Matching Theorems for Systems of a Finitely Generated Coxeter Group

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1 Introduction

The isomorphism problem for finitely generated Coxeter groups is the problem of deciding if two finite Coxeter matrices define isomorphic Coxeter groups. Coxeter [3] solved this problem for finite irreducible Coxeter groups. Recently there has been considerable interest and activity on the isomorphism problem for arbitrary finitely generated Coxeter groups. For a recent survey, see Mühlherr [10].

The isomorphism problem for finitely generated Coxeter groups is equivalent to the problem of determining all the automorphism equivalence classes of sets of Coxeter generators for an arbitrary finitely generated Coxeter group. In this paper, we prove a series of matching theorems for two sets of Coxeter generators of a finitely generated Coxeter group that identify common features of the two sets of generators. As an application, we describe an algorithm for finding a set of Coxeter generators of maximum rank for a finitely generated Coxeter group.

In §2, we state some preliminary results. In §3, we prove a matching theorem for two systems of a finite Coxeter group. In §4, we prove our Basic Matching Theorem between the sets of maximal noncyclic irreducible spherical subgroups of two systems of a finitely generated Coxeter group. In §5, we study nonisomorphic basic matching. In §6, we prove a matching theorem between the sets of noncyclic irreducible spherical subgroups of two systems of a finitely generated Coxeter group. As an application, we prove the Edge Matching Theorem. In §7, we discuss twisting and flattening visual graph of groups decompositions of Coxeter systems. In §8, we prove the

Decomposition Matching Theorem. In §9, we prove the Simplex Matching Theorem. In §10, we describe our algorithm for finding a set of Coxeter generators of maximum rank for a finitely generated Coxeter group.

2 Preliminaries

A *Coxeter matrix* is a symmetric matrix $M = (m(s,t))_{s,t\in S}$ with $m(s,t)$ either a positive integer or infinity and $m(s,t) = 1$ if and only if $s = t$. A *Coxeter system* with Coxeter matrix $M = (m(s,t))_{s,t\in S}$ is a pair (W,S) consisting of a group W and a set of generators S for W such that W has the presentation

$$W = \langle S \mid (st)^{m(s,t)} : \ s,t \in S \text{ and } m(s,t) < \infty \rangle$$

If (W,S) is a Coxeter system with Coxeter matrix $M = (m(s,t))_{s,t\in S}$, then the order of st is $m(s,t)$ for each s,t in S by Prop. 4, p. 92 of Bourbaki [1], and so a Coxeter system (W,S) determines its Coxeter matrix; moreover, any Coxeter matrix $M = (m(s,t))_{s,t\in S}$ determines a Coxeter system (W,S) where W is defined by the above presentation. If (W,S) is a Coxeter system, then W is called a *Coxeter group* and S is called a set of *Coxeter generators* for W, and the cardinality of S is called the *rank* of (W,S).

Proposition 2.1 *A Coxeter system (W,S) has finite rank if and only if W is finitely generated.*

Proof: This follows Theorem 2 (iii), p. 20 of Bourbaki [1]. □

Let (W,S) be a Coxeter system. A *visual subgroup* of (W,S) is a subgroup of W of the form $\langle A\rangle$ for some $A \subset S$. If $\langle A\rangle$ is a visual subgroup of (W,S), then $(\langle A\rangle, A)$ is also a Coxeter system by Theorem 2 (i), p. 20 of Bourbaki [1].

When studying a Coxeter system (W,S) with Coxeter matrix M it is helpful to have a visual representation of (W,S). There are two graphical ways of representing (W,S) and we shall use both depending on our needs.

The *Coxeter diagram* (C-*diagram*) of (W,S) is the labeled undirected graph $\Delta = \Delta(W,S)$ with vertices S and edges

$$\{(s,t) : s,t \in S \text{ and } m(s,t) > 2\}$$

such that an edge (s,t) is labeled by $m(s,t)$. Coxeter diagrams are useful for visually representing finite Coxeter groups. If $A \subset S$, then $\Delta(\langle A\rangle, A)$ is the subdiagram of $\Delta(W,S)$ induced by A.

A Coxeter system (W,S) is said to be *irreducible* if its C-diagram Δ is connected. A visual subgroup $\langle A\rangle$ of (W,S) is said to be *irreducible* if $(\langle A\rangle, A)$ is irreducible. A subset A of S is said to be *irreducible* if $\langle A\rangle$ is irreducible.

A subset A of S is said to be a *component* of S if A is a maximal irreducible subset of S or equivalently if $\Delta(\langle A\rangle, A)$ is a connected component of $\Delta(W,S)$. The connected components of the $\Delta(W,S)$ represent the factors of a direct product decomposition of W.

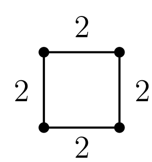
The *presentation diagram* (P-*diagram*) of (W,S) is the labeled undirected graph $\Gamma = \Gamma(W,S)$ with vertices S and edges

$$\{(s,t) : s,t \in S \text{ and } m(s,t) < \infty\}$$

such that an edge (s,t) is labeled by $m(s,t)$. Presentation diagrams are useful for visually representing infinite Coxeter groups. If $A \subset S$, then $\Gamma(\langle A\rangle, A)$ is the subdiagram of $\Gamma(W,S)$ induced by A. The connected components of $\Gamma(W,S)$ represent the factors of a free product decomposition of W.

Example Consider the Coxeter group W generated by the four reflections in the sides of a rectangle in E^2. The C-diagram of (W,S) is the disjoint union of two edges labeled by ∞.

Therefore W is the direct product of two infinite dihedral groups. The P-diagram of W is a square with edge labels 2.



Let (W,S) and (W',S') be Coxeter systems with P-diagrams Γ and Γ', respectively. An *isomorphism* $\phi : (W,S) \to (W',S')$ of Coxeter systems is an isomorphism $\phi : W \to W'$ such that $\phi(S) = S'$. An *isomorphism* $\psi : \Gamma \to \Gamma'$ of P-diagrams is a bijection from S to S' that preserves edges and their labels.

Proposition 2.2 *Let* (W,S) *and* (W',S') *be Coxeter systems with P-diagrams* Γ *and* Γ'*, respectively. Then*

1. $(W,S) \cong (W',S')$ *if and only if* $\Gamma \cong \Gamma'$,
2. $W \cong W'$ *if and only if* W *has a set of Coxeter generators* S'' *such that* $(W,S'') \cong (W',S')$,
3. $W \cong W'$ *if and only if* W *has a P-diagram* Γ'' *such that* $\Gamma'' \cong \Gamma'$.

Proof: (1) If $\phi : (W,S) \to (W',S')$ is an isomorphism, then ϕ restricts to an isomorphism $\overline{\phi} : \Gamma \to \Gamma'$ and if $\psi : \Gamma \to \Gamma'$ is an isomorphism, then ψ extends to a unique isomorphism $\hat{\psi} : (W,S) \to (W',S')$.

(2) If $\phi : W \to W'$ is an isomorphism, then $S'' = \phi^{-1}(S')$ is a set of Coxeter generators for W and $\phi : (W,S'') \to (W',S')$ is an isomorphism.

(3) Statement (3) follows from (1) and (2). □

Proposition 2.3 *Let* W *be a Coxeter group and let* $\mathcal{S}$ *be the collection of sets of Coxeter generators for* W*. Then*

1. *The group* $\mathrm{Aut}(W)$ *acts on* $\mathcal{S}$.
2. *Sets of Coxeter generators* S *and* S' *for* W *are in the same* $\mathrm{Aut}(W)$*-orbit if and only if* $(W,S) \cong (W,S')$.
3. *The set of* $\mathrm{Aut}(W)$*-orbits* $\mathcal{S}/\mathrm{Aut}(W)$ *is in one-to-one correspondence with the set of isomorphism classes of P-diagrams for* W.

Proof: (1) and (2) are obvious. (3) follows from (2) and Prop. 2.2(1). □

A Coxeter group W is said to be *rigid* if for any two sets of Coxeter generators S and S' for W, there is an automorphism $\alpha : (W,S) \to (W,S')$ or equivalently any two sets of Coxeter generators S and S' for W determine isomorphic P-diagrams for W. A Coxeter group W is said to be *strongly rigid* if any two sets of Coxeter generators for W are conjugate.

A Coxeter system (W,S) is said to be *complete* if the underlying graph of the P-diagram of (W,S) is complete. A Coxeter system (W,S) is said to be *finite* (resp. *infinite*) if W is finite (resp. infinite).

Theorem 2.4 (Caprace, Franzsen, Haglund, Howlett, and Mühlherr [2],[7]) *If* (W,S) *is an infinite, complete, irreducible Coxeter system of finite rank, then* W *is strongly rigid.*

3 Coxeter Systems of Finite Coxeter Groups

We shall use Coxeter's notation on p. 297 of [4] for the irreducible spherical Coxeter simplex reflection groups except that we denote the dihedral group $\mathbf{D}_2^k$ by $\mathbf{D}_2(k)$. Subscripts denote the rank of a Coxeter system in Coxeter's notation. Coxeter's notation partly agrees with but differs from Bourbaki's notation on p.193 of [1].

Coxeter [3] proved that every finite irreducible Coxeter system is isomorphic to exactly one of the Coxeter systems $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_2(k)$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_3$, $\mathbf{G}_4$ described below. Each of these Coxeter groups, of rank n, is a finite group of orthogonal $n \times n$ matrices. The center of each of these Coxeter groups is either $\{I\}$ or $\{\pm I\}$. We denote the center of a group G by $Z(G)$. If G is a group of orthogonal matrices, we denote the subgroup of determinant 1 matrices in G by G^+.

The *type* of a finite irreducible Coxeter system (W, S) is the isomorphism type of (W, S) represented by one of the systems $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_2(k)$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_3$, $\mathbf{G}_4$. The *type* of an irreducible subset A of S is the type of $(\langle A \rangle, A)$.

The Coxeter group $\mathbf{A}_n$ is the group of symmetries of a regular n-simplex for each $n \geq 1$, and so $\mathbf{A}_n$ is isomorphic to the symmetric group S_{n+1} for each $n \geq 1$. The C-diagram of $\mathbf{A}_n$ is the following linear diagram with n vertices and all edge labels 3:

The Coxeter generators $a_1, \ldots, a_n$ of $\mathbf{A}_n$, indexed so that $m(a_i, a_{i+1}) = 3$ for $i = 1, \ldots, n$, correspond to the transpositions $(12), (23), \ldots, (n\, n{+}1)$ of S_{n+1}. The group $\mathbf{A}_n$ has order $(n+1)!$ for all $n \geq 1$. The center of $\mathbf{A}_n$ is trivial for all $n \geq 2$.

The Coxeter group $\mathbf{C}_n$ is the group of symmetries of an n-cube for each $n \geq 2$, and $\mathbf{C}_n$ is represented by the group of all $n \times n$ orthogonal matrices in which each column has all zero entries except for one, which is ± 1. Thus we have a split short exact sequence

$$1 \longrightarrow D_n \longrightarrow \mathbf{C}_n \xrightarrow{\pi} S_n \longrightarrow 1$$

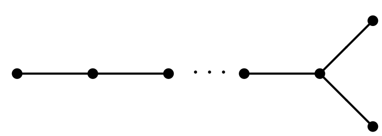
where $D_n = \{\mathrm{diag}(\pm 1, \pm 1, \ldots, \pm 1)\}$ and π maps a permutation matrix to the corresponding permutation. The C-diagram of $\mathbf{C}_n$ is the following linear diagram with n vertices:

The Coxeter generators $c_1, \dots, c_n$ of $\mathbf{C}_n$ are indexed so that $m(c_i, c_{i+1}) = 3$ for $i = 1, \dots, n-2$ and $m(c_{n-1}, c_n) = 4$. The generators $c_1, \dots, c_{n-1}$ are represented by the permutation matrices corresponding to the transpositions $(12), (23), \dots, (n{-}1\, n)$ and c_n is represented by the matrix $\mathrm{diag}(1, \dots, 1, -1)$. The order of the group $\mathbf{C}_n$ is $2^n n!$ and $Z(\mathbf{C}_n) = \{\pm I\}$.

