On the Rank of a Coxeter Group

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Abstract In this paper, we determine the set of all possible ranks of an arbitrary finitely generated Coxeter group.

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1 Introduction

This paper is a continuation of our previous paper with Steven Tschantz [5] in which we studied the relationship between two sets S and S' of Coxeter generators of a finitely generated Coxeter group W. A *basic subset* of S is a maximal subset B of S such that B generates an irreducible, noncyclic, finite subgroup of W. In [5], we proved the Basic Matching Theorem which says that there is a natural bijection (matching) between the basic subsets of S and the basic subsets of S'. A basic subset B of S matches a basic subset B' of S' if and only if $[\langle B\rangle, \langle B\rangle]$ is conjugate to $[\langle B'\rangle, \langle B'\rangle]$ in W. Usually matching basic subsets generate isomorphic groups, in which case, we say that the *basic subsets match isomorphically*; however, there are exceptions, due to well known isomorphisms between irreducible and reducible finite Coxeter groups (for instance the dihedral group $\mathbf{D}_2(6)$ of order 12 and $\mathbf{A}_1 \times \mathbf{A}_2$). We showed that nonisomorphic matching of basic subsets can be understood by *blowing up* Coxeter generating sets. This is a procedure to replace a given Coxeter generating set S by a Coxeter generating set R such that $|R \cap S| = |S| - 1$ and $|R| = |S| + 1$. In [5], we proved that there exists a set of Coxeter generators S' of W such that a basic subset B of S matches a basic subset B' of S' with $|\langle B\rangle| > |\langle B'\rangle|$ if and only if S can be blown up. We proved that S has maximum rank over all sets of Coxeter generators of W if and only if S can not be blown up.

In this paper, we study the procedure of *blowing down* Coxeter generating sets. We first determine necessary and sufficient conditions on (W, S) such that there exists a set of Coxeter generators S' of W such that a basic subset B of S matches a basic subset B' of S' with $|\langle B\rangle| < |\langle B'\rangle|$. We then determine necessary and sufficient conditions on (W, S) such that W has a set of Coxeter generators S' such that $|S'| < |S|$. As an application, we determine the set of all possible ranks of an arbitrary finitely generated Coxeter group.

2 Preliminaries

We continue with the terminology of [5]. In particular, we use Coxeter's notation on p. 297 of [2] for the irreducible spherical simplex reflection groups except that we denote the dihedral group $\mathbf{D}_2^k$ by $\mathbf{D}_2(k)$. Subscripts denote the rank of a Coxeter system in Coxeter's notation. Coxeter's notation partly agrees with but differs from Bourbaki's notation on p.193 of [1]. Coxeter proved that every finite irreducible Coxeter system is isomorphic to exactly one of the Coxeter systems $\mathbf{A}_n$, $n \geq 1$, $\mathbf{B}_n$, $n \geq 4$, $\mathbf{C}_n$, $n \geq 2$, $\mathbf{D}_2(k)$, $k \geq 5$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_3$, $\mathbf{G}_4$. See §3 of [5] for definitions. For uniformity of notation, we define $\mathbf{B}_3 = \mathbf{A}_3$, $\mathbf{D}_2(3) = \mathbf{A}_2$ and $\mathbf{D}_2(4) = \mathbf{C}_2$.

Let (W, S) be a Coxeter system. A *basic subset* of S is a maximal irreducible subset B of S such that $\langle B\rangle$ is a noncyclic finite group. If B is a basic subset of S, we call B a *base* of (W, S) and $\langle B\rangle$ a *basic subgroup* of W.

Theorem 2.1 (Basic Matching Theorem, Theorem 4.18 [5]) *Let W be a finitely generated Coxeter group with two sets of Coxeter generators S and S'. Let B be a base of (W, S). Then there is a unique irreducible subset B' of S' such that $[\langle B\rangle, \langle B\rangle]$ is conjugate to $[\langle B'\rangle, \langle B'\rangle]$ in W. Moreover,*

1. *the set B' is a base of (W, S'), and we say that B and B' match,*
2. *if $|\langle B\rangle| = |\langle B'\rangle|$, then B and B' have the same type and there is an isomorphism $\phi : \langle B\rangle \to \langle B'\rangle$ that restricts to conjugation on $[\langle B\rangle, \langle B\rangle]$ by an element of W, and we say that B and B' match isomorphically,*
3. *if $|\langle B\rangle| < |\langle B'\rangle|$, then either B has type $\mathbf{B}_{2q+1}$ and B' has type $\mathbf{C}_{2q+1}$ for some $q \geq 1$ or B has type $\mathbf{D}_2(2q+1)$ and B' has type $\mathbf{D}_2(4q+2)$ for some $q \geq 1$. Moreover, there is a monomorphism $\phi : \langle B\rangle \to \langle B'\rangle$ that restricts to conjugation on $[\langle B\rangle, \langle B\rangle]$ by an element of W.*

3 Blowing Down Coxeter Systems

Let (W,S) be a Coxeter system of finite rank. In this section, we determine necessary and sufficient conditions for a base B of (W,S) to match a base B' of (W,S') with $|\langle B\rangle| < |\langle B'\rangle|$. If a base B of (W,S) matches a base B' of (W,S') with $|\langle B\rangle| < |\langle B'\rangle|$, then either B is of type $\mathbf{B}_{2q+1}$ and B' is of type $\mathbf{C}_{2q+1}$ for some $q \geq 1$ or B is of type $\mathbf{D}_2(2q+1)$ and B' is of type $\mathbf{D}_2(4q+2)$ for some $q \geq 1$ by the Basic Matching Theorem.

If $a \in S$, the *neighborhood* of a in P-diagram of (W,S) is defined to be the set $N(a) = \{s \in S : m(s,a) < \infty\}$.

If $A \subseteq S$, define $A^\perp = \{s \in S : m(s,a) = 2 \text{ for all } a \in A\}$.

Lemma 3.1 *Let $B = \{x,y\}$ be a base of (W,S) of type $\mathbf{D}_2(2q+1)$ that matches a base B' of (W,S') of type $\mathbf{D}_2(4q+2)$ for some $q \geq 1$. Then $N(x) \cap N(y) = B \cup B^\perp$.*

Proof: Suppose $s \in S - B$ with $m(s,x), m(s,y) < \infty$. Let $M \subseteq S$ be a maximal simplex containing $\{s,x,y\}$. Then there is a unique maximal simplex $M' \subseteq S'$ such that $\langle M\rangle$ is conjugate to $\langle M'\rangle$ by Prop. 4.21 of [5]. By conjugating S', we may assume that $\langle M\rangle = \langle M'\rangle$. Then M' contains B' and $[\langle B\rangle, \langle B\rangle]$ is conjugate to $[\langle B'\rangle, \langle B'\rangle]$ in $\langle M'\rangle$ by the Basic Matching Theorem.

Let $B' = \{a,b\}$. Then $m(s',a) = m(s',b) = 2$ for all $s' \in M' - B'$ by Theorem 8.7 of [5]. Hence B' is a component of M'. Therefore $[\langle B'\rangle, \langle B'\rangle]$ is a normal subgroup of $\langle M'\rangle$. Hence $[\langle B\rangle, \langle B\rangle]$ is a normal subgroup of $\langle M\rangle$. Therefore $s\{x,y\}s = \{x,y\}$ by Lemma 4.17 of [5], and so $sxs = x$ and $sys = y$ by the deletion condition. Hence $N(x) \cap N(y) = B \cup B^\perp$. $\square$

Lemma 3.2 *Let $\phi : \mathbf{B}_n \to \mathbf{C}_n$ be a monomorphism with n odd and $n \geq 3$. Then ϕ maps $b_{n-1}b_n$ to a conjugate of $(c_{n-1}c_n)^2$ in $\mathbf{C}_n$.*

Proof: See §3 of [5] for the definition of the generators of $\mathbf{B}_n$ and $\mathbf{C}_n$. Now $\phi(\mathbf{B}_n)$ does not contain the center of $\mathbf{C}_n$, since $Z(\mathbf{B}_n) = \{1\}$. Therefore either $\phi(\mathbf{B}_n) = \mathbf{B}_n$ or $\phi(\mathbf{B}_n) = \theta(\mathbf{B}_n)$ where θ is the automorphism of $\mathbf{C}_n$ defined by $\theta(c_i) = -c_i$, for $i = 1, \ldots, n-1$ and $\theta(c_n) = c_n$. Now θ restricts to the identity on $[\mathbf{C}_n, \mathbf{C}_n]$, and so by composing ϕ with θ in the latter case, we may assume that $\phi(\mathbf{B}_n) = \mathbf{B}_n$. Every automorphism of $\mathbf{B}_n$ is inner by Theorem 31 of [3]. Hence ϕ restricts to conjugation on $[\mathbf{B}_n, \mathbf{B}_n]$ by an element of $\mathbf{B}_n$.

