$ (and hence each Γ_i) has semistable first homology at ∞ . By Theorem 3.2 there is $D_i(C)$ compact such that if α is a proper edge path line or edge path loop in $\Gamma - D_i$, there is a 2-manifold M_α with single boundary component a line/circle S_α , and proper map $L_\alpha : M_\alpha \rightarrow Y - C$ such that L_α restricted to S_α is α . Let $D(C) = D_1(C) \cup \dots \cup D_m(C)$

Choose compact sets $C_0 \subset C_1 \subset \dots$ in Y so that C_i is a subset of the interior of C_{i+1} , $D(C_i) \subset C_{i+1}$ for $i \geq 1$ and $\cup_{i=1}^\infty C_i = Y$. Let $C_0 = \emptyset$. If B is a boundary component of $W_1 \subset [0, \infty) \times [0, \infty)$ then let $N(B)$ be the largest integer such that $L_1(B)$ has image in $Y - C_{N(B)}$. There is a 2-manifold M_B with single boundary component S_B and a proper map $L_B : M_B \rightarrow Y - C_{N(B)-1}$ so that L_B on S_B agrees with L_1 on B . By this we mean, there is a homeomorphism of circles/lines $h_B : S_B \rightarrow B$ such that $L_1 h_B = L_B$. Attach M_B to W_1 by the attaching homeomorphism h_B and extend L_1 on M_B by L_B . After attaching all M_B the result is a 2-manifold M_1 and L_1 extends to $\bar{L} : M_1 \rightarrow Y$. All that is left to show is that \bar{L} is proper.

Let C be compact in Y , and ∂W_1 be the set of boundary components of the 2-manifold W_1 . Then

$$\bar{L}^{-1}(C) = L_1^{-1}(C) \cup_{B \in \partial W_1} L_B^{-1}(C)$$

Say $C \subset C_i$. If $B \in \partial W_1$ and $L_1(B) \cap C_{i+1} = \emptyset$, then $L_B^{-1}(C) = \emptyset$. Since L_1 is proper, only finitely many $B \in \partial W_1$ are such that $L_1(B) \cap C_{i+1} \neq \emptyset$. As each L_B is proper, $\bar{L}^{-1}(C) = L_1^{-1}(C) \cup_{B \in \partial W_1} L_B^{-1}(C)$ is a finite union of compact sets in M_1 . So \bar{L} is proper, r_1 is properly homologous to s_1 and Y has semistable first homology at ∞ .

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