The Coxeter group $\mathbf{B}_n$, with $n \geq 4$, is a subgroup of $\mathbf{C}_n$ of index 2 with Coxeter generators $b_i = c_i$, for $i = 1, \dots, n-1$, and $b_n = c_n c_{n-1} c_n$. We have $b_{n-1} b_n = \mathrm{diag}(1, \dots, 1, -1, -1)$ and $m(b_{n-2}, b_n) = 3$. The group $\mathbf{B}_n$ contains D_n^+ and the group of permutation matrices, and so we have a split short exact sequence

$$1 \longrightarrow D_n^+ \longrightarrow \mathbf{B}_n \xrightarrow{\pi} S_n \longrightarrow 1.$$

The C-diagram of $\mathbf{B}_n$ is the following Y-shaped diagram with n vertices and all edge labels 3:

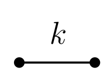


In order to have uniformity of notation, we extend the above definition of $\mathbf{B}_n$ to include the rank $n = 3$. The group $\mathbf{B}_3$ is of type $\mathbf{A}_3$ and represents the degenerate case when there are no horizontal edges in the above diagram. The order of the group $\mathbf{B}_n$ is $2^{n-1} n!$ for each $n \geq 3$. The center of the group $\mathbf{B}_n$ is trivial if n is odd and is $\{\pm I\}$ if n is even.

If $n \neq 4$, we call the two right most vertices, b_{n-1} and b_n, of the above C-diagram of $\mathbf{B}_n$ the *split ends* of the diagram. We call any two endpoints of the C-diagram of $\mathbf{B}_4$ a pair of *split ends* of the diagram.

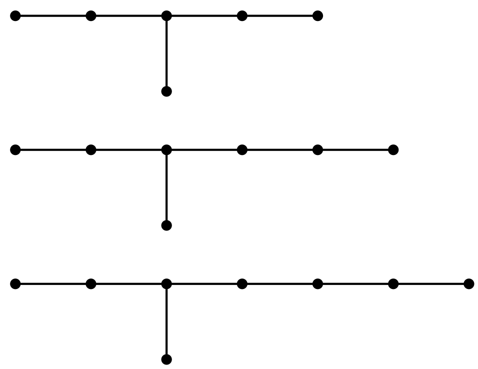
The group $\mathbf{D}_2(k)$ is the group of symmetries of a regular k-gon for each $k \geq 5$. In order to have uniformity of notation, we extend the definition of $\mathbf{D}_2(k)$ to include the cases $k = 3, 4$, and so $\mathbf{D}_2(k)$ is a dihedral group of order $2k$ for each $k \geq 3$. Note that $\mathbf{D}_2(3)$ is of type $\mathbf{A}_2$ and $\mathbf{D}_2(4)$ is of type $\mathbf{C}_2$.

The C-diagram of $\mathbf{D}_2(k)$ is an edge with label k:

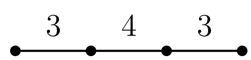


Let a and b be Coxeter generators for $\mathbf{D}_2(k)$. The center of $\mathbf{D}_2(k)$ is trivial if k is odd and is generated by $(ab)^{k/2}$ if k is even.

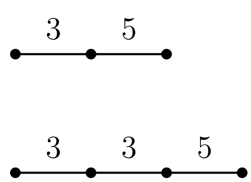
The orders of $\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$ are $72 \cdot 6!, 8 \cdot 9!, 192 \cdot 10!$, respectively. The center of $\mathbf{E}_6$ is trivial while the centers of $\mathbf{E}_7$ and $\mathbf{E}_8$ are $\{\pm I\}$. The C-diagrams of $\mathbf{E}_6$, $\mathbf{E}_7, \mathbf{E}_8$ are the following diagrams with all edge labels 3:



The Coxeter group $\mathbf{F}_4$ is the group of symmetries of a regular 24-cell. The order of $\mathbf{F}_4$ is 1152 and $Z(\mathbf{F}_4) = \{\pm I\}$. The C-diagram of $\mathbf{F}_4$ is the linear diagram:



The Coxeter group $\mathbf{G}_3$ is the group of symmetries of a regular dodecahedron. The order of $\mathbf{G}_3$ is 120 and the center of $\mathbf{G}_3$ has order two. The Coxeter group $\mathbf{G}_4$ is the group of symmetries of a regular 120-cell. The order of $\mathbf{G}_4$ is 120^2 and $Z(\mathbf{G}_4) = \{\pm I\}$. The C-diagrams of $\mathbf{G}_3$ and $\mathbf{G}_4$ are the linear diagrams:



Lemmas 3.1 through 3.7 are either elementary or well known.

Lemma 3.1 *The Coxeter groups* $\mathbf{A}_n$ *and* $\mathbf{B}_n$ *are indecomposable with respect to direct products for all* n.

Lemma 3.2 *The Coxeter group* $\mathbf{C}_n$ *is decomposable with respect to direct products if and only if* n *is odd. If* n *is odd and* $\mathbf{C}_n = H \times K$ *with* $1 < |H| \leq |K|$*, then* $H = \{\pm I\}$ *and* $K = \mathbf{B}_n$ *or* $\theta(\mathbf{B}_n)$ *where* θ *is the automorphism of* $\mathbf{C}_n$ *defined by* $\theta(c_i) = -c_i$*, for* $i = 1, \ldots, n-1$*, and* $\theta(c_n) = c_n$*.*

Lemma 3.3 *The Coxeter group* $\mathbf{D}_2(n)$*, with Coxeter generators* a *and* b*, is decomposable with respect to direct products if and only if* $n \equiv 2 \bmod 4$*. If* $n \equiv 2 \bmod 4$ *and* $\mathbf{D}_2(n) = H \times K$ *with* $1 < |H| \leq |K|$*, then* $H = \langle (ab)^{n/2} \rangle$ *and* $K = \langle a, bab \rangle$ *or* $\langle b, aba \rangle$*, moreover* $K \cong \mathbf{D}_2(n/2)$*.*

Lemma 3.4 *The Coxeter groups* $\mathbf{E}_6$ *and* $\mathbf{E}_8$ *are indecomposable with respect to direct products.*

Lemma 3.5 *The Coxeter group* $\mathbf{E}_7$ *is decomposable with respect to direct products. If* $\mathbf{E}_7 = H \times K$ *with* $1 < |H| \leq |K|$*, then* $H = \{\pm I\}$ *and* $K = \mathbf{E}_7^+$*, moreover* $\mathbf{E}_7^+$ *is a nonabelian simple group.*

Lemma 3.6 *The Coxeter groups* $\mathbf{F}_4$ *and* $\mathbf{G}_4$ *are indecomposable with respect to direct products.*

Lemma 3.7 *The Coxeter group* $\mathbf{G}_3$ *is decomposable with respect to direct products. If* $\mathbf{G}_3 = H \times K$ *with* $1 < |H| \leq |K|$*, then* $H = \{\pm I\}$ *and* $K = \mathbf{G}_3^+$*, moreover* $\mathbf{G}_3^+$ *is a nonabelian simple group.*

The next lemma follows from the Krull-Remak-Schmidt Theorem (KRS-Theorem), Theorem 4.8 in [13].

Lemma 3.8 *Let* G *be a finite group with direct product decompositions*

$$G = H_1 \times H_2 \times \cdots \times H_r \quad \text{and} \quad G = K_1 \times K_2 \times \cdots \times K_s$$

such that H_i *and* K_j *are nontrivial and indecomposable with respect to direct products for each* i *and* j*. Let* $\iota_i : H_i \to G$ *be the inclusion map for each* i *and let* $\pi_j : G \to K_j$ *be the projection map for each* j*. Suppose* H_p *is nonabelian. Then there is a unique* q *such that* $H_p \cap K_q \neq \{1\}$*. Moreover* $\pi_q \iota_p : H_p \to K_q$ *is an isomorphism and* $Z(G)H_p = Z(G)K_q$*. Furthermore* $[H_p, H_p] = [K_q, K_q]$ *and* $\pi_q \iota_p : H_p \to K_q$ *restricts to the identity on* $[H_p, H_p]$*.*

Theorem 3.9 (Matching Theorem for Systems of a Finite Coxeter Group) *Let W be a finite Coxeter group with two sets of Coxeter generators S and S'. Let*

$$(W, S) = (W_1, S_1) \times \cdots \times (W_m, S_m)$$

and

$$(W, S') = (W_1', S_1') \times \cdots \times (W_n', S_n')$$

be the factorizations of (W, S) and (W, S') into irreducible factors. Let k be such that W_k is noncyclic. Then there is a unique ℓ such that W_ℓ' is noncyclic and $[W_k, W_k] = [W_\ell', W_\ell']$. Moreover,

1. *$Z(W)W_k = Z(W)W_\ell'$.*
2. *If $|W_k| = |W_\ell'|$, then $(W_k, S_k) \cong (W_\ell', S_\ell')$ and there is an isomorphism $\phi : W_k \to W_\ell'$ that restricts to the identity on $[W_k, W_k]$.*
3. *If $|W_k| < |W_\ell'|$, then either (W_k, S_k) has type $\mathbf{B}_{2q+1}$ and (W_ℓ', S_ℓ') has type $\mathbf{C}_{2q+1}$ for some $q \geq 1$ or (W_k, S_k) has type $\mathbf{D}_2(2q+1)$ and (W_ℓ', S_ℓ') has type $\mathbf{D}_2(4q+2)$ for some $q \geq 1$, and there is a monomorphism $\phi : W_k \to W_\ell'$ that restricts to the identity on $[W_k, W_k]$.*

Proof: By Lemmas 3.1-3.7, we can refine the decomposition $W = W_1 \times \cdots \times W_m$ to a decomposition $W = H_1 \times \cdots \times H_r$, with H_i nontrivial and indecomposable with respect to direct products, by replacing each W_i that factors into a direct product $W_i = H_{j-1} \times H_j$, with $|H_{j-1}| = 2$, by $H_{j-1} \times H_j$. Likewise refine the decomposition $W = W_1' \times \cdots \times W_\ell'$ to a decomposition $W = K_1 \times \cdots \times K_s$, with K_i nontrivial and indecomposable with respect to direct products, by replacing each W_i' that factors into a direct product $W_i' = K_{j-1} \times K_j$, with $|K_{j-1}| = 2$, by $K_{j-1} \times K_j$. Then $r = s$ by the KRS-Theorem.