As $b_{n-1}b_n$ is in $[\mathbf{B}_n, \mathbf{B}_n]$ and $b_{n-1}b_n = (c_{n-1}c_n)^2$, we conclude that $\phi(b_{n-1}b_n)$ is conjugate to $(c_{n-1}c_n)^2$ in $\mathbf{C}_n$. $\square$

Let W be a finitely generated Coxeter group with two sets S and S' of Coxeter generators, and let A be a subset of S. Let $\overline{A}$ be the intersection of all subsets B of S such that B contains A and $\langle B\rangle$ is conjugate to $\langle B'\rangle$ for some $B' \subseteq S'$. Then $\overline{A}$ is the smallest subset B of S such that B contains A and $\langle B\rangle$ is conjugate to $\langle B'\rangle$ for some $B' \subseteq S'$ by Prop. 4.14 of [5]. If A is a spherical simplex, then $\overline{A}$ is a spherical simplex, since for any maximal spherical simplex $M \subseteq S$ that contains A, there exists a unique maximal spherical simplex $M' \subseteq S'$ such that $\langle M\rangle$ is conjugate to $\langle M'\rangle$ by Prop. 4.13 of [5].

Lemma 3.3 *Let B be a base of (W, S) of type $\mathbf{B}_{2q+1}$ that matches a base B' of (W, S') of type $\mathbf{C}_{2q+1}$ for some $q \geq 1$. Let x, y be the split ends of the C-diagram of $(\langle B\rangle, B)$. Then $\overline{\{x, y\}} = \overline{B}$ and $N(x) \cap N(y) = B \cup B^{\perp}$.*

Proof: Let $C = \overline{\{x, y\}}$. Then C is a spherical simplex of (W, S) and $\langle C\rangle$ is conjugate to $\langle C'\rangle$ for some $C' \subseteq S'$. By conjugating S', we may assume that $\langle C\rangle = \langle C'\rangle$. Let a, b, c be the elements of B' such that $m(a, b) = 4$ and $m(b, c) = 3$. Now $xy \in [\langle B\rangle, \langle B\rangle]$, and so xy is conjugate to $(ab)^2$ by the Basic Matching Theorem and Lemma 3.2. Hence there is a $w \in W$ such that $w(ab)^2w^{-1} \in \langle C'\rangle$. Now $\langle (ab)^2\rangle = [\langle a, b\rangle, \langle a, b\rangle]$. Let u be the shortest element of $\langle C'\rangle w\langle a, b\rangle$. Then $u\{a, b\}u^{-1} \subseteq C'$ by Lemma 4.17 of [5]. As $m(a, b) = 4$, we deduce that $\{a, b\} \subseteq C'$ by Lemma 4.9 of [5]. Hence $B' \subseteq C'$ by Lemma 8.1 of [5]. By the Basic Matching Theorem, $B \subseteq C$. Hence $\overline{B} \subseteq C$. As $\{x, y\} \subset B$, we have $C \subseteq \overline{B}$. Thus $C = \overline{B}$.

Suppose $s \in S - B$ with $m(s, x), m(s, y) < \infty$. Let $M \subseteq S$ be a maximal simplex containing $\{s, x, y\}$. Then there is a maximal simplex $M' \subseteq S'$ such that $\langle M\rangle$ is conjugate to $\langle M'\rangle$ by Prop. 4.21 of [5]. By conjugating S', we may assume that $\langle M\rangle = \langle M'\rangle$. Now $\overline{\{x, y\}} \subseteq M$, and so $B \subseteq M$. Hence $B' \subseteq M'$ by the Basic Matching Theorem. Moreover $m(s', t') = 2$ for all $(s', t') \in (M' - B') \times B'$ by Theorem 8.2 of [5]. Hence B' is a component of M'. Therefore $[\langle B'\rangle, \langle B'\rangle]$ is a normal subgroup of $\langle M'\rangle$. By the Basic Matching Theorem, $[\langle B\rangle, \langle B\rangle]$ is conjugate to $[\langle B'\rangle, \langle B'\rangle]$ in $\langle M\rangle$. Therefore $[\langle B\rangle, \langle B\rangle]$ is a normal subgroup of $\langle M\rangle$. Then $sBs = B$ by Lemma 4.17 of [5], and so $sts = t$ for all $t \in B$ by the deletion condition. Hence $N(x) \cap N(y) = B \cup B^{\perp}$. $\square$

The *odd diagram* of W is the labeled undirected diagram $\Omega(W, S)$ obtained from the P-diagram of (W, S) be deleting the even labeled edges. If $a \in S$, we define $\mathrm{Odd}(a)$ to be the vertex set of the connected component of $\Omega(W, S)$ containing a. By Prop. 3 on p. 12 of Bourbaki [1], we have that

$$\mathrm{Odd}(a) = \{s \in S : s \text{ is conjugate to } a \text{ in } W\}.$$

We define

$$\mathrm{EOdd}(a) = \mathrm{Odd}(a) \cup \{s \in S : m(s, b) \text{ is even for some } b \in \mathrm{Odd}(a)\}.$$

Lemma 3.4 *Let B be a base of (W, S) that matches a base B' of (W, S') with $|\langle B\rangle| < |\langle B'\rangle|$. Then there exists $r \in \overline{B} - B$ such that $N(r) = B \cup B^{\perp}$ and $\mathrm{Odd}(r) = \{r\}$, and if K is the component of $B^{\perp}$ containing r, then K is of type $\mathbf{A}_1$, $\mathbf{C}_{2q+1}$, or $\mathbf{D}_2(4q + 2)$ for some $q \geq 1$; moreover, if $K \neq \{r\}$, then K is a basic subset of S and if K' is the basic subset of S' that matches K, then K' is a component of $(B')^{\perp}$ and $K' \cup (K')^{\perp} = B' \cup (B')^{\perp}$.*

Proof: Let $C = \overline{B}$. Then C is a spherical simplex of (W, S) and $\langle C\rangle$ is conjugate to $\langle C'\rangle$ for some $C' \subseteq S'$. By conjugating S', we may assume that $\langle C\rangle = \langle C'\rangle$. Then C' contains B' by the Basic Matching Theorem. Hence B is a proper subset of C, since otherwise $\langle B'\rangle \subseteq \langle C'\rangle = \langle C\rangle = \langle B\rangle$ which is not the case, since $|\langle B\rangle| < |\langle B'\rangle|$.

Let $M \subseteq S$ be a maximal spherical simplex containing B, and let $M' \subseteq S'$ be the maximal spherical simplex such that $\langle M\rangle$ is conjugate to $\langle M'\rangle$. Then M' contains B' by the Basic Matching Theorem. Let w be an element of W such that $w\langle M\rangle w^{-1} = \langle M'\rangle$. Now $B' \subseteq \langle C\rangle \subseteq \langle M\rangle$. Hence $wB'w^{-1} \subset \langle M'\rangle$. Let u be the shortest element of $\langle M'\rangle w\langle B'\rangle$. Then $uB'u^{-1} \subseteq M'$ by Lemma 4.3 of [5]. As B' is a base of (W, S'), we have that $uB'u^{-1} = B'$ by Lemma 4.10 of [5], and so u acts as a graph automorphism on $\langle B'\rangle$. Let z' be the longest element of $\langle B'\rangle$. Then $uz'u^{-1} = z'$. Now $w = xuy$ with $x \in \langle M'\rangle$ and $y \in \langle B'\rangle$. Hence $wz'w^{-1} = z'$, since z' is in the center of $\langle M'\rangle$. Therefore z' is in the center of $\langle M\rangle = w^{-1}\langle M'\rangle w$. As $B \cup B^{\perp}$ is the union of all the maximal spherical simplices of (W, S) that contain B, we deduce the z' is in the center of $B^{\perp}$. Hence, there are distinct components $K_1, \ldots, K_n$ of $B^{\perp}$, with nontrivial center, such that $z' = z_1 \cdots z_n$ with z_i the longest element of $\langle K_i\rangle$ for each $i = 1, \ldots, n$. As $z' \in \langle C\rangle$, we have that $K_i \subseteq C$ for each $i = 1, \ldots, n$ by Prop. 7 on p. 19 of [1], since every reduced form of z_i involves every element of K_i for each $i = 1, \ldots, n$.