Suppose that W_k is noncyclic. Then W_k is nonabelian, since (W_k, S_k) is irreducible. Now $W_k = H_p$ or $H_{p-1} \times H_p$, with $|H_{p-1}| = 2$, for some p. In either case H_p is nonabelian by Lemmas 3.1-3.7. By Lemma 3.8, there is a unique q such that $H_p \cap K_q \neq \{1\}$. Moreover $[H_p, H_p] = [K_q, K_q]$ and $\pi_q \iota_p : H_p \to K_q$ restricts to the identity on $[H_p, H_p]$. Then K_q is nonabelian. Now there is an ℓ such that $W_\ell' = K_q$ or $K_{q-1} \times K_q$ with $|K_{q-1}| = 2$. Then W_ℓ' is noncyclic and

$$[W_k, W_k] = [H_p, H_p] = [K_q, K_q] = [W_\ell', W_\ell'].$$

Now suppose W_i' is noncyclic and $[W_k, W_k] = [W_i', W_i']$. Then W_i' is nonabelian, since W_k is nonabelian. Now $W_i' = K_j$ or $K_{j-1} \times K_j$, with $|K_{j-1}| = 2$, for some j. Then we have

$$[H_p, H_p] = [W_k, W_k] = [W_i', W_i'] = [K_j, K_j].$$

Hence $H_p \cap K_j \neq \{1\}$, and so $j = q$ by the uniqueness of q. Therefore $K_q \subset W_\ell' \cap W_i'$, and so $i = \ell$ and ℓ is unique.

(1) By Lemma 3.8, we have $Z(W)W_k = Z(W)H_p = Z(W)H_q = Z(W)W_\ell'$.

(2) Suppose $|W_k| = |W_\ell'|$. As $H_p \cong H_q$, we have either $W_k = H_p$ and $W_\ell' = K_q$ or $W_k = H_{p-1} \times H_p$, with $|H_{p-1}| = 2$, and $W_\ell' = K_{q-1} \times K_q$, with $|K_{q-1}| = 2$. Hence $W_k \cong W_\ell'$ and $(W_k, S_k) \cong (W_\ell', S_\ell')$, since (W_k, S_k) and (W_ℓ', S_ℓ') are irreducible. Moreover $\pi_q\iota_p : H_p \to K_q$ is an isomorphism that restricts to the identity on $[H_p, H_p]$. If $W_k = H_p$ and $W_\ell' = K_q$, let $\phi = \pi_q\iota_p$. If $W_k = H_{p-1} \times H_p$ and $W_\ell' = H_{q-1} \times H_q$ extend $\pi_q\iota_p : H_p \to K_q$ to an isomorphism $\phi : W_k \to W_\ell'$ by mapping the generator of H_{p-1} to the generator of H_{q-1}. Then $\phi : W_k \to W_\ell'$ is an isomorphism that restricts to the identity on $[W_k, W_k] = [H_p, H_p]$.

(3) Suppose $|W_k| < |W_\ell'|$. As $H_p \cong H_q$, we have $W_k = H_p$ and $W_\ell' = K_{q-1} \times K_q$, with $|K_{q-1}| = 2$. By Lemmas 3.1-3.7, either $(W_k, S_k) \cong \mathbf{B}_{2q+1}$ and $(W_\ell', S_\ell') \cong \mathbf{C}_{2q+1}$ for some $q \geq 1$ or $(W_k, S_k) \cong \mathbf{D}_2(2q+1)$ and $(W_\ell', S_\ell') \cong \mathbf{D}_2(4q+2)$ for some $q \geq 1$. Moreover $\pi_q\iota_p : H_p \to K_q$ is an isomorphism that restricts to the identity on $[H_p, H_p]$. Hence $\pi_q\iota_p : H_p \to K_q$ extends to a monomorphism $\phi : W_k \to W_\ell'$ that restricts to the identity on $[W_k, W_k]$. $\square$

4 The Basic Matching Theorem

Let (W, S) be a Coxeter system. The undirected *Cayley graph* of (W, S) is graph $\mathrm{K} = \mathrm{K}(W, S)$ with vertices W and edges unordered pairs (v, w) such that $w = vs$ for some element s of S. The element $s = v^{-1}w$ of S is called the *label* of the edge (v, w). We represent an edge path in K beginning at vertex v by "$\alpha = (s_1, \ldots, s_n)$ at v" where s_i is the label of the ith edge of the path. The *length of an edge path* $\alpha = (s_1, \ldots, s_n)$ is $|\alpha| = n$. The *distance* between vertices v and w in K is the minimal length $d(v, w)$ of an edge path from v to w. A *geodesic* in K is an edge path α from a vertex v to a vertex w such that $|\alpha| = d(v, w)$ in K. The *length of an element* w of W is $l(w) = d(1, w)$. A word $w = s_1 s_2 \cdots s_n$, with s_i in S, is said to be *reduced* if $n = l(w)$.

Lemma 4.1 *If $A \subset S$, then for any pair of vertices v, w in* K*, there is a unique element x of the coset $w\langle A\rangle$ nearest to v and for any geodesic α from v to x and geodesic β at x in $w\langle A\rangle$ (i.e. all edge labels of β are in A), the path $\alpha\beta$ is geodesic. Moreover an element x of $w\langle A\rangle$ is the nearest element of $w\langle A\rangle$ to v if and only if for any geodesic α from v to x the path (α, a) is a geodesic for each a in A.*

Proof: Suppose x and y are distinct elements of $w\langle A\rangle$ that are nearest to v. Let α and γ be geodesics from v to x and y, respectively. Then $|\alpha| = |\gamma|$. Let β be a geodesic, with labels in A, from x to y. The path $\alpha\beta$ is not geodesic, since $|\alpha\beta| > |\gamma|$, and so a letter of β deletes with a letter of α by the deletion condition. This defines a path from v to $w\langle A\rangle$ shorter than α, which is impossible. A proof of the second assertion of the lemma is analogous.

Now suppose y is an element of $w\langle A\rangle$ such that for any geodesic γ from v to y the path (γ, a) is geodesic for each a in A. Then y is the nearest element x of $w\langle A\rangle$ to v otherwise there would be a geodesic $\alpha\beta$ from v to y with α a geodesic from v to x and β a nontrivial geodesic, with labels in A, from x to y, but β ends in some element a of A, and so the path $(\alpha\beta, a)$ would not be geodesic. $\square$

Lemma 4.2 (Bourbaki [1], Ch. IV, §1, Ex. 3) *If $A, B \subset S$ and w is an element of W, then there is a unique shortest representative x of the double coset $\langle A\rangle w\langle B\rangle$.*

Lemma 4.3 *Let $A, B \subset S$ and let w in W be such that $w\langle A\rangle w^{-1} \subset \langle B\rangle$. If u is the shortest element of $\langle B\rangle w\langle A\rangle$, then $uAu^{-1} \subset B$.*

Proof: Certainly we have $u\langle A\rangle u^{-1} \subset \langle B\rangle$. Let $u = u_1 \cdots u_n$ be reduced. For any a in A, the word $ua = u_1 \cdots u_n a$ is reduced by Lemma 4.1. Now uau^{-1} is in $\langle B\rangle$. Write $uau^{-1} = b_1 \cdots b_k$ with $b_1 \cdots b_k$ reduced in $\langle B\rangle$. Now u is a shortest element of $\langle B\rangle u$, and so u^{-1} is the shortest element of $u^{-1}\langle B\rangle$. Hence $u_n \cdots u_1 b_k \cdots b_1$ is reduced by Lemma 4.1, and so $b_1 \cdots b_k u_1 \cdots u_n$ is reduced. As $u_1 \cdots u_n a = b_1 \cdots b_k u_1 \cdots u_n$, we have $k = 1$ and $uau^{-1} = b_1$. $\square$

Lemma 4.4 (Bourbaki [1], Ch. IV, §1, Ex. 22) *Let w_0 be an element of W. Then the following are equivalent.*

1. *$l(w_0 s) < l(w_0)$ for all s in S.*

2. *$l(w_0 w) = l(w_0) - l(w)$ for all w in W.*

Such an element w_0 is unique and exists if and only if W is finite. If W is finite, then w_0 is the unique element of maximal length in W. Moreover $w_0^2 = 1$ and $w_0 S w_0 = S$.

Let (W, S) be a Coxeter system. The *quasi-center* of (W, S) is the subgroup

$$QZ(W, S) = \{w \in W : wSw^{-1} = S\}.$$

Lemma 4.5 (Bourbaki [1], Ch. V, §4, Ex. 3) *Let (W, S) be an irreducible Coxeter system with a nontrivial quasi-center. Then W is a finite group and $QZ(W) = \{1, w_0\}$ with w_0 the longest element of (W, S).*

Let V be a real vector space having a basis $\{e_s : s \in S\}$ in one-to-one correspondence with S. Let B be the symmetric bilinear form on V defined by

$$B(e_s, e_t) = \begin{cases} -\cos(\pi/m(s,t)) & \text{if } m(s,t) < \infty, \\ -1 & \text{if } m(s,t) = \infty. \end{cases}$$

There is an action of W on V defined by

$$s(x) = x - 2B(x, e_s)e_s \quad \text{for all } s \in S \text{ and } x \in V.$$

The *root system* of (W, S) is the set of vectors

$$\Phi = \{w(e_s) : w \in W \text{ and } s \in S\}.$$

The elements of Φ are called *roots*. By Prop. 2.1 of Deodhar [5], every root ϕ can be written uniquely in the form $\phi = \sum_{s \in S} a_s e_s$ with $a_s \in \mathbb{R}$ where either $a_s \geq 0$ for all s or $a_s \leq 0$ for all s. In the former case, we say ϕ is *positive* and write $\phi > 0$. Let Φ^+ be the set of positive roots.

The set of *reflections* of (W, S) is the set

$$T = \{wsw^{-1} : w \in W \text{ and } s \in S\}.$$

Proposition 4.6 (Deodhar [5], Prop. 3.1) *The function $\rho : \Phi^+ \to T$ defined by $\rho(w(e_s)) = wsw^{-1}$ is well defined and a bijection.*

Proposition 4.7 (Deodhar [5], Prop. 2.2) *Let $w \in W$ and $s \in S$. Then $l(ws) > l(w)$ if and only if $w(e_s) > 0$.*

If $A \subset S$, set $E_A = \{e_s : s \in A\}$. The next lemma follows from Lemma 4.1 and Propositions 4.6 and 4.7.

Lemma 4.8 *Let $A, B \subset S$ and let $w \in W$. Then the following are equivalent:*

1. *$w(E_A) = E_B$.*
2. *$wAw^{-1} = B$ and $l(wa) > l(w)$ for all $a \in A$.*
3. *$wAw^{-1} = B$ and w is the shortest element of $w\langle A\rangle$.*

The next lemma follows from Lemma 4.3 and Lemma 4.8.

Lemma 4.9 *Let $A, B \subset S$ and let w in W be such that $w\langle A\rangle w^{-1} = \langle B\rangle$. If u is the shortest element of $\langle B\rangle w\langle A\rangle$, then $u(E_A) = E_B$.*

Suppose $A \subset S$. If $\langle A\rangle$ is finite, we denote the longest element of $\langle A\rangle$ by w_A. Suppose $s \in S - A$. Let $K \subset S$ be the irreducible component of $A \cup \{s\}$ containing s. We say that s is A-*admissible* if $\langle K\rangle$ is finite. If s is A-admissible, define

$$\nu(s, A) = w_K w_{K-\{s\}}.$$

Then $\nu(s, A)$ is the shortest element of $w_K\langle A\rangle$ by Lemma 4.4; moreover, if $t = w_K s w_K$ and $B = (A \cup \{s\}) - \{t\}$, then $\nu(s, A)(E_A) = E_B$ by Lemma 4.8.