Define a homomorphism $\rho : \langle C' \rangle \to \langle z' \rangle$ as follows. To begin with, define $\rho(s') = 1$ if $s' \in C' - B'$. By the Basic Matching Theorem, B' is of type $\mathbf{C}_{2p+1}$ or $\mathbf{D}_2(4p+2)$ for some $p \geq 1$. If B' is of type $\mathbf{C}_{2p+1}$, let $a' \in B'$ be such that $B' \cap \mathrm{Odd}(a') = \{a'\}$, and define $\rho(a') = z'$ and $\rho(s') = 1$ for each $s' \in B' - \{a'\}$. Suppose B' is of type $\mathbf{D}_2(4p+2)$ and $B' = \{a', b'\}$. By Lemma 8.6 of [5], one of a' or b', say a', has the property that if $a' \in A' \subseteq S'$ and $\langle A' \rangle$ is conjugate to $\langle A \rangle$ for some $A \subseteq S$, then $B' \subseteq A'$. Define $\rho(a') = z'$ and $\rho(b') = 1$. In both cases, ρ is well defined and $\rho(z') = z'$.

As $z' = z_1 \cdots z_n$, there is an i such that $\rho(z_i) = z'$. By reindexing, we may assume $i = 1$. Let $K = K_1$. Then there exists $r \in K$ such that $\rho(r) = z'$. As $r \in \overline{B}$, we have that $\overline{\{r\}} \subseteq \overline{B}$. Let $A = \overline{\{r\}}$. Then $\langle A \rangle$ is conjugate in $\langle C' \rangle$ to $\langle A' \rangle$ for some $A' \subseteq C'$ by Prop. 4.14 of [5]. Now $\rho(\langle A' \rangle) = \rho(\langle A \rangle) = \langle z' \rangle$. Hence $a' \in A'$. Then $B' \subseteq A'$ by Lemma 8.1 of [5] or the choice of a'. Hence $B \subseteq A$ by the Basic Matching Theorem, and so $\overline{B} \subseteq A$. Hence $\overline{\{r\}} = \overline{B}$.

As K is a component of $B^\perp$, we have $B \cup B^\perp \subseteq N(r)$. Suppose $s \in N(r)$. Let $M \subseteq S$ be a maximal spherical simplex containing $\{r, s\}$. Then there is a maximal spherical simplex $M' \subseteq S'$ such that $\langle M \rangle$ is conjugate to $\langle M' \rangle$. Then $\overline{\{r\}} \subseteq M$, and so $B \subseteq M$. Therefore $s \in B \cup B^\perp$, since B is a basic subset of M. Thus $N(r) = B \cup B^\perp$.

As $[\langle K \rangle, \langle K \rangle]$ is in the kernel of ρ, we have that z_1 is not in $[\langle K \rangle, \langle K \rangle]$. Therefore K is of type $\mathbf{A}_1$, $\mathbf{C}_{2q+1}$, $\mathbf{D}_2(4q+2)$, $\mathbf{E}_7$, or $\mathbf{G}_3$ for some $q \geq 1$. Suppose K is of type $\mathbf{C}_{2q+1}$ for some $q \geq 1$. Let $a \in K$ be such that $K \cap \mathrm{Odd}(a) = \{a\}$. Then $a[\langle K \rangle, \langle K \rangle] = z_1[\langle K \rangle, \langle K \rangle]$. As the restriction of ρ to $\langle K \rangle$ factors through $\langle K \rangle / [\langle K \rangle, \langle K \rangle]$, we may assume that $r = a$. Then $\mathrm{Odd}(r) = \{r\}$.

If $K = \{r\}$, then we are done. Suppose $K \neq \{r\}$. Then K is a basic subset of S, since $N(r) = B \cup B^\perp$ and K is a component of $B^\perp$. Let K' is the basic subset of C' that matches K. Then K' is the basic subset of S' that matches K by the Basic Matching Theorem. Let $M' \subseteq S'$ be a maximal spherical simplex that contains B', and let $M \subseteq S$ be the maximal spherical simplex such that $\langle M \rangle$ is conjugate to $\langle M' \rangle$. Then M contains B by the Basic Matching Theorem. Now $K \subseteq C \subseteq M$ and K is a basic subset of M. Therefore K' is a basic subset of M' by the Basic Matching Theorem. Hence K' is a component of M'. Therefore K' is a component of $(B')^\perp$.

Suppose $s' \in (K')^\perp$. Let $M' \subseteq S'$ be a maximal spherical simplex that contains $K' \cup \{s'\}$, and let $M \subseteq S$ be the maximal spherical simplex such that $\langle M \rangle$ is conjugate to $\langle M' \rangle$. Then $K \subseteq M$ by the Basic Matching Theorem. Hence M contains r, and so $B \subseteq \overline{\{r\}} \subseteq M$. Therefore M' contains B' by the

Basic Matching Theorem. Hence $s' \in B' \cup (B')^\perp$. Therefore $(K') \cup (K')^\perp = B' \cup (B')^\perp$. If K is of type $\mathbf{C}_{2q+1}$ or $\mathbf{D}_2(4q+2)$, we are done.

Suppose K is of type $\mathbf{E}_7$ or $\mathbf{G}_3$. Then $\rho(s) = z'$ for each s in K, since s and r are conjugate in $\langle K \rangle$ for each $s \in K$. Hence $\overline{\{s\}} = \overline{B}$ and $N(s) = B \cup B^\perp$ for each $s \in K$ by the above argument. Therefore $\mathrm{Odd}(r) = K$ and $\mathrm{EOdd}(r) = B \cup B^\perp$. Suppose $n > 1$. By Lemma 28 of [4], there is an automorphism θ of W such that $\theta(s) = s$ for all $s \in S - K$, and $\theta(s) = sz_2 \cdots z_n$ for all $s \in K$. The longest element of $\langle \theta(K) \rangle$ is $z_1 \cdots z_n = z'$. Now replace S by $\theta(S)$. Let $\overline{\theta(B)}$ denote $\overline{B}$ with respect to the Coxeter generators $\theta(S)$ and S'. Then $\overline{\theta(B)} \subseteq \theta(C)$, since $\langle \theta(C) \rangle = \langle C \rangle = \langle C' \rangle$. If $\overline{\theta(B)}$ is a proper subset of $\theta(C)$, we return to the start of the proof. As C is finite, we will eventually be done or have $\overline{\theta(B)} = \theta(C)$. Thus we may assume without loss of generality that $n = 1$ and $z' = z_1$.

Let ℓ' be the longest element of $\langle K' \rangle$. Define a homomorphism $\eta : \langle C' \rangle \to \langle \ell' \rangle$ as follows. Define $\eta(s') = 1$ for all $s' \in C' - K'$ and define $\eta(s') = \ell'$ for all $s' \in K'$. Then η is well defined and $\eta(\ell') = \ell'$. By the same argument as above, ℓ' is in the center of $B^\perp$ and there is a component L of $B^\perp$ such that $L \subseteq C$ and L has nontrivial center, and if ℓ is the longest element of $\langle L \rangle$, then $\eta(\ell) = \ell'$ and $\eta(t) = \ell'$ for some $t \in L$. As $z' \in B' \subseteq C' - K'$, we have that $\eta(z') = 1$, and so $L \neq K$.

Let $A = \overline{\{t\}}$. Then $\langle A \rangle$ is conjugate in $\langle C' \rangle$ to $\langle A' \rangle$ for some $A' \subseteq C'$ by Prop. 4.14 of [5]. Now $\eta(\langle A' \rangle) = \eta(\langle A \rangle) = \langle \ell' \rangle$. Hence $s' \in A'$ for some $s' \in K'$. Now killing s' in $\langle C' \rangle$ kills K'. As $[\langle K \rangle, \langle K \rangle] = [\langle K' \rangle, \langle K' \rangle]$ and $\langle A \rangle$ is conjugate to $\langle A' \rangle$ in $\langle C \rangle$, the group $\langle A \rangle$ contains an element that kills $[\langle K \rangle, \langle K \rangle]$ in $\langle C \rangle$. Therefore A contains an element $s \in K$. Now $\overline{\{s\}} \subseteq A$. As $\overline{\{s\}} = \overline{B}$, we have $B \subseteq A$. Therefore $\overline{\{t\}} = \overline{B}$. As before, $N(t) = B \cup B^\perp$.

By the same argument as above, L is of type $\mathbf{A}_1$, $\mathbf{C}_{2q+1}$, $\mathbf{D}_2(4q+2)$, $\mathbf{E}_7$, or $\mathbf{G}_3$, and if L is of type $\mathbf{A}_1$, $\mathbf{C}_{2q+1}$, or $\mathbf{D}_2(4q+2)$, we are done. Suppose L is of type $\mathbf{E}_7$ or $\mathbf{G}_3$. Then by the same argument as above, $\mathrm{Odd}(t) = L$ and $\mathrm{EOdd}(t) = B \cup B^\perp$, and we may assume that $\ell = \ell'$. By Lemma 38 of [4], there is an automorphism β of W such that $\beta(s) = s$ for all $s \in S - (K \cup L)$, and $\beta(s) = s\ell z'$ for each $s \in K \cup L$. Then $\beta(\ell) = z'$ and $\beta(z') = \ell$. As β fixes each element of $S - (K \cup L)$ and β leaves $\langle C \rangle$ invariant, we may replace S by $\beta(S)$. Then K is replaced by $\beta(L)$ and $\beta(K)$ is removed as a possibility for replacing L, since $\langle \beta(K) \rangle = \langle K' \rangle$. In the above procedure only Coxeter generators of components of $B^\perp$ of type $\mathbf{E}_7$ or $\mathbf{G}_3$ are replaced. By repeating this procedure a finite number of times, we can remove the possibility that L is of type $\mathbf{E}_7$ or $\mathbf{G}_3$, and we are done. $\square$

A *cycle* of S is a sequence $\{c_1, \ldots, c_n\}$ of distinct elements of S so that $m(c_i, c_{i+1}) < \infty$ for $i = 1, \ldots, n-1$ and $m(c_n, c_1) < \infty$. A *chord* of cycle $C = \{c_1, \ldots, c_n\}$ of S is a pair of distinct elements c_i, c_j of C such that $m(c_i, c_j) < \infty$ and c_i, c_j are neither consecutive terms of the cycle nor the end terms of the cycle.

Lemma 3.5 *Let B be a base of (W, S) that matches a base B' of (W, S') with $|\langle B\rangle| < |\langle B'\rangle|$. If B is of type $\mathbf{D}_2(2q+1)$, let $B = \{x, y\}$. If B is of type $\mathbf{B}_{2q+1}$, let $\{x, y\}$ be the set of split ends of the C-diagram of $(\langle B\rangle, B)$. Then $\{x, y\}$ is not part of a chord-free cycle of S of length at least 4.*

Proof: On the contrary, suppose $C \subseteq S$ is a chord-free cycle of length at least 4 that contains $\{x, y\}$. We may assume that $|S|$ is as small as possible. By Lemmas 3.1 and 3.3, we have that $\overline{\{x, y\}} = \overline{B}$ and $N(x) \cap N(y) = B \cup B^{\perp}$. By Lemma 3.4, there is an $r \in \overline{B} - B$ such that $N(r) = B \cup B^{\perp}$ and $\mathrm{Odd}(r) = \{r\}$. Now $C \cap (B \cup B^{\perp}) = \{x, y\}$, since C is cord-free of length at least 4. Let $a \in C - B$. Then $(B \cup B^{\perp}) - \{r\}$ is an (a, r)-separator of S, that is, every path from a to r in the P-diagram of (W, S) passes through $(B \cup B^{\perp}) - \{r\}$.

Let S_0 be a c-minimal separator (see §6 of [5]) of S such that S_0 is conjugate to a subset of $(B \cup B^{\perp}) - \{r\}$. By Lemma 4.9 of [5], we have that $S_0 = S_1 \cup S_2$ with S_1 a spherical simplex, $S_2 \subseteq S_1^{\perp}$, and $wS_2w^{-1} \subseteq S$ if and only if $w = 1$. Then $S_2 \subseteq (B \cup B^{\perp}) - \{r\}$. By Theorem 6.1 of [5], there exists $S_0' \subseteq S'$, a reduced visual graph of groups decomposition Λ for (W, S), and a reduced visual graph of groups decomposition Λ' for (W, S') such that $\langle S_0\rangle$ is conjugate to $\langle S_0'\rangle$, and the edge groups of Λ and Λ' are conjugate to $\langle S_0\rangle$, and there is a 1-1 correspondence between the vertices of Λ and the vertices of Λ' such that each vertex group of Λ is conjugate to the corresponding vertex group of Λ'.

Now r is not in an edge group of Λ, since r is not conjugate to an element of $(B \cup B^{\perp}) - \{r\}$. Let V be the vertex group of Λ that contains r. Then $N(r) \subset V$, and so $B \cup B^{\perp} \subset V$. Now $\{x, y\}$ is not contained in an edge group E of Λ, since otherwise $r \in \overline{\{x, y\}} \subset E$. We claim that $C \subset V$. On the contrary, suppose $C \not\subset V$. Let $C = \{c_1, \ldots, c_n\}$ with $x = c_1$ and $c_n = y$, and $m(c_i, c_{i+1}) < \infty$ for each $i = 1, \ldots, n-1$. Let k be the first index such that $c_k \notin V$ and let ℓ be the last index such that $c_\ell \notin V$. Then c_{k-1} is in an edge group E of Λ that is a subgroup of V and $c_{\ell+1}$ is in an edge group F of Λ that is a subgroup of V. Now $E = F$, since the graph of Λ is a tree. As

$\{x,y\} \not\subset E$, we have that $\{c_{k-1}, c_{\ell+1}\} \neq \{x,y\}$. Now E is conjugate to $\langle S_0\rangle$, and so there exists $S_3 \subseteq S_2^\perp$ such that $E = \langle S_3 \cup S_2\rangle$ and S_3 is conjugate to S_1. Hence S_3 is a spherical simplex. Now $(C - B) \cap (B \cup B^\perp) = \emptyset$, since C is chord-free. As $S_2 \subseteq B \cup B^\perp$, we deduce that $\{c_{k-1}, c_{\ell+1}\} \cap S_3 \neq \emptyset$, and so c_{k-1} and $c_{\ell+1}$ are joined by a chord, which is a contradiction. Thus $C \subset V$.

By conjugating S', we may assume that V is a vertex group V' of Λ'. Then $B' \subset V'$ be the Basic Matching Theorem. Now $\mathrm{rank}(V) < |S|$, and so we have a contradiction to the minimality of $|S|$. □

Lemma 3.6 (Blow-Down Lemma) *Let (W,S) be a Coxeter system of finite rank, and let B be a base of (W,S) of type $\mathbf{B}_{2q+1}$ or $\mathbf{D}_2(2q+1)$ for some $q \geq 1$. If $|B| = 2$, let $B = \{x,y\}$. If B is of type $\mathbf{B}_{2q+1}$, let $\{x,y\}$ be the set of split ends of the C-diagram of $(\langle B\rangle, B)$. Let $r \in B^\perp$ such that $N(r) = B \cup B^\perp$ and $\{r\}$ is a component of $B^\perp$. Suppose $N(y) = B \cup B^\perp$. Let ℓ be the longest element of $\langle B\rangle$, let $a = r\ell$, let $S' = (S - \{r,y\}) \cup \{a\}$, and let $B' = (B - \{y\}) \cup \{a\}$. Then S' is a set of Coxeter generators for W such that*

1. *the set B' is a base of (W,S') that matches B with $|\langle B\rangle| < |\langle B'\rangle|$,*
2. *$(B')^\perp = B^\perp - \{r\}$,*
3. *the neighborhood of a satisfies $N(a) = B' \cup (B')^\perp$,*
4. *the basic subsets of S and S' are the same except for B and B'.*

Proof: Consider the Coxeter presentation

$$W = \langle S \mid (st)^{m(s,t)} : s,t \in S \text{ and } m(s,t) < \infty\rangle.$$

Let ℓ be the longest element of $\langle B\rangle$. Then $\ell^2 = 1$. Regard ℓ as a reduced word in the elements of B. Add the generator a and the relation $a = r\ell$ to the above presentation of W. Then we can add the relators a^2 and $(sa)^2$ and $(as)^2$ for all $s \in B^\perp - \{r\}$. As $\ell s \ell = s$ for all $s \in B - \{x,y\}$, we can add the relators $(sa)^2$ and $(as)^2$ for all $s \in B - \{x,y\}$.

Next delete the generator r and the relation $a = r\ell$ and replace r by $a\ell$ in the remaining relators. The relator r^2 is replaced by $(a\ell)^2$. We delete the relators $(sa\ell)^2$ and $(a\ell s)^2$ for $s \in B \cup B^\perp - \{r,x,y\}$, since they are equivalent to $(a\ell)^2$.