Proposition 4.10 (Deodhar [5], Prop. 5.5) *Let $A, B \subset S$, and let $w \in W$. If $w(E_A) = E_B$ and $w \neq 1$, then there exists a sequence $A_1, A_2, \ldots, A_{n+1}$ of subsets of A, and a sequence $s_1, s_2, \ldots, s_n$ of elements of S such that*

1. *$A_1 = A$ and $A_{n+1} = B$,*
2. *$s_i \in S - A_i$ and s_i is A_i-admissible for $i = 1, \ldots, n$,*
3. *$\nu(s_i, A_i)(E_{A_i}) = E_{A_{i+1}}$ for $i = 1, \ldots, n$,*
4. *$w = \nu(s_n, A_n) \cdots \nu(s_2, A_2)\nu(s_1, A_1)$,*
5. *$l(w) = l(\nu(s_1, A_1)) + l(\nu(s_2, A_2)) + \cdots + l(\nu(s_n, A_n))$.*

The next lemma follows from Proposition 4.10.

Lemma 4.11 *Let $A \subset S$. Then there exists $B \subset S$ such that $A \neq B$ and $\langle A \rangle$ is conjugate to $\langle B \rangle$ in W if and only if there exists $s \in S - A$ such that*

1. *$m(s, a) > 2$ for some $a \in A$,*
2. *the element s is A-admissible,*
3. *if K is the component of $A \cup \{s\}$ containing s, then $w_K s w_K \neq s$.*

Lemma 4.12 *Let $A, B \subset S$. If $\langle A \rangle$ is a maximal finite visual subgroup of (W, S) and $\langle A \rangle$ and $\langle B \rangle$ are conjugate, then $A = B$.*

Proof: If $s \in S - A$, then the irreducible component of $\langle A \cup \{s\} \rangle$ containing s is infinite, since $\langle A \rangle$ is a maximal finite visual subgroup of (W, S). Hence no $s \in S - A$ is A-admissible, and so $A = B$ by Lemma 4.11. □

Proposition 4.13 (Bourbaki [1], Ch. V, §4. Ex. 2) *If H is a finite subgroup of W, then there is a subset A of S such that $\langle A \rangle$ is finite and H is conjugate to a subgroup of $\langle A \rangle$.*

Lemma 4.14 *Every maximal finite visual subgroup of (W, S) is a maximal finite subgroup of W.*

Proof: Let $M \subset S$ be such that $\langle M \rangle$ is a maximal finite visual subgroup of (W, S). Suppose H is a finite subgroup of W containing $\langle M \rangle$. Then $wHw^{-1} \subset \langle A \rangle$ for some $w \in W$ and some $A \subset S$ such that $\langle A \rangle$ is finite by Prop. 4.13. Then $w\langle M \rangle w^{-1} \subset \langle A \rangle$. Let u be the shortest element of $\langle A \rangle w \langle M \rangle$. Then $uMu^{-1} \subset A$ by Lemma 4.3. As no element of $S - M$ is M-admissible, $uMu^{-1} = M$ by Prop. 4.10. Therefore $M = A$, since M is a maximal finite visual subgroup. Hence $w\langle M \rangle w^{-1} = \langle A \rangle$, and so $\langle M \rangle = H$. Thus $\langle M \rangle$ is a maximal finite subgroup of W. □

A *simplex* C of (W, S) is a subset C of S such that $(\langle C \rangle, C)$ is a complete Coxeter system. A simplex C of (W, S) is said to be *spherical* if $\langle C \rangle$ is finite. The next proposition follows from Proposition 4.13 and Lemmas 4.12 and 4.14.

Proposition 4.15 *Let W be a finitely generated Coxeter group with two sets of Coxeter generators S and S', and let M be a maximal spherical simplex of (W, S). Then there is a unique maximal spherical simplex M' of (W, S') such that $\langle M \rangle$ and $\langle M' \rangle$ are conjugate in W.*

The next lemma follows from Lemma 4.11.

Lemma 4.16 *Let $A, B \subset S$. If $\langle A \rangle$ is a maximal finite irreducible subgroup of (W, S) and $\langle A \rangle$ and $\langle B \rangle$ are conjugate, then $A = B$.*

Lemma 4.17 *Let $x, y \in S$ be distinct, let $B \subset S$, and let $w \in W$ such that $wxyw^{-1} \in \langle B \rangle$. If u is the shortest element of the double coset $\langle B \rangle w \langle x, y \rangle$, then $u\{x, y\}u^{-1} \subset B$.*

Proof: Let $b = uxyu^{-1}$. Then b is in $\langle B \rangle$. Write $b = b_1 \cdots b_k$ with $b_1 \cdots b_k$ reduced in $\langle B \rangle$ and let $u = u_1 \cdots u_n$ be reduced. Observe that the left and right sides of $b_1 \cdots b_k u_n \cdots u_1 = u_1 \cdots u_n xy$ are reduced, and so $k = 2$. Consider the bigon in K with geodesic sides $(u_1, \ldots, u_n, x, y)$ and $(b_1, b_2, u_1, \ldots, u_n)$. The path $(b_1, u_1, \ldots, u_n, x, y)$ is not geodesic. If b_1 deletes with x, then $uxu^{-1} = b_1$; otherwise, b_1 deletes with y and $uxu^{-1} = b_2$. Now $(u_1, \ldots, u_n, y, x)$ is geodesic, but $(b_2, u_1, \ldots, u_n, y, x)$ is not. If b_2 deletes with y, then $uyu^{-1} = b_2$; otherwise, b_2 deletes with x and $uyu^{-1} = b_1$. □

Lemma 4.18 *Let $A, B \subset S$ with no $a \in A$ central in $\langle A \rangle$. Suppose*

$$w[\langle A \rangle, \langle A \rangle]w^{-1} \subset \langle B \rangle \quad \text{for some } w \in W.$$

Let u be the shortest element of $\langle B \rangle w \langle A \rangle$. Then $uAu^{-1} \subset B$.

Proof: Let $x \in A$. Then there exists $y \in A$ such that $m(x, y) > 2$. Assume $m(x, y)$ is odd. Then $xy \in [\langle A \rangle, \langle A \rangle]$. Hence $uxu^{-1} \in B$ by Lemma 4.17.

Assume now that $m(x, y)$ is even. Then $xyxy \in [\langle A \rangle, \langle A \rangle]$. Let $b = uxyxyu^{-1}$. Then b is in $\langle B \rangle$. Write $b = b_1 \cdots b_k$ with $b_1 \cdots b_k$ reduced in $\langle B \rangle$ and let $u = u_1 \cdots u_n$ be reduced. Observe that the left and right sides of $u_1 \cdots u_n xyxy = b_1 \cdots b_k u_1 \cdots u_n$ are reduced, and so $k = 4$.

Assume that $m(x, y) > 4$. Consider the bigon in K with geodesic sides $(u_1, \ldots, u_n, x, y, x, y)$ and $(b_1, \ldots, b_4, u_1, \ldots, u_n)$. See Figure 1. The word $yxyxu_n \cdots u_1 b_1$ is not reduced and b_1 must delete with one of the first four letters. Also $xyxyxu_n \cdots u_1$ is reduced, so $xyxyxu_n \cdots u_1 b_1$ has length $n+4$. But then, in this last word, b_1 cannot delete with the second, third, or fourth letter. Hence in the word, $yxyxu_n \cdots u_1 b_1$, the letter b_1 must delete with the fourth letter. This shows that $xu^{-1}b_1 = u^{-1}$, and so $uxu^{-1} = b_1$, as desired.

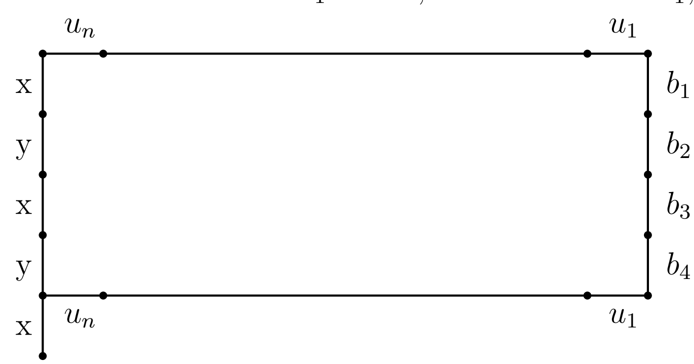


Figure 1

Assume now that $m(x,y) = 4$. The word $yxyxu_n \cdots u_1 b_1$ is not reduced and represents an element of length $n+3$. See Figure 1. In particular, b_1 does not delete with the second or third letter of this word. Consequently, b_1 deletes with the first or fourth letter of the word. If b_1 deletes with the fourth letter, then $uxu^{-1} = b_1$ and we are done. Suppose b_1 deletes with the first letter. Then $(uxyx)y(xyxu^{-1}) = b_1$, and so $uyu^{-1} = b_1$. Now

$$b_1 b_2 b_3 b_4 = uxyxyu^{-1} = uyxyxu^{-1} = uyu^{-1}uxyxu^{-1} = b_1 uxyxu^{-1}.$$

Hence $uxyxu^{-1} = b_2 b_3 b_4$. Combining this last fact with the fact that the word $u_1 \cdots u_n xyxy$ is reduced allows us to use the technique of the previous case to show that $uxu^{-1} = b_2$. See Figure 2. □

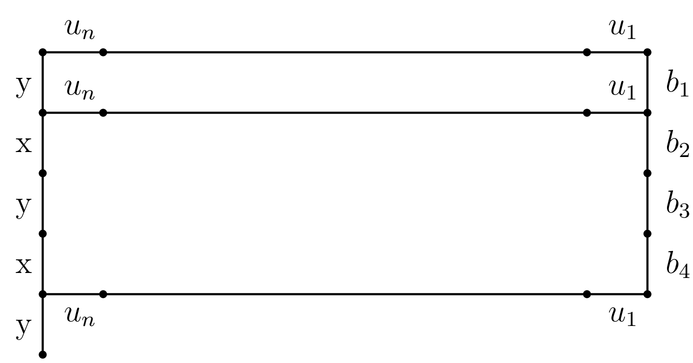


Figure 2

Let (W,S) be a Coxeter system. A *basic subgroup* of (W,S) is a noncyclic, maximal, finite, irreducible, visual subgroup of (W,S). A *base* of (W,S) is a subset B of S such that $\langle B\rangle$ is a basic subgroup of (W,S).

Theorem 4.19 (Basic Matching Theorem) *Let W be a finitely generated Coxeter group with two sets of Coxeter generators S and S'. Let B be a base of (W,S). Then there is a unique irreducible subset B' of S' such that $[\langle B\rangle,\langle B\rangle]$ is conjugate to $[\langle B'\rangle,\langle B'\rangle]$ in W. Moreover,*

1. *The set B' is a base of (W,S'),*

2. *If $|\langle B\rangle| = |\langle B'\rangle|$, then B and B' have the same type and there is an isomorphism $\phi:\langle B\rangle\to\langle B'\rangle$ that restricts to conjugation on $[\langle B\rangle,\langle B\rangle]$ by an element of W.*

3. *If $|\langle B\rangle| < |\langle B'\rangle|$, then either B has type $\mathbf{B}_{2q+1}$ and B' has type $\mathbf{C}_{2q+1}$ for some $q\geq 1$ or B has type $\mathbf{D}_2(2q+1)$ and B' has type $\mathbf{D}_2(4q+2)$ for some $q\geq 1$. Moreover, there is a monomorphism $\phi:\langle B\rangle\to\langle B'\rangle$ that restricts to conjugation on $[\langle B\rangle,\langle B\rangle]$ by an element of W.*

Proof: Let $M\subset S$ be a maximal spherical simplex containing B. Then there is a unique maximal spherical simplex M' of (W,S') and an element u of W such that $\langle M'\rangle = u\langle M\rangle u^{-1}$ by Proposition 4.15. By the Matching Theorem for systems of a finite Coxeter group applied to $(\langle M'\rangle, uMu^{-1})$ and $(\langle M'\rangle, M')$, there is a base B' of $(\langle M'\rangle, M')$ such that

$$[\langle B'\rangle,\langle B'\rangle] = [\langle uBu^{-1}\rangle,\langle uBu^{-1}\rangle] = u[\langle B\rangle,\langle B\rangle]u^{-1}.$$

Moreover, B' satisfies conditions 2 and 3, and so $|B|=|B'|$.

Let C' be a base of (W,S') that contains B'. Then by the above argument, there is a $C\subset S$ and a $v\in W$ such that $\langle C\rangle$ is a finite irreducible subgroup of (W,S), and $|C|=|C'|$, and $[\langle C\rangle,\langle C\rangle] = v[\langle C'\rangle,\langle C'\rangle]v^{-1}$. Then we have $vu[\langle B\rangle,\langle B\rangle]u^{-1}v^{-1}\subset[\langle C\rangle,\langle C\rangle]$. By Lemma 4.18, there is a $w\in W$ such that $wBw^{-1}\subset C$. As B is a base of (W,S), we have that $wBw^{-1}=C=B$ by Lemma 4.16. Therefore $B'=C'$ and B' is a base of (W,S').

Suppose $D'\subset S'$ is irreducible and $x\in W$ such that $[\langle D'\rangle,\langle D'\rangle] = x[\langle B\rangle,\langle B\rangle]x^{-1}$. Then $xu^{-1}[\langle B'\rangle,\langle B'\rangle]ux^{-1} = [\langle D'\rangle,\langle D'\rangle]$. By Lemma 4.18, there is a $y\in W$ such that $yB'y^{-1}\subset D'$. As B' is a base of (W,S') and D' is irreducible, $yB'y^{-1}=D'=B'$ by Lemma 4.16. Thus B' is unique. $\square$

5 Nonisomorphic Basic Matching

Let W be a finitely generated Coxeter group with two sets of Coxeter generators S and S'. A base B of (W,S) is said to *match* a base B' of (W,S') if $[\langle B\rangle,\langle B\rangle]$ is conjugate to $[\langle B'\rangle,\langle B'\rangle]$ in W. In this section, we determine some necessary and some sufficient conditions for a base $B\subset S$ to match a base $B'\subset S'$ of a different type.

Proposition 5.1 (Solomon [12], Lemma 2) *If $A,B\subset S$ and u is the shortest element of $\langle A\rangle u\langle B\rangle$, then*

$$\langle A\rangle\cap u\langle B\rangle u^{-1}=\langle C\rangle \quad \textit{where}\;\; C=A\cap uBu^{-1}.$$

The next proposition follows from Proposition 5.1.

Proposition 5.2 *Let $A,B\subset S$ and $w\in W$. Write $w=xuy$ with $x\in\langle A\rangle$, $y\in\langle B\rangle$, and u the shortest element of $\langle A\rangle w\langle B\rangle$. Then*

$$\langle A\rangle\cap w\langle B\rangle w^{-1}=x\langle C\rangle x^{-1} \quad \textit{where}\;\; C=A\cap uBu^{-1}.$$

Lemma 5.3 *Suppose B is a base of (W,S) of type $\mathbf{C}_{2q+1}$ that matches a base B' of (W,S') of type $\mathbf{B}_{2q+1}$ for some $q\geq 1$. Let a,b,c be the elements of B such that $m(a,b)=4$ and $m(b,c)=3$, and let $A\subset S$ such that $a\in A$. If $\langle A\rangle$ is conjugate to $\langle A'\rangle$ for some $A'\subset S'$, then $B\subset A$.*

Proof: Let $M\subset S$ be a maximal spherical simplex containing B. Then there is a unique maximal spherical simplex $M'\subset S'$ such that $\langle M\rangle$ is conjugate to $\langle M'\rangle$ by Proposition 4.15. By conjugating S', we may assume that $\langle M\rangle=\langle M'\rangle$. Then M' contains B' by the Basic Matching Theorem. Let w be an element of W such that $\langle A\rangle = w\langle A'\rangle w^{-1}$. By Proposition 5.2, there is an element x of $\langle M'\rangle$ and a subset C of M' such that

$$\langle M\cap A\rangle=\langle M\rangle\cap\langle A\rangle=\langle M'\rangle\cap w\langle A'\rangle w^{-1}=x\langle C\rangle x^{-1}.$$

Hence, we may assume that W is finite by restricting to $\langle M\rangle$. Furthermore, by conjugating S', we may assume that $\langle A\rangle=\langle A'\rangle$.

Let C be a base of (W,S) other than B. Then each element of C commutes with each element of B, and so $\langle B\rangle$ injects into the quotient of W by the commutator subgroup of $\langle C\rangle$. Hence, by Theorem 3.9, we may assume that W is the direct product of $\langle B\rangle$ and copies of $\mathbf{A}_1$. The center Z of W

is generated by $S - B$ and the longest element z of $\langle B\rangle$. The center Z is also generated by $S' - B'$. Let K be the kernel of the homomorphism of Z to $\{\pm 1\}$ that maps $S - B$ to 1 and z to -1. Then W/K is a Coxeter group. P-diagrams for W/K are obtained from the P-diagram of (W, S) by removing the vertices $S - B$ and from the P-diagram of (W, S') by removing the vertices in $(S' - B') \cap K$ and identifying the remaining vertices of $S' - B'$ to a single vertex. By passing to the quotient W/K, we may assume that $S = B$ and $S' = B' \cup \{z'\}$ and z' commutes with each element of B'. Then $z = z'$, since $\langle z\rangle = Z(W) = \langle z'\rangle$.

Now as $a \in A$ and $\langle A\rangle = \langle A'\rangle$, we have that $a \in \langle A'\rangle$. The element a is represented by the matrix $\mathrm{diag}(1, \ldots, 1, -1)$ in $\mathbf{C}_{2q+1}$. Observe that

$$\mathrm{diag}(1, \ldots, 1, -1) = \mathrm{diag}(-1, \ldots, -1, -1)\mathrm{diag}(-1, \ldots, -1, 1).$$

The matrix $-I$ represents z. The matrix $\mathrm{diag}(-1, \ldots, -1, 1)$ is the longest element of $\mathbf{B}_{2q+1}$ which is in $[\mathbf{C}_{2q+1}, \mathbf{C}_{2q+1}]$. Hence $\mathrm{diag}(-1, \ldots, -1, 1)$ represents an element ℓ of $[\langle B\rangle, \langle B\rangle]$, with $a = \ell z$. As $[\langle B\rangle, \langle B\rangle] = [\langle B'\rangle, \langle B'\rangle]$, we have $\ell \in \langle B'\rangle$. Hence every reduced word in the generators $S' = B' \cup \{z\}$ representing $a = \ell z$ involves z by Prop. 7 on p. 19 of Bourbaki[1]. Therefore $z \in A'$, since $A' \subset B' \cup \{z\}$ and $a \in \langle A'\rangle$. Hence $z \in \langle A\rangle$. As z involves all the elements of B, we deduce that $B \subset A$. □

Theorem 5.4 *Suppose B is a base of (W, S) of type $\mathbf{C}_{2q+1}$ that matches a base B' of (W, S') of type $\mathbf{B}_{2q+1}$ for some $q \geq 1$. Let a, b, c be the elements of B such that $m(a, b) = 4$ and $m(b, c) = 3$. If $s \in S - B$ and $m(s, a) < \infty$, then $m(s, t) = 2$ for all t in B.*

Proof: Let $A \subset S$ be a maximal spherical simplex containing $\{a, s\}$. Then there is a maximal spherical simplex $A' \subset S'$ such that $\langle A\rangle$ is conjugate to $\langle A'\rangle$. Hence $B \subset A$ by Lemma 5.3. As B is a base of $(\langle A\rangle, A)$, we deduce that s commutes with each element of B. □

Lemma 5.5 *Let $w = s_1 \cdots s_n$ be a reduced word in (W, S) and let $s \in S$ such that $s \neq s_i$ for each $i = 1, \ldots, n$. If sw has finite order in W, then $m(s, s_i) < \infty$ for each $i = 1, \ldots, n$.*

Proof: On the contrary, suppose $m(s, s_i) = \infty$ for some i. We may assume $S = \{s, s_1, \ldots, s_n\}$. Then

$$W = \langle s, s_1, \ldots, \hat{s}_i, \ldots, s_n\rangle *_{\langle s_1, \ldots, \hat{s}_i, \ldots, s_n\rangle} \langle s_1, \ldots, s_n\rangle$$

is a free product with amalgamation decomposition. Observe that $(sw)^k = swsw \cdots sw$ is a normal form for $(sw)^k$ for each $k \geq 1$ with respect to the amalgamated product, and so $(sw)^k \neq 1$ for each $k \geq 1$. □

Theorem 5.6 *Let B be a base of (W, S) of type $\mathbf{C}_{2q+1}$ for some $q \geq 1$, and let a, b, c be the elements of B such that $m(a, b) = 4$ and $m(b, c) = 3$. Suppose that $m(s, t) = 2$ for all $(s, t) \in (S - B) \times B$ such that $m(s, a) < \infty$. Let $d = aba$, and let z be the longest element of $\langle B\rangle$. Let $S' = (S - \{a\}) \cup \{d, z\}$ and $B' = (B - \{a\}) \cup \{d\}$. Then S' is a set of Coxeter generators for W such that*

1. *The set B' is a base of (W, S') of type $\mathbf{B}_{2q+1}$ that matches B,*
2. *$m(z, t) = 2$ for all $t \in B'$,*
3. *If $(s, t) \in (S-B)\times\{d, z\}$, then $m(s, t) < \infty$ if and only if $m(s, a) < \infty$, moreover if $m(s, t) < \infty$, then $m(s, t) = 2$.*

Proof: Consider the Coxeter presentation

$$W = \langle S \mid (st)^{m(s,t)} : s, t \in S \text{ and } m(s, t) < \infty\rangle$$

Now $(\langle B'\rangle, B')$ is a finite Coxeter system of type $\mathbf{B}_{2q+1}$. Let ℓ be the longest element of $(\langle B'\rangle, B')$. Regard ℓ as a reduced word in the elements of B'. Add generators d and z and relations $d = aba$ and $z = a\ell$ to the above presentation for W. Now add the relators $(st)^{m(s,t)}$ for (s, t) in $\{d, z\} \times S'$ or in $S' \times \{d, z\}$ where $m(s, t)$ is the order of st in W and $m(s, t) < \infty$. This includes all the relators of $(\langle B'\rangle, B')$. As $\langle z\rangle$ is the center of $\langle B\rangle$, we have that $m(z, t) = 2$ for all t in B'.

Next delete the generator a and the relation $z = a\ell$ and replace a by $z\ell$ in the remaining relations. As z commutes with each element of B', we can replace the relation $d = z\ell bz\ell$ by the relation $d = \ell b\ell$.