As $\ell x\ell = y$ and $\ell a\ell = a$, we have $xa\ell xa\ell = xa\ell x\ell a\ell\ell = xaya$, and so we can replace $(xa\ell)^2$ by $xaya$. Likewise $(a\ell x)^2$ can be replaced by $ayax$, and $(ya\ell)^2$ can be replaced by $yaxa$, and $(a\ell y)^2$ can be replaced by $axay$.

Next we delete the generator y and the relators $xaya$, $ayax$, $ayax$, $axay$, and replace y by axa in the remaining relators. We have eliminated all the relators originally involving r except for $(a\ell)^2$.

We delete the relators $(saxa)^2$ and $(axas)^2$ for all $s \in B \cup B^\perp - \{r, x, y\}$, since that are equivalent to $(axa)^2 = ax^2a$. The relator y^2 is replaced by $(axa)^2$ which we can delete.

Assume first that B is of type $\mathbf{B}_{2q+1}$. The relators $(xy)^2$ and $(yx)^2$ are replaced by $(xa)^4$ and $(ax)^4$. Let $t \in B$ be such that $m(t, y) = 3$. Then $m(t, x) = 3$. Now $(ty)^3 = (taxa)^3 = (atxa)^3 = a(tx)^3a$, and so $(ty)^3$ can be deleted. Likewise $(yt)^3$ can be deleted. The relator $(a\ell)^2$ can be deleted, since it is redundant. Then we obtain a Coxeter presentation for W with Coxeter generators S' and $(\langle B'\rangle, B')$ of type $\mathbf{C}_{2q+1}$.

Now assume that B is of type $\mathbf{D}_2(2q + 1)$. The relators $(xy)^{2q+1}$ and $(yx)^{2q+1}$ are replaced by $(xa)^{4q+2}$ and $(ax)^{4q+2}$. The relator $(a\ell)^2$ can be deleted, since it is redundant. Then we obtain a Coxeter presentation for W with Coxeter generators S' and $(\langle B'\rangle, B')$ of type $\mathbf{D}_2(4q + 2)$.

As $a = r\ell$ and $\{r\}$ is a component of $B^\perp$, we have that $B^\perp - \{r\} \subseteq (B')^\perp$. Suppose $s \in (B')^\perp$. Then $s \in S - (B \cup \{r\})$. As s commutes with $a = r\ell$, we have that $s \in N(r) = B \cup B^\perp$ by Lemma 8.3 of [5]. Hence $s \in B^\perp - \{r\}$. Therefore $(B')^\perp = B^\perp - \{r\}$.

Clearly, we have $B' \cup (B')^\perp \subseteq N(a)$. Suppose $s \in N(a) - B'$. Then $s \in N(r) = B \cup B^\perp$ by Lemma 8.3 of [5]. Hence $s \in B^\perp - \{r\} = (B')^\perp$, and so $N(a) = B' \cup (B')^\perp$. Therefore B' is a base of (W, S') and B' is the only base of (W, S') that contains a. The base B' matches B, since $[\langle B'\rangle, \langle B'\rangle] = [\langle B\rangle, \langle B\rangle]$. As $N(y) = B \cup B^\perp$, we have that B is the only base of (W, S) that contains y. Therefore the basic subsets of S and S' are the same except for B and B'. $\square$

Let B and $r \in B^\perp$ be as in the Blow-Down Lemma. We call r a *sink* for the base B. Let B' and S' be as in the Blow-Down Lemma. We say that (W, S') is obtained by *blowing down* (W, S) along the base B. We also say that B' has been obtained by *blowing down* B.

Theorem 3.7 (Blow-Down Theorem) *Let (W, S) be a Coxeter system of finite rank, and let B be a base of (W, S) of type $\mathbf{B}_{2p+1}$ or $\mathbf{D}_2(2p+1)$ for some $p \geq 1$. If $|B| = 2$, let $B = \{x, y\}$. If B is of type $\mathbf{B}_{2p+1}$, let $\{x, y\}$ be the set of split ends of the C-diagram of $(\langle B\rangle, B)$. Then W has a set of Coxeter generators S' such that B matches a base B' of (W, S') with $|\langle B\rangle| < |\langle B'\rangle|$ if and only if*

1. *the neighborhoods of x and y satisfy $N(x) \cap N(y) = B \cup B^{\perp}$,*
2. *the set $\{x, y\}$ is not part of a chord-free cycle of S of length at least 4,*
3. *there exists $r \in B^{\perp}$ such that $N(r) = B \cup B^{\perp}$ and $\mathrm{Odd}(r) = \{r\}$, and if K is the component of $B^{\perp}$ containing r, then K is of type $\mathbf{A}_1$, $\mathbf{C}_{2q+1}$, or $\mathbf{D}_2(4q+2)$ for some $q \geq 1$.*

Proof: Suppose W has a set of Coxeter generators S' such that B matches a base B' of (W, S') with $|\langle B\rangle| < |\langle B'\rangle|$. Then condition (1) follows from Lemmas 3.1 and 3.3, condition (2) follows from Lemma 3.5, and condition (3) follows from Lemma 3.4.

Conversely, suppose conditions (1), (2), and (3) are satisfied. Let $S_0 = B \cup B^{\perp}$, and let $T = S - S_0$. Let T_x be the set of all $t \in T$ such that there is a sequence $t_1, \ldots, t_n$ in T such that $m(x, t_1) < \infty$, $m(t_i, t_{i+1}) < \infty$ for each $i = 1, \ldots, n-1$, and $t_n = t$. Define T_y similarly. We claim that $T_x \cap T_y = \emptyset$. On the contrary suppose that $T_x \cap T_y \neq \emptyset$. Then there is a cycle C of S such that $C \cap S_0 = \{x, y\}$. Assume that C is as short as possible. Then C is chord-free. By condition (1), we deduce that C has length at least 4, but this contradicts condition (2). Therefore $T_x \cap T_y = \emptyset$.

Let $S_1 = S - T_y$ and $S_2 = S_0 \cup T_y$. Then $S = S_1 \cup S_2$ and $S_1 \cap S_2 = S_0$, and $m(a, b) = \infty$ for all $a \in S_1 - S_0$ and $b \in S_2 - S_0$. Let ℓ be the longest element of $\langle B\rangle$. Then $\ell S_0 \ell^{-1} = S_0$ and the triple (S_1, ℓ, S_2) determines an elementary twist (see §5 of [5]) of (W, S) giving a new Coxeter generating set $S_* = S_1 \cup \ell S_2 \ell^{-1}$. As $\ell y \ell^{-1} = x$, we have $\ell T_y \ell^{-1} \subseteq T_x$ with respect to S_*, and so by replacing S by S_*, we may assume $T_y = \emptyset$. Then $N(y) = B \cup B^{\perp}$.

If $K = \{r\}$, then we can blow down B. Hence W has a set of Coxeter generators S' such that B matches a base B' of (W, S') with $|\langle B\rangle| < |\langle B'\rangle|$ by Lemma 3.6. If $K \neq \{r\}$, we can blow up S along K by Theorems 8.4 and 8.8 of [5]. This creates a sink for B, which allows us to blow down B. Therefore W has a set of Coxeter generators S' such that B matches a base B' of (W, S') with $|\langle B\rangle| < |\langle B'\rangle|$ by Lemma 3.6. $\square$

The proof of the Blown-Down Theorem indicates that we may have to blow up along one base in order to create a sink before we can blow down along another base. For example, the base $\mathbf{D}_2(3)$ of the Coxeter system $\mathbf{C}_3 \times \mathbf{D}_2(3)$ can be blown down only after the system is blown up along the base $\mathbf{C}_3$ to yield the system $\mathbf{B}_3 \times \mathbf{A}_1 \times \mathbf{D}_2(3)$. Then $\mathbf{A}_1$ is a sink for the base $\mathbf{D}_2(3)$, and so now we can blow down $\mathbf{D}_2(3)$ to obtain the system $\mathbf{B}_3 \times \mathbf{D}_2(6)$. If we blow up a Coxeter system and then blow down the resulting Coxeter system, the initial and final systems have the same rank. For example, the initial system $\mathbf{C}_3 \times \mathbf{D}_2(3)$ has the same rank as the final system $\mathbf{B}_3 \times \mathbf{D}_2(6)$.

4 Contracting Coxeter Systems

In this section, we determine necessary and sufficient conditions on (W, S) such that W has a set of Coxeter generators S' such that $|S'| < |S|$.

Theorem 4.1 (Contracting Theorem) *Let (W, S) be a Coxeter system of finite rank. Then W has a set of Coxeter generators S' such that $|S'| < |S|$ if and only if there is a base B of (W, S) of type $\mathbf{B}_{2p+1}$ or $\mathbf{D}_2(2p+1)$ for some $p \geq 1$ satisfying conditions (1), (2), (3) of the Blow-Down Theorem with $K = \{r\}$.*

Proof: Suppose there is a base B of (W, S) of type $\mathbf{B}_{2p+1}$ or $\mathbf{D}_2(2p+1)$ for some $p \geq 1$ satisfying conditions (1), (2), (3) of the Blow-Down Theorem with $K = \{r\}$. By twisting (W, S) as in the proof of the Blow-Down Theorem, leaving $B \cup B^\perp$ invariant, we may assume that $N(y) = B \cup B^\perp$. Then W has a set of Coxeter generators S' such that $|S'| = |S| - 1$ by Lemma 3.6.

Conversely, suppose W has a set of Coxeter generators S' such that $|S'| < |S|$. We may assume that S' has the maximum possible number of basic subsets that isomorphically match basic subsets of S. Now S has a basic subset B that nonisomorphically matches a basic subset B' of S' by the Simplex Matching Theorem (Theorem 7.7 of [5]). Then $|\langle B\rangle| \neq |\langle B'\rangle|$ by the Basic Matching Theorem.

Assume first that $|\langle B\rangle| < |\langle B'\rangle|$. Then B satisfies conditions (1), (2), (3) of the Blow-Down Theorem. Let r and K be as in Lemma 3.4. If $K = \{r\}$, we are done. Suppose that K is of type $\mathbf{C}_{2q+1}$ or $\mathbf{D}_2(4q+2)$ for some $q \geq 1$. Then K is a basic subset of S. Let K' be the basic subset of S' that matches K. We claim that K isomorphically matches K'. On the contrary, suppose that K nonisomorphically matches K'. By Lemma 3.4, we have that K' is a component of $(B')^\perp$ and $(K') \cup (K')^\perp = B' \cup (B')^\perp$. By Theorems 8.4-8.8 of [5], we can blow up S' along B', and after twisting as in the proof of the Blow-Down Theorem, leaving $(K') \cup (K')^\perp$ invariant, we can blow down K' to obtain a set of Coxeter generators S'' such that $|S''| = |S'|$ and S'' has two more basic subsets than S' isomorphically matching basic subsets of S, which contradicts the choice of S'. Thus K isomorphically matches K'.

Let z' be the longest element of $\langle B' \rangle$. As in the proof of Lemma 3.4, the element z' is in the center of $B^\perp$. By applying the automorphism θ of W defined by $\theta(s) = s$ for all $s \in S - \{r\}$ and $\theta(r) = rz_2 \cdots z_n$ as in the proof of Lemma 3.4, if necessary, we may assume that z' is the longest element of $\langle K \rangle$. By Theorems 8.5 and 8.7 of [5], the group W has a set of Coxeter generators S'' such that K matches a base K'' of (W, S'') with $|\langle K \rangle| > |\langle K'' \rangle|$. Therefore W has a set of Coxeter generators S'' such that K' matches a base K'' of (W, S'') with $|\langle K' \rangle| > |\langle K'' \rangle|$. Let ℓ' be the longest element of $\langle K' \rangle$. If K' is of type $\mathbf{C}_{2q+1}$, let $a' \in K'$ be such that $K' \cap \mathrm{Odd}(a') = \{a'\}$. If K' is of type $\mathbf{D}_2(4q+2)$, let $a' \in K'$ be as in Lemma 8.6 of [5]. As in the proof of Lemma 3.4, define a homomorphism $\eta : \langle C' \rangle \to \langle \ell' \rangle$ as follows. Define $\eta(s') = 1$ for all $s' \in C' - \{a'\}$ and define $\eta(a') = \ell'$. Then η is well defined and $\eta(\ell') = \ell'$. By the argument in the proof of Lemma 3.4, the element ℓ' is in the center of $B^\perp$ and there is a component L of $B^\perp$ such that $L \subseteq C$ and L has nontrivial center, and if ℓ is the longest element of $\langle L \rangle$, then $\eta(\ell) = \ell'$ and $\eta(t) = \ell'$ for some $t \in L$. Moreover L is of type $\mathbf{A}_1$, $\mathbf{C}_{2q+1}$, $\mathbf{D}_2(4q+2)$, $\mathbf{E}_7$, or $\mathbf{G}_3$ for some $q \geq 1$, and if L is of type $\mathbf{C}_{2q+1}$, then $L \cap \mathrm{Odd}(t) = \{t\}$. As $\eta(z') = 1$, we have that $L \neq K$.

Let $A = \overline{\{t\}}$. As $t \in \overline{B}$, we have that $\overline{\{t\}} \subseteq \overline{B}$. Then $\langle A \rangle$ is conjugate in $\langle C' \rangle$ to $\langle A' \rangle$ for some $A' \subseteq C'$ by Prop. 4.14 of [5]. Now $\eta(\langle A' \rangle) = \eta(\langle A \rangle) = \langle \ell' \rangle$. Hence $a' \in A'$. Therefore $B' \subseteq A'$ by Lemmas 8.1 and 8.6 of [5]. Hence $B \subseteq A$ by the Basic Matching Theorem, and so $\overline{B} \subseteq A$. Therefore $\overline{\{t\}} = \overline{B}$. As before, $N(t) = B \cup B^\perp$, and so $\mathrm{Odd}(t) \subseteq L$. If $L = \{t\}$, we are done, and so we may assume that L is not of type $\mathbf{A}_1$. As before, by applying an automorphism, we may assume that $\ell = \ell'$.

By Lemma 38 of [4], there is an automorphism β of W such that $\beta(s) = s$ for all $s \in S - (\{r\} \cup \mathrm{Odd}(t))$, and $\beta(r) = r\ell z'$, and $\beta(s) = s\ell z'$ for all $s \in \mathrm{Odd}(t)$. Then $\beta(\ell) = z'$ and $\beta(z') = \ell$. As β fixes each element of $S - (K \cup L)$ and β leaves $\langle C \rangle$ invariant, we may replace S by $\beta(S)$. Then K is replaced by $\beta(L)$, and $\beta(K)$ is removed as a possibility for replacing L, since $\beta(K)$ matches K' and $\langle \beta(K) \rangle$ and $\langle K' \rangle$ have the same longest element. By the argument in Lemma 3.4, we may assume that $\beta(L)$ is of type $\mathbf{C}_{2q+1}$

or $\mathbf{D}_2(4q+2)$.

In the above procedure only Coxeter generators of components of $B^\perp$ of type $\mathbf{C}_{2q+1}$, $\mathbf{D}_2(4q+2)$, $\mathbf{E}_7$, or $\mathbf{G}_3$ are replaced. By repeating this procedure a finite number of times, we can remove that possibility that L is of type $\mathbf{C}_{2q+1}$, $\mathbf{D}_2(4q+2)$, $\mathbf{E}_7$, or $\mathbf{G}_3$. Then $L = \{t\}$, and we are done.

Assume now that $|\langle B\rangle| > |\langle B'\rangle|$. Then B' satisfies conditions (1), (2), (3) of the Blown-Down Theorem. Let r' and K' be as in Lemma 3.4. If $K' = \{r'\}$, we can blow down B' to obtain a set of Coxeter generators S'' such that $|S''| = |S'| - 1$ and S'' has one more basic subset than S' isomorphically matching basic subsets of S, which contradicts the choice of S'. Therefore K' is of type $\mathbf{C}_{2q+1}$ or $\mathbf{D}_2(4q+2)$ for some $q \geq 1$. As $N(r') = B' \cup (B')^\perp$, we have that K' is a basic subset of S' and $K' \cup (K')^\perp = B' \cup (B')^\perp$.

Let K be the basic subset of S that matches K'. Then K isomorphically matches K', since otherwise by Theorems 8.4-8.8 of [5], we can blow up S' along K' and then blow down B' to obtain a set of Coxeter generators S'' such that $|S''| = |S'|$ and S'' has two more basic subsets than S' isomorphically matching basic subsets of S, which contradicts the choice of S'.