The relators $(z\ell b)^4$ and $(bz\ell)^4$ can be replaced by $(\ell b)^4$ and $(b\ell)^4$ which in turn can be replaced by $(db)^2$ and $(bd)^2$ using the relation $d = \ell b\ell$. The relators $(db)^2$ and $(bd)^2$ are redundant and so we delete them. The relation $d = \ell b \ell$ is derivable from the relators of $(\langle B' \rangle, B')$ and so we delete it. The relators $(z\ell s)^2$ and $(sz\ell)^2$ for $s \in B - \{a, b\}$ can be replaced by $(\ell s)^2$ and $(s\ell)^2$. The relators $(\ell s)^2$ and $(s\ell)^2$ are derivable from the relators of $(\langle B' \rangle, B')$ and so we delete them.

Suppose $s \in S - B$ and $m(s, a) < \infty$. Then $m(s, t) = 2$ for all $t \in B$ by hypothesis. Hence $m(s, t) = 2$ for all $t \in B' \cup \{z\}$. Now the relators $(z\ell s)^2$ and $(sz\ell)^2$ can be replaced by $(\ell s)^2$ and $(s\ell)^2$. The relators $(\ell s)^2$ and $(s\ell)^2$ are derivable from the relators $(st)^2$ for $t \in B'$ and the relation $\ell^2 = 1$. Hence we may delete the relators $(\ell s)^2$ and $(s\ell)^2$. This leaves the Coxeter presentation

$$W = \langle S' \mid (st)^{m(s,t)} : s, t \in S' \text{ and } m(s, t) < \infty \rangle$$

Thus S' is a set of Coxeter generators for W.

Statement 3 follows from Theorem 5.4, Lemma 5.5, and the hypothesis that $m(s, t) = 2$ for all $(s, t) \in (S - B) \times B$ such that $m(s, a) < \infty$. The set B' is a base of (W, S'), since B is a base of (W, S) and if $s \in S' - B'$ and $m(s, d) < \infty$, then $m(s, d) = 2$. The base B matches the base B', since $[\langle B \rangle, \langle B \rangle] = [\langle B' \rangle, \langle B' \rangle]$. □

The next theorem follows from Theorems 5.4 and 5.6.

Theorem 5.7 *Let (W, S) be a Coxeter system of finite rank. Let B be a base of (W, S) of type $\mathbf{C}_{2q+1}$ for some $q \geq 1$, and let a, b, c be the elements of B such that $m(a, b) = 4$ and $m(b, c) = 3$. Then W has a set of Coxeter generators S' such that B matches a base B' of (W, S') of type $\mathbf{B}_{2q+1}$ if and only if $m(s, t) = 2$ for all $(s, t) \in (S - B) \times B$ such that $m(s, a) < \infty$.*

We next consider the analogue of Theorem 5.7 in the dihedral case.

Theorem 5.8 *Let (W, S) be a Coxeter system of finite rank, and let $B = \{a, b\}$ be a base of (W, S) of type $\mathbf{D}_2(4q + 2)$ for some $q \geq 1$. Then W has a set of Coxeter generators S' such that B matches a base B' of (W, S') of type $\mathbf{D}_2(2q + 1)$ if and only if either $v = a$ or $v = b$ has the property that if $s \in S - B$ and $m(s, v) < \infty$, then $m(s, a) = m(s, b) = 2$.*

Proof: Suppose that W has a set of Coxeter generators S' such that B matches a base B' of (W, S') of type $\mathbf{D}_2(2q + 1)$. Let $v = a$ or b with the choice specified below. Suppose $s \in S - B$ and $m(s,v) < \infty$. Let $A \subset S$ be a maximal spherical simplex containing $\{s,v\}$. Then there is a maximal spherical simplex $A' \subset S'$ such that $\langle A\rangle$ is conjugate to $\langle A'\rangle$. We claim that $B \subset A$. As in the proof of Lemma 5.3, we may assume that $\langle A\rangle = \langle A'\rangle$ and reduce W so that $S = B$ and $S' = B' \cup \{z\}$ where $\langle z\rangle$ is the center of $\langle B\rangle$.

Now a and b are not both in $\langle B'\rangle$. Choose v so that v is not in $\langle B'\rangle$. Then every reduced S'-word representing v involves z. Now as $v \in A$, we have that $v \in \langle A'\rangle$. Therefore $z \in A'$. Hence $z \in \langle A\rangle$. Therefore $B \subset A$ as claimed. Now return to the original state of W. As B is a base of $\langle A\rangle$, we have $m(s,a) = m(s,b) = 2$. The converse follows from the next theorem. □

Theorem 5.9 *Let $B = \{a,b\}$ be a base of (W,S) of type $\mathbf{D}_2(4q+2)$ for some $q \geq 1$. Suppose that if $s \in S - B$ and $m(s,a) < \infty$, then $m(s,a) = m(s,b) = 2$. Let $c = aba$ and let z be the longest element of $\langle B\rangle$. Let $S' = (S - \{a\}) \cup \{c,z\}$ and $B' = \{b,c\}$. Then S' is a set of Coxeter generators of W such that*

1. *The set B' is a base of (W,S') of type $\mathbf{D}_2(2q+1)$ that matches B,*
2. *$m(z,b) = m(z,c) = 2$,*
3. *if $(s,t) \in (S-B)\times\{c,z\}$, then $m(s,t) < \infty$ if and only if $m(s,a) < \infty$, moreover if $m(s,t) < \infty$, then $m(s,t) = 2$.*

Proof: Consider the Coxeter presentation

$$W = \langle S \mid (st)^{m(s,t)} : \ s,t \in S \text{ and } m(s,t) < \infty\rangle$$

Now $(\langle B'\rangle, B')$ is a finite Coxeter system of type $\mathbf{D}_2(2q+1)$. Let ℓ be the longest element of $(\langle B'\rangle, B')$. Regard ℓ as the reduced word $(bc)^q b$ in the elements of B'. Add generators c and z and relations $c = aba$ and $z = a\ell$ to the above presentation for W. Now add the relators $(st)^{m(s,t)}$ for (s,t) in $\{c,z\} \times S'$ or in $S' \times \{c,z\}$ where $m(s,t)$ is the order of st in W and $m(s,t) < \infty$. This includes all the relators of $(\langle B'\rangle, B')$. As $\langle z\rangle$ is the center of $\langle B\rangle$, we have that $m(z,b) = m(z,c) = 2$.

Next delete the generator a and the relation $z = a\ell$ and replace a by $z\ell$ in the remaining relations. As z commutes with each element of B', we can replace the relation $c = z\ell bz\ell$ by the relation $c = \ell b\ell$.

The relators $(z\ell b)^{2(2q+1)}$ and $(bz\ell)^{2(2q+1)}$ can be replaced by $(\ell b)^{2(2q+1)}$ and $(b\ell)^{2(2q+1)}$ which in turn can be replaced by $(cb)^{2q+1}$ and $(bc)^{2q+1}$ using the relation $c = \ell b\ell$. The relators $(cb)^{2q+1}$ and $(bc)^{2q+1}$ are redundant and so we delete them. The relation $c = \ell b \ell$ is derivable from the relators of $(\langle B'\rangle, B')$ and so we delete it.

Suppose $s \in S - B$ and $m(s,a) < \infty$. Then $m(s,a) = m(s,b) = 2$ by hypothesis. Hence $m(s,t) = 2$ for $t \in \{b, c, z\}$. Now the relators $(z\ell s)^2$ and $(sz\ell)^2$ can be replaced by $(\ell s)^2$ and $(s\ell)^2$. The relators $(\ell s)^2$ and $(s\ell)^2$ are derivable from the relators $(st)^2$ for $t \in B'$ and the relation $\ell^2 = 1$. Hence we may delete the relators $(\ell s)^2$ and $(s\ell)^2$. This leaves the Coxeter presentation

$$W = \langle S' \mid (st)^{m(s,t)} : s, t \in S' \text{ and } m(s,t) < \infty\rangle$$

Thus S' is a set of Coxeter generators for W.

Statement 3 follows from Lemma 5.5 and the hypothesis that $m(s,t) = m(s,b) = 2$ for all $s \in S - B$ such that $m(s,a) < \infty$. The set B' is a base of (W, S'), since B is a base of (W, S) and if $s \in S' - B'$ and $m(s,c) < \infty$, then $m(s,c) = 2$. The base B matches the base B', since $[\langle B\rangle, \langle B\rangle] = [\langle B'\rangle, \langle B'\rangle]$. $\square$

A group G has property FA if for every tree on which G acts without inversions, the set of fixed points of G in the tree is nonempty. Let (W, S) be a Coxeter system, and let $A \subset S$. We say that $\langle A\rangle$ is a *complete* visual subgroup of (W, S) if $(\langle A\rangle, A)$ is a complete Coxeter system.

Proposition 5.10 (Mihalik and Tschantz [9]) *Let (W, S) be a Coxeter system of finite rank. The maximal* FA *subgroups of W are the conjugates of the maximal complete visual subgroups of (W, S).*

Lemma 5.11 *Let $A, B \subset S$. If $\langle A\rangle$ is a maximal complete visual subgroup of (W, S) and $\langle A\rangle$ and $\langle B\rangle$ are conjugate, then $A = B$.*

Proof: If $s \in S - A$, then the irreducible component of $\langle A \cup \{s\}\rangle$ containing s is incomplete and therefore infinite, since $\langle A\rangle$ is a maximal complete visual subgroup of (W, S). Hence no $s \in S - A$ is A-admissible, and so $A = B$ by Lemma 4.11. $\square$

The next proposition follows form Proposition 5.10 and Lemma 5.11.

Proposition 5.12 *Let W be a finitely generated Coxeter group with two sets of Coxeter generators S and S', and let M be a maximal simplex of (W,S). Then there is a unique maximal simplex M' of (W,S') such that $\langle M\rangle$ and $\langle M'\rangle$ are conjugate in W.*

Let W be a finitely generated Coxeter group with two Coxeter systems S and S', and let A be a subset of S. Let $\overline{A}$ be the intersection of all subsets B of S such that B contains A and $\langle B\rangle$ is conjugate to $\langle B'\rangle$ for some $B'\subset S'$. Then $\overline{A}$ is the smallest subset B of S such that B contains A and $\langle B\rangle$ is conjugate to $\langle B'\rangle$ for some $B'\subset S'$ by Prop. 5.2. If A is a spherical simplex, then $\overline{A}$ is a spherical simplex, since for any maximal spherical simplex M of (W,S) that contains A, there exists $M'\subset S'$ such that $\langle M\rangle$ is conjugate to $\langle M'\rangle$ by Prop. 4.15.

Theorem 5.13 *Suppose $B=\{x,y\}$ is a base of (W,S) of type $\mathbf{D}_2(2q+1)$ that matches a base B' of (W,S') of type $\mathbf{D}_2(4q+2)$ for some $q\geq 1$. Then there exists $r\in\overline{B}-B$ such that $\overline{\{r\}}=\overline{B}$. Moreover, if $s\in S-B$ and $m(s,x),m(s,y)<\infty$, then $m(s,x)=m(s,y)=2$.*

Proof: Let $C=\overline{B}$. Then C is a spherical simplex of (W,S) and $\langle C\rangle$ is conjugate to $\langle C'\rangle$ for some $C'\subset S'$. By conjugating S', we may assume that $\langle C\rangle=\langle C'\rangle$. Then C' contains B' by the Basic Matching Theorem. Hence B is a proper subset of C, since otherwise $\langle B'\rangle\subset\langle C'\rangle=\langle C\rangle=\langle B\rangle$ which is not the case, since $|\langle B\rangle|<|\langle B'\rangle|$.