Let z be the longest element of $\langle B\rangle$. As in the proof of Lemma 3.4, the element z is in the center of $(B')^\perp$, and by applying an automorphism, we may assume that z is the longest element of $\langle K'\rangle$. Let ℓ be the longest element of $\langle K\rangle$. If K is of type $\mathbf{C}_{2q+1}$, let $a \in K$ be such that $K \cap \mathrm{Odd}(a) = \{a\}$. If K is of type $\mathbf{D}_2(4q+2)$, let $a \in K$ be as in Lemma 8.6 of [5]. As in the proof of Lemma 3.4, define a homomorphism $\eta : \langle C\rangle \to \langle \ell\rangle$ as follows. Define $\eta(s) = 1$ for all $s \in C - \{a\}$ and define $\eta(a) = \ell$. Then η is well defined and $\eta(\ell) = \ell$. By the argument in the proof of Lemma 3.4, the element ℓ is in the center of $(B')^\perp$ and there is a component L' of $(B')^\perp$ such that $L' \subseteq C'$ and L' has nontrivial center, and if ℓ' is the longest element of $\langle L'\rangle$, then $\eta(\ell') = \ell$ and $\eta(t') = \ell$ for some $t' \in L'$. Moreover L' is of type $\mathbf{A}_1$, $\mathbf{C}_{2q+1}$, $\mathbf{D}_2(4q+2)$, $\mathbf{E}_7$, or $\mathbf{G}_3$ for some $q \geq 1$, and if L' is of type $\mathbf{C}_{2q+1}$, then $L' \cap \mathrm{Odd}(t') = \{t'\}$. As $\eta(z) = 1$, we have that $L' \neq K'$.

Let $A' = \overline{\{t'\}}$. As $t' \in \overline{B'}$, we have that $\overline{\{t'\}} \subseteq \overline{B'}$. Then $\langle A'\rangle$ is conjugate in $\langle C\rangle$ to $\langle A\rangle$ for some $A \subseteq C$ by Prop. 4.14 of [5]. Now $\eta(\langle A\rangle) = \eta(\langle A'\rangle) = \langle \ell\rangle$. Hence $a \in A$. Therefore $B \subseteq A$ by Lemmas 8.1 and 8.6 of [5]. Hence $B' \subseteq A'$ by the Basic Matching Theorem, and so $\overline{B'} \subseteq A'$. Therefore $\overline{\{t'\}} = \overline{B'}$. As before, $N(t') = B' \cup (B')^\perp$, and so $\mathrm{Odd}(t') \subseteq L'$. If $L' = \{t'\}$, we derive a contradiction as before. Therefore L' is not of type $\mathbf{A}_1$. By applying an automorphism, we may assume that $\ell = \ell'$.

By Lemma 38 of [4], there is an automorphism β of W such that $\beta(s') = s'$ for all $s' \in S' - (\{r'\} \cup \mathrm{Odd}(t'))$, and $\beta(r') = r'\ell z$, and $\beta(s') = s'\ell z$ for all $s' \in \mathrm{Odd}(t')$. Then $\beta(\ell) = z$ and $\beta(z) = \ell$. As β fixes each element of $S' - (K' \cup L')$ and β leaves $\langle C' \rangle$ invariant, we may replace S' by $\beta(S')$. Then K' is replaced by $\beta(L')$, and $\beta(K')$ is removed as a possibility for replacing L', since $\beta(K')$ matches K and $\langle \beta(K') \rangle$ and $\langle K \rangle$ have the same longest element. By the argument in Lemma 3.4, we may assume that $\beta(L')$ is of type $\mathbf{C}_{2q+1}$ or $\mathbf{D}_2(4q+2)$.

In the above procedure only Coxeter generators of components of $(B')^{\perp}$ of type $\mathbf{C}_{2q+1}$, $\mathbf{D}_2(4q+2)$, $\mathbf{E}_7$, or $\mathbf{G}_3$ are replaced. By repeating this procedure a finite number of times, we can remove the possibility that L' is of type $\mathbf{C}_{2q+1}$, $\mathbf{D}_2(4q+2)$, $\mathbf{E}_7$, or $\mathbf{G}_3$. Then L' is of type $\mathbf{A}_1$ and we have a contradiction as before. Thus the case $|\langle B \rangle| > |\langle B' \rangle|$ leads to a contradiction. $\square$

5 The Rank Spectrum of a Coxeter Group

In this section, we describe how to determine the set of all possible ranks of an arbitrary finitely generated Coxeter group W by inspection of any presentation diagram for W.

Let (W, S) be a Coxeter system of finite rank. Suppose that $S_1, S_2 \subseteq S$, with $S = S_1 \cup S_2$ and $S_0 = S_1 \cap S_2$, are such that $m(a, b) = \infty$ for all $a \in S_1 - S_0$ and $b \in S_2 - S_0$. Let $\ell \in \langle S_0 \rangle$ such that $\ell S_0 \ell^{-1} = S_0$. The triple (S_1, ℓ, S_2) determines an elementary twist of (W, S) giving a new set of Coxeter generators $S_* = S_1 \cup \ell S_2 \ell^{-1}$ for W such that $S_1 \cap \ell S_2 \ell^{-1} = S_0$.

Let B be a base of (W, S) of type $\mathbf{B}_{2p+1}$ or $\mathbf{D}_2(2p+1)$ for some $p \geq 1$ that satisfies the conditions (1), (2), (3) of the Blown-Down Theorem with $K = \{r\}$. As B is a simplex, either $B \subseteq S_1$ or $B \subseteq S_2$. If $B \subseteq S_1$, define $B_* = B$. If $B \subseteq S_2$, define $B_* = \ell B \ell^{-1}$. If $B \subseteq S_0$, then $\ell B \ell^{-1} = B$ by Lemma 4.10 of [5], and so B_* is well defined. If $r \in S_1$, define $r_* = r$. If $r \in S_2$, define $r_* = \ell r \ell^{-1}$. If $r \in S_0$, then $\ell r \ell^{-1} = r$ by Lemma 4.8 of [5], and so r_* is well defined.

Lemma 5.1 *The set B_* is a base of (W, S_*) of type $\mathbf{B}_{2p+1}$ or $\mathbf{D}_2(2p+1)$ for some $p \geq 1$ that satisfies the conditions (1), (2), (3) of the Blown-Down Theorem with $\{r_*\}$ a component of $B_*^{\perp}$.*

Proof: As B_* is conjugate to B, we deduce that B_* is a base of (W, S_*) of type $\mathbf{B}_{2p+1}$ or $\mathbf{D}_2(2p+1)$ for some $p \geq 1$ by the Basic Matching Theorem. By the Blow-Down Theorem, W has a set of Coxeter generators S' such that B matches a base B' of (W, S') with $|\langle B\rangle| < |\langle B'\rangle|$. Hence B_* matches B' with $|\langle B_*\rangle| < |\langle B'\rangle|$. Therefore B_* satisfies conditions (1) and (2) of the Blow-Down Theorem.

As $B \cup \{r\}$ is a simplex, either $B \cup \{r\} \subseteq S_1$ or $B \cup \{r\} \subseteq S_2$. Hence $r_* \in (B_*)^\perp$. Let $s_* \in N(r_*) - (B_* \cup \{r_*\})$. Assume first that $B \cup \{r\} \subseteq S_1$. Then $B_* = B$ and $r_* = r$. Suppose $s_* \in S_1$. Then $s_* \in N(r) = B \cup B^\perp$. Hence $s_* \in B^\perp$, and so $s_* \in (B_*)^\perp$ and $m(r_*, s_*) = 2$. Now suppose that $s_* \in \ell S_2 \ell^{-1} - S_0$. Then $r \in S_0$ and $r = \ell r \ell^{-1}$ and $s_* = \ell s \ell^{-1}$ for some $s \in S_2 - S_0$. Hence $s \in N(r) = B \cup B^\perp$. As $B \subset S_1$, we have that $s \in B^\perp$. Therefore $m(r, s) = 2$, and so $m(r_*, s_*) = 2$. Moreover, $B \subseteq S_0$, since $s \in S_2 - S_0$. Hence $B_* = \ell B \ell^{-1}$, and so $s_* \in (B_*)^\perp$.

Assume now that $B \cup \{r\} \subseteq S_2$. Then $B_* = \ell B \ell^{-1}$ and $r_* = \ell r \ell^{-1}$. Suppose $s_* \in S_1 - S_0$. Then $r_* \in S_0$, and so $r_* = r$. Hence $s_* \in N(r) = B \cup B^\perp$. Now $s_* \in B^\perp$, since $s_* \in S_1 - S_0$. Hence $m(r_*, s_*) = 2$. Moreover $B \subseteq S_0$, and so $B_* = B$. Hence $s_* \in (B_*)^\perp$. Now suppose $s_* \in \ell S_2 \ell^{-1}$. Then $s_* = \ell s \ell^{-1}$ for some $s \in S_2$. As $1 < m(r_*, s_*) < \infty$, we have that $1 < m(r, s) < \infty$. Hence $s \in N(r) = B \cup B^\perp$. Moreover $s \in B^\perp$, since $s_* \notin B_*$. Hence $m(r, s) = 2$, and so $m(r_*, s_*) = 2$. Moreover $s_* \in (B_*)^\perp$. Thus, in all cases, $N(r_*) = B_* \cup (B_*)^\perp$ and $\{r_*\}$ is a component of $(B_*)^\perp$. Therefore B_* satisfies condition (3) of the Blow-Down Theorem with $K = \{r_*\}$. $\square$

Let B be a base of (W, S) of type $\mathbf{B}_{2p+1}$ or $\mathbf{D}_2(2p+1)$ for some $p \geq 1$ satisfying conditions (1), (2), (3) of The Blow-Down Theorem with $K = \{r\}$. We call r a *sink* for B. A element r of S may be a sink for more than one base of (W, S). For example, $\mathbf{A}_1$ is a sink for the two bases of $\mathbf{A}_1 \times \mathbf{D}_2(3) \times \mathbf{D}_2(3)$. A base may have more than one sink. For example, the base of $\mathbf{A}_1 \times \mathbf{A}_1 \times \mathbf{D}_2(3)$ has two sinks.

Lemma 5.2 *Let B be a base of (W, S) of type $\mathbf{B}_{2p+1}$ or $\mathbf{D}_2(2p+1)$ for some $p \geq 1$ that satisfies the hypothesis of the Blow-Down Lemma with sink r. Let S' be the set of Coxeter generators obtained by blowing down S along B. Let C be a base of (W, S) of type $\mathbf{B}_{2q+1}$ or $\mathbf{D}_2(2q+1)$ for some $q \geq 1$ that satisfies the conditions (1), (2), (3) of the Blow-Down Theorem with sink s. If $B \neq C$ and $r \neq s$, then C is a base of (W, S') that satisfies the conditions (1), (2), (3) of the Blow-Down Theorem with sink s.*

Proof: Let B' be the base of (W, S') obtained by blowing down B. Then S and S' have the same basic subsets except for B and B'. Therefore C is a base of (W, S'). By the Blow-Down Theorem, W has a set of Coxeter generators S'' such that C matches a base C'' of (W, S'') with $|\langle C\rangle| < |\langle C''\rangle|$. Therefore C satisfies the conditions (1) and (2) of the Blow-Down Theorem. As $\mathrm{Odd}(s) = \{s\}$, we have that $s \notin B$. Therefore $s \in S'$.

Let a be the element of S' that is not in S. Then $a = r\ell$ with ℓ the longest element of $\langle B\rangle$. If $a \notin N(s)$, then $N(s) = C \cup C^{\perp}$ and $\{s\}$ is a component of $C^{\perp}$ with respect to S', since s is a sink for C with respect to S. Suppose $a \in N(s)$. Then $s \in N(r)$ by Lemma 8.3 of [5]. Hence $s \in B^{\perp}$. Therefore $m(a, s) = 2$. As $B \cup \{r\} \subseteq N(s)$, we have that $B \cup \{r\} \subseteq C \cup C^{\perp}$. As $B \neq C$, we have that $B \subseteq C^{\perp}$. As $\mathrm{Odd}(r) = \{r\}$, we have that $r \notin C$, and so $r \in C^{\perp}$. Therefore $a \in C^{\perp}$. Hence $N(s) = C \cup C^{\perp}$ and $\{s\}$ is a component of $C^{\perp}$ with respect to S'. Thus C satisfies condition (3) of the Blow-Down Theorem, with sink s, with respect to S'. □

Lemma 5.3 *Let B be a base of (W, S) of type $\mathbf{C}_{2p+1}$ or $\mathbf{D}_2(4p+2)$ for some $p \geq 1$ along which (W, S) can be blown up. Let S' be the set of Coxeter generators obtained by blowing up S along B. Let C be a base of (W, S) of type $\mathbf{B}_{2q+1}$ or $\mathbf{D}_2(2q+1)$ for some $q \geq 1$ that satisfies the conditions (1), (2), (3) of the Blow-Down Theorem with sink s. If $B \neq C$, then C is a base of (W, S') that satisfies the conditions (1), (2), (3) of the Blow-Down Theorem with sink s.*

Proof: Let B' be the base of (W, S') obtained by blowing up B. Then S and S' have the same basic subsets except for B and B'. Therefore C is a base of (W, S'). By the Blow-Down Theorem, W has a set of Coxeter generators S'' such that C matches a base C'' of (W, S'') with $|\langle C\rangle| < |\langle C''\rangle|$. Therefore C satisfies the conditions (1) and (2) of the Blow-Down Theorem. As $\mathrm{Odd}(s) = \{s\}$, we have that $s \notin B$. Therefore $s \in S'$.

Let z be the longest element of $\langle B\rangle$. If B is of type $\mathbf{C}_{2p+1}$, let a, b, c be the elements of B such that $m(a, b) = 4$ and $m(b, c) = 3$. Then $N(a) = B \cup B^{\perp}$ by Theorem 8.2 of [5]. If B is of type $\mathbf{D}_2(4p + 2)$, let $B = \{a, b\}$ with $N(a) = B \cup B^{\perp}$. In either case, let $d = aba$. Then d and z are the elements of S' that are not in S by Theorems 8.4 and 8.8 of [5].

If d and z are not elements of $N(s)$, then $N(s) = C \cup C^{\perp}$ and $\{s\}$ is a component of $C^{\perp}$ with respect to S', since s is a sink for C with respect to

S. Suppose d or z is an element of $N(s)$. Then $s \in N(a) = B \cup B^\perp$ by Lemma 8.3 of [5]. Hence $s \in B^\perp$. Therefore $m(d,s) = 2$ and $m(z,s) = 2$. As $B \subseteq N(s)$, we have that $B \subseteq C \cup C^\perp$. As $B \neq C$, we have that $B \subseteq C^\perp$. Therefore $\{d, z\} \subseteq C^\perp$. Hence $N(s) = C \cup C^\perp$ and $\{s\}$ is a component of $C^\perp$ with respect to S'. Thus C satisfies condition (3) of the Blow-Down Theorem, with sink s, with respect to S'. $\square$

Theorem 5.4 (Rank Spectrum Theorem) *Let $\{B_1, \ldots, B_k\}$, $k \geq 0$, be a maximal set of bases of (W,S) of type $\mathbf{B}_{2p+1}$ or $\mathbf{D}_2(2p+1)$ for some $p \geq 1$ that satisfy the conditions of the Blown-Down Theorem with distinct sinks $\{s_1, \ldots, s_k\}$. Let $C_1, \ldots, C_\ell$, $\ell \geq 0$, be the bases of (W,S) of type $\mathbf{C}_{2q+1}$ or $\mathbf{D}_2(4q+2)$ for some $q \geq 1$ along which (W,S) can be blown up. Then the set of all possible ranks of W is $|S| - k, \ldots, |S| + \ell$.*

Proof: By Lemmas 5.1 and 5.2, we get a sequence of Coxeter generators $S_0, \ldots, S_k$ for W such that $S = S_0$ and S_i is obtained from S_{i-1} by twisting S_i, as in Theorem 3.7, and then blowing down along a base conjugate to B_i for each $i = 1, \ldots, k$. Then S_k has minimum rank over all sets of Coxeter generators of W by Theorem 4.1 and Lemma 5.3. In particular, the mininum rank of W is $|S| - k$.

Let $a_i \in C_i$ be the element of C_i, for $i = 1, \ldots, \ell$ that is removed in the blowing up process. As $N(a_i) = C_i \cup C_i^\perp$ for each i, we have that C_i is the only base of (W,S) that contains a_i for each i. Then we have a sequence of Coxeter generators $S^{(0)}, \ldots, S^{(\ell)}$ for W such that $S = S^{(0)}$ and $S^{(i)}$ is obtained from $S^{(i-1)}$ by blowing up $S^{(i)}$ along C_i for each $i = 1, \ldots, \ell$. Then $S^{(\ell)}$ has maximum rank over all sets of Coxeter generators of W by Theorem 9.1 of [5]. In particular, the maximum rank of W is $|S| + \ell$. Thus the set of all possible ranks of W is $|S| - k, \ldots, |S| + \ell$. $\square$

The numbers k and ℓ in the Rank Spectrum Theorem can be determined by inspecting the presentation diagram of (W,S). For example, $k = \ell = 1$ for the system $\mathbf{A}_1 \times \mathbf{D}_2(3) \times \mathbf{D}_2(6)$.

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