Let r be an element of $C-B$ which will be specified below. As $r\in\overline{B}$, we have that $\overline{\{r\}}\subset\overline{B}$. Let $A=\overline{\{r\}}$. We claim that $B\subset A$. Now $\langle A\rangle$ is conjugate in $\langle C'\rangle$ to $\langle A'\rangle$ for some $A'\subset C'$ by Prop. 5.2. Hence we may assume that $S=C$. Then W is a finite group. By quotienting out the commutator subgroups of all the bases of (W,C) other than B, we may assume that W is the direct product of $\langle B\rangle$ and copies of $\mathbf{A}_1$.

The center Z of W is generated by $C-B$. The center Z is also generated by $C'-B'$ and z'. Let K be the kernel of the homomorphism $\phi: Z\to\{\pm 1\}$ that maps $C'-B'$ to 1 and z' to -1. Choose $r\in C-B$ so that $\phi(r)=-1$.

By quotienting out K, we may assume that $C=B\cup\{r\}$ and $C'=B'$. Then $r=z'$, and so $z'\in\langle A\rangle$. Hence $z'\in\langle A'\rangle$. Therefore $A'=B'$, and so $B\subset A$ as claimed. Now return to the original state of W. As $B\subset A$, we have $\overline{B}\subset\overline{\{r\}}$. Thus $\overline{\{r\}}=\overline{B}$.

Suppose $s\in S-B$ with $m(s,x),m(s,y)<\infty$. Let $M\subset S$ be a maximal simplex containing $\{s,x,y\}$. Then there is a maximal simplex $M'\subset S'$ such that $\langle M\rangle$ is conjugate to $\langle M'\rangle$ by Prop. 5.12. By conjugating S', we may assume that $\langle M\rangle = \langle M'\rangle$. Then M' contains B' and $[B,B]$ is conjugate to $[B',B']$ in $\langle M'\rangle$ by the Basic Matching Theorem.

Let $B' = \{a,b\}$. Then $m(s',a) = m(s',b) = 2$ for all $s' \in M' - B'$ by Theorem 5.8. Hence B' is an irreducible component of M'. Therefore $[B',B']$ is a normal subgroup of $\langle M'\rangle$. Hence $[B,B]$ is a normal subgroup of $\langle M\rangle$. As $\langle xy\rangle = [B,B]$, we have that $\langle xy\rangle$ is a normal subgroup of $\langle M\rangle$. Therefore $s\{x,y\}s = \{x,y\}$ by Lemma 4.17, and $sxs = x$ and $sys = y$ by the deletion condition. □

Lemma 5.14 *Let $\phi : \mathbf{B}_n \to \mathbf{C}_n$ be a monomorphism with n odd and $n \geq 3$. Then ϕ maps $b_{n-1}b_n$ to a conjugate of $(c_{n-1}c_n)^2$ in $\mathbf{C}_n$.*

Proof: Now $\phi(\mathbf{B}_n)$ does not contain the center of $\mathbf{C}_n$, since $Z(\mathbf{B}_n) = \{1\}$. Therefore either $\phi(\mathbf{B}_n) = \mathbf{B}_n$ or $\phi(\mathbf{B}_n) = \theta(\mathbf{B}_n)$ where θ is the automorphism of $\mathbf{C}_n$ defined by $\theta(c_i) = -c_i$, for $i = 1,\ldots,n-1$ and $\theta(c_n) = c_n$. Now θ restricts to the identity on $[\mathbf{C}_n,\mathbf{C}_n]$, and so by composing ϕ with θ in the latter case, we may assume that $\phi(\mathbf{B}_n) = \mathbf{B}_n$. Now every automorphism of $\mathbf{B}_n$ is inner according to Franzsen [7]. Hence ϕ restricts to conjugation on $[\mathbf{B}_n,\mathbf{B}_n]$ by an element of $\mathbf{B}_n$. As $b_{n-1}b_n$ is in $[\mathbf{B}_n,\mathbf{B}_n]$ and $b_{n-1}b_n = (c_{n-1}c_n)^2$, we conclude that $\phi(b_{n-1}b_n)$ is conjugate to $(c_{n-1}c_n)^2$ in $\mathbf{C}_n$. □

Theorem 5.15 *Suppose B is a base of (W,S) of type $\mathbf{B}_{2q+1}$ that matches a base B' of (W,S') of type $\mathbf{C}_{2q+1}$ for some $q \geq 1$. Let x,y be the split ends of the C-diagram of $(\langle B\rangle, B)$. Then there exists $r \in \overline{B} - B$ such that $\overline{\{r\}} = \overline{B}$. Moreover if $s \in S - B$ and $m(s,x), m(s,y) < \infty$, then $m(s,t) = 2$ for all $t \in B$.*

Proof: The proof that there exists $r \in \overline{B} - B$ such that $\overline{\{r\}} = \overline{B}$ is the same as for Theorem 5.13. Suppose $s \in S - B$ with $m(s,x), m(s,y) < \infty$. Let $M \subset S$ be a maximal simplex containing $\{s,x,y\}$. Then there is a unique maximal simplex $M' \subset S'$ such that $\langle M\rangle$ is conjugate to $\langle M'\rangle$ by Prop. 5.12. By conjugating S', we may assume that $\langle M\rangle = \langle M'\rangle$.

Let a,b,c be the elements of B' such that $m(a,b) = 4$ and $m(b,c) = 3$. Now xy is in $[\langle B\rangle, \langle B\rangle]$, and so xy is conjugate to $(ab)^2$ by the Basic Matching Theorem and Lemma 5.14. Hence there is a $w \in W$ such that

$w(ab)^2w^{-1} \in \langle M'\rangle$. Now $\langle (ab)^2\rangle = [\langle a, b\rangle, \langle a, b\rangle]$. Let u be the shortest element of $\langle M'\rangle w\langle a, b\rangle$. Then $u\{a, b\}u^{-1} \subset M'$ by Lemma 4.18. As $m(a, b) = 4$, we deduce that $\{a, b\} \subset M'$ by Lemma 4.11. Hence $B' \subset M'$ by Lemma 5.3. Moreover $m(s', t') = 2$ for all $(s', t') \in (M' - B') \times B'$ by Theorem 5.4. Hence B' is an irreducible component of M'. Therefore $[\langle B'\rangle, \langle B'\rangle]$ is a normal subgroup of $\langle M'\rangle$.

Now M contains B and $[B, B]$ is conjugate to $[B', B']$ in $\langle M\rangle$ by the Basic Matching Theorem. Therefore $[\langle B\rangle, \langle B\rangle] = [\langle B'\rangle, \langle B'\rangle]$, since $[\langle B'\rangle, \langle B'\rangle]$ is a normal subgroup of $\langle M'\rangle$. Hence $[\langle B\rangle, \langle B\rangle]$ is a normal subgroup of $\langle M\rangle$. Then $sBs = B$ by Lemma 4.18, and $sts = t$ for all $t \in B$ by the deletion condition. □

6 Matching of Finite Irreducible Subgroups

As a reference for the automorphism groups of finite irreducible Coxeter groups, see Chapter 2 of Franzsen [6] or §6 of Franzsen and Howlett [8]. An automorphism of a Coxeter system (W, S) is called a *graph automorphism*. The graph automorphisms of (W, S) correspond to the automorphisms of the P-diagram of (W, S).

Lemma 6.1 *Let $\alpha : \mathbf{B}_n \to \mathbf{B}_n$ be an automorphism. Then there is an inner automorphism ι of $\mathbf{B}_n$ and a graph automorphism γ of $\mathbf{B}_n$ such that $\alpha|_{[\mathbf{B}_n,\mathbf{B}_n]} = \iota\gamma|_{[\mathbf{B}_n,\mathbf{B}_n]}$ with γ the identity map if n is odd.*

Proof: If n is odd, then every automorphism of $\mathbf{B}_n$ is inner. Assume that n is even. Let ψ be the automorphism of $\mathbf{B}_n$ defined by $\psi(w) = (-1)^{l(w)}w$. All the elements of $[\mathbf{B}_n, \mathbf{B}_n]$ have even length. Therefore ψ restricts to the identity on $[\mathbf{B}_n, \mathbf{B}_n]$. Now there is an inner automorphism ι of $\mathbf{B}_n$ and a graph automorphism γ of $\mathbf{B}_n$ such that $\alpha = \iota\gamma$ or $\alpha = \iota\gamma\psi$. Hence $\alpha|_{[\mathbf{B}_n,\mathbf{B}_n]} = \iota\gamma|_{[\mathbf{B}_n,\mathbf{B}_n]}$. □

Lemma 6.2 *Let $\alpha : \mathbf{C}_n \to \mathbf{C}_n$ be an automorphism. Then there is an inner automorphism ι of $\mathbf{C}_n$ such that $\alpha|_{[\mathbf{C}_n,\mathbf{C}_n]} = \iota|_{[\mathbf{C}_n,\mathbf{C}_n]}$.*

Proof: This is clear if α is inner, so suppose α is outer. Let θ be the automorphism of $\mathbf{C}_n$ defined by $\theta(c_i) = -c_i$, for $i = 1, \ldots, n-1$, and $\theta(c_n) =$

c_n. Then θ restricts to the identity on $[\mathbf{C}_n, \mathbf{C}_n]$. If n is odd, then there is an inner automorphism ι of $\mathbf{C}_n$ such that $\alpha = \iota\theta$. Hence $\alpha|_{[\mathbf{C}_n,\mathbf{C}_n]} = \iota|_{[\mathbf{C}_n,\mathbf{C}_n]}$.

Suppose now that n is even. Let ψ be the automorphism of $\mathbf{C}_n$ defined by $\psi(w) = (-1)^{l(w)}w$. All the elements of $[\mathbf{C}_n, \mathbf{C}_n]$ have even length. Therefore ψ restricts to the identity on $[\mathbf{C}_n, \mathbf{C}_n]$. Now there is an inner automorphism ι of $\mathbf{C}_n$ such that either $\alpha = \iota\theta$, $\iota\psi$, or $\iota\theta\psi$. Hence $\alpha|_{[\mathbf{C}_n,\mathbf{C}_n]} = \iota|_{[\mathbf{C}_n,\mathbf{C}_n]}$. □

Lemma 6.3 *Let $\alpha : \mathbf{F}_4 \to \mathbf{F}_4$ be an automorphism. Then there is an inner automorphism ι of $\mathbf{F}_4$ and a graph automorphism γ of $\mathbf{F}_4$ such that $\alpha|_{[\mathbf{F}_4,\mathbf{F}_4]} = \iota\gamma|_{[\mathbf{F}_4,\mathbf{F}_4]}$.*

Proof: Let f_1, f_2, f_3, f_4 be the Coxeter generators of $\mathbf{F}_4$ with $m(f_1, f_2) = 3$, $m(f_2, f_3) = 4$, and $m(f_3, f_4) = 3$. Let ψ_ℓ be the automorphism of $\mathbf{F}_4$ defined by $\psi_\ell(f_i) = -f_i$ for $i = 1, 2$ and $\psi_\ell(f_i) = f_i$ for $i = 3, 4$. Then ψ_ℓ restricts to the identity on $[\mathbf{F}_4, \mathbf{F}_4]$. Now there is an inner automorphism ι of $\mathbf{F}_4$ and a graph automorphism γ of $\mathbf{F}_4$ such that $\alpha = \iota\gamma$ or $\alpha = \iota\gamma\psi_\ell$. Hence $\alpha|_{[\mathbf{F}_4,\mathbf{F}_4]} = \iota\gamma|_{[\mathbf{F}_4,\mathbf{F}_4]}$. □

Lemma 6.4 *Let $\alpha : \mathbf{G}_4 \to \mathbf{G}_4$ be an automorphism. Then there is a reflection preserving automorphism β of $\mathbf{G}_4$ such that $\alpha|_{[\mathbf{G}_4,\mathbf{G}_4]} = \beta|_{[\mathbf{G}_4,\mathbf{G}_4]}$.*

Proof: Let g_1, g_2, g_3, g_4 be the Coxeter generators of $\mathbf{G}_4$ with $m(g_1, g_2) = 3$, $m(g_2, g_3) = 3$, and $m(g_3, g_4) = 5$. According to Franzsen [6], the group $\mathbf{G}_4$ has an outer automorphism ξ such that $\xi(g_i) = g_i$ for $i = 1, 2, 3$ and $\xi(g_4)$ is conjugate to g_4. Let ψ be the automorphism of $\mathbf{G}_4$ defined by $\psi(w) = (-1)^{l(w)}w$. All the elements of $[\mathbf{G}_4, \mathbf{G}_4]$ have even length. Therefore ψ restricts to the identity on $[\mathbf{G}_4, \mathbf{G}_4]$. Now there is an inner automorphism ι of $\mathbf{G}_4$ such that $\alpha = \beta$ or $\beta\psi$ where $\beta = \iota$ or $\iota\xi$. Hence $\alpha|_{[\mathbf{G}_4,\mathbf{G}_4]} = \beta|_{[\mathbf{G}_4,\mathbf{G}_4]}$. □

Proposition 6.5 (Franzsen and Howlett [8], Prop. 32) *Let (W, S) be a finite Coxeter system, and let α be an automorphism of W that preserves reflections. Then α maps each visual subgroup of (W, S) to a conjugate of a visual subgroup.*

Every automorphism of $\mathbf{A}_n$ is inner, except when $n = 5$. The group $\mathrm{Out}(\mathbf{A}_5)$ has order two. The outer automorphisms of $\mathbf{A}_5$ behave badly with respect to visual subgroups because of the next proposition.

Proposition 6.6 (Franzsen and Howlett [8], Prop. 35) *Let (W, S) be a finite Coxeter system of type $\mathbf{A}_5$, and let α be an automorphism of W. If there are proper subsets A, B of S and $w \in W$ such that $\alpha(\langle A\rangle) = w\langle B\rangle w^{-1}$, then α is inner.*

Lemma 6.7 *Let (W, S) be a finite irreducible Coxeter system which is not of type $\mathbf{A}_5$. Let α be an automorphism of W, and let $A \subset S$. Then there is a $B \subset S$ such that $(\langle A\rangle, A) \cong (\langle B\rangle, B)$ and the group $\alpha([\langle A\rangle, \langle A\rangle])$ is conjugate to $[\langle B\rangle, \langle B\rangle]$ in W.*

Proof: If (W, S) is of type $\mathbf{A}_n$, then α maps $\langle A\rangle$ to a conjugate of itself, since every automorphism of $\mathbf{A}_n$ is inner for all $n \neq 5$.

Suppose (W, S) is of type $\mathbf{B}_n$. By Lemma 6.1 there is an inner automorphism ι of W and a graph automorphism γ of (W, S) such that $\alpha|_{[W,W]} = \iota\gamma|_{[W,W]}$. Let $B = \gamma(A)$. Then $\alpha([\langle A\rangle, \langle A\rangle])$ is conjugate to $[\langle B\rangle, \langle B\rangle]$.

If (W, S) is of type $\mathbf{C}_n$, then α maps $[\langle A\rangle, \langle A\rangle]$ to a conjugate of itself by Lemma 6.2. If (W, S) is of type $\mathbf{D}_2(k)$, then α maps $\langle A\rangle$ to a conjugate of itself, since α preserves reflections. If (W, S) is of type $\mathbf{E}_6$ or $\mathbf{E}_7$, then α maps $\langle A\rangle$ to a conjugate of itself, since every automorphism of $\mathbf{E}_6$ or $\mathbf{E}_7$ is inner.

Suppose (W, S) is of type $\mathbf{E}_8$. Let ψ be the automorphism of W defined by $\psi(w) = (w_0)^{l(w)}w$ where w_0 is the longest element of (W, S). All the elements of $[W, W]$ have even length. Therefore ψ restricts to the identity on $[W, W]$. Now there is an inner automorphism ι of W such that $\alpha = \iota$ or $\iota\psi$. Hence $\alpha|_{[W,W]} = \iota|_{[W,W]}$. Therefore α maps $[\langle A\rangle, \langle A\rangle]$ to a conjugate of itself.

Suppose (W, S) is of type $\mathbf{F}_4$. By Lemma 6.3 there is an inner automorphism ι of W and a graph automorphism γ of (W, S) such that $\alpha|_{[W,W]} = \iota\gamma|_{[W,W]}$. Let $B = \gamma(A)$. Then $\alpha([\langle A\rangle, \langle A\rangle])$ is conjugate to $[\langle B\rangle, \langle B\rangle]$.

Suppose (W, S) is of type $\mathbf{G}_3$. Then every automorphism of W preserves reflections. Hence α maps $\langle A\rangle$ to a conjugate of itself by Prop. 6.5.

Suppose (W, S) is of type $\mathbf{G}_4$. By Lemma 6.4 there is a reflection preserving automorphism β of W such that $\alpha|_{[W,W]} = \beta|_{[W,W]}$. Therefore α maps $[\langle A\rangle, \langle A\rangle]$ to a conjugate of itself by Prop. 6.5. □

The next proposition follows easily from Lemma 6.7.

Proposition 6.8 *Let (W, S) and (W', S') be finite irreducible Coxeter systems which are not of type $\mathbf{A}_5$. Let $\alpha : W \to W'$ be an isomorphism, and* *let $A \subset S$. Then there is an $A' \subset S'$ such that $(\langle A\rangle, A) \cong (\langle A'\rangle, A')$ and the group $\alpha([\langle A\rangle, \langle A\rangle])$ is conjugate to $[\langle A'\rangle, \langle A\rangle]$ in W'.*

Lemma 6.9 *Let n be odd with $n \geq 3$, and let k be such that $3 \leq k \leq n$. Identify $\mathbf{B}_k$ with $\langle b_{n-k+1}, \ldots, b_n\rangle$ in $\mathbf{B}_n$ and $\mathbf{C}_k$ with $\langle c_{n-k+1}, \ldots, c_n\rangle$ in $\mathbf{C}_n$. Let $\phi : \mathbf{B}_n \to \mathbf{C}_n$ be a monomorphism. Then ϕ maps $[\mathbf{B}_k, \mathbf{B}_k]$ to a conjugate of $[\mathbf{C}_k, \mathbf{C}_k]$ for each $k = 3, \ldots, n$.*

Proof: This follows from the proof of Lemma 5.14, since $[\mathbf{B}_k, \mathbf{B}_k] = [\mathbf{C}_k, \mathbf{C}_k]$ for each $k = 3, \ldots, n$. □

A *subbase* of a Coxeter system (W, S) is a subset A of S such that $\langle A\rangle$ is a noncyclic, nonmaximal, finite, irreducible subgroup of (W, S).

Theorem 6.10 (Subbase Matching Theorem) *Let W be a finitely generated Coxeter group with two sets of Coxeter generators S and S'. Let A be a subbase of (W, S). Let B be a base of (W, S) containing A, and let B' be the base of (W, S') that matches B. Suppose that B is not of type $\mathbf{A}_5$ and if $|\langle B\rangle| > |\langle B'\rangle|$, suppose that A is not of type $\mathbf{C}_2$. Then B' contains a subbase A' of (W, S') such that $[\langle A\rangle, \langle A\rangle]$ is conjugate to $[\langle A'\rangle, \langle A'\rangle]$ in W. Moreover*

1. *If $|\langle A\rangle| = |\langle A'\rangle|$, then A and A' have the same type.*
2. *If $|\langle A\rangle| < |\langle A'\rangle|$, then $|\langle B\rangle| < |\langle B'\rangle|$ and A is of type $\mathbf{B}_k$ and A' is of type $\mathbf{C}_k$ for some $k \geq 3$.*
3. *If $|\langle A\rangle| > |\langle A'\rangle|$, then $|\langle B\rangle| > |\langle B'\rangle|$ and A is of type $\mathbf{C}_k$ and A' is of type $\mathbf{B}_k$ for some $k \geq 3$.*

Furthermore, if $|\langle B\rangle| > |\langle B'\rangle|$ and A is of type $\mathbf{C}_2$, then B' is of type $\mathbf{B}_{2q+1}$ for some $q \geq 1$ and $[\langle A\rangle, \langle A\rangle]$ is conjugate to $\langle xy\rangle$ in W where $\{x, y\}$ is the set of split ends of the C-diagram of $(\langle B'\rangle, B')$.

Proof: Suppose $|\langle B\rangle| = |\langle B'\rangle|$. By the Basic Matching Theorem, B and B' have the same type and there is an isomorphism $\phi : \langle B\rangle \to \langle B'\rangle$ that restricts to conjugation on $[\langle B\rangle, \langle B\rangle]$ by an element u of W. By Prop. 6.8, there is a $A' \subset B'$ such that $(\langle A\rangle, A) \cong (\langle A'\rangle, A')$ and $\phi([\langle A\rangle, \langle A\rangle])$ is conjugate to $[\langle A'\rangle, \langle A'\rangle]$ by an element v of $\langle B'\rangle$. Then A' is a subbase of (W, S') of the same type as A and $vu[\langle A\rangle, \langle A\rangle]u^{-1}v^{-1} = [\langle A'\rangle, \langle A'\rangle]$.

Now suppose $|\langle B\rangle| < |\langle B'\rangle|$. By the Basic Matching Theorem, B is of type $\mathbf{B}_{2q+1}$ and B' is of type $\mathbf{C}_{2q+1}$ for some $q \geq 1$ and there is an monomorphism $\phi : \langle B\rangle \to \langle B'\rangle$ that restricts to conjugation on $[\langle B\rangle, \langle B\rangle]$ by an element u of W. Let $\alpha : (\langle B\rangle, B) \to \mathbf{B}_{2q+1}$ and $\beta : (\langle B'\rangle, B') \to \mathbf{C}_{2q+1}$ be isomorphisms of Coxeter systems. If A is of type $\mathbf{A}_k$, we may assume, if necessary, by conjugating $\langle A\rangle$ by the longest element of $(\langle B\rangle, B)$, that $\alpha(A) \subset \langle b_1, \ldots, b_{2q}\rangle$. Now $b_i = c_i$ for $i = 1, \ldots, 2q$. If A is of type $\mathbf{A}_k$, let $A' = \beta^{-1}\alpha(A)$. Then $(\langle A\rangle, A) \cong (\langle A'\rangle, A')$. If A is of type $\mathbf{B}_k$, let $\langle A'\rangle = \beta^{-1}(\mathbf{C}_k)$ where $\mathbf{C}_k$ is as in Lemma 6.9 and $A' \subset S'$. Then A' is of type $\mathbf{C}_k$.

By the proof of Lemma 5.14, we deduce that $\beta\phi\alpha^{-1} : \mathbf{B}_{2q+1} \to \mathbf{C}_{2q+1}$ maps $[\langle\alpha(A)\rangle, \langle\alpha(A)\rangle]$ to $g[\langle\beta(A')\rangle, \langle\beta(A')\rangle]g^{-1}$ for some g in $\mathbf{C}_{2q+1}$. Let $v = \beta^{-1}(g^{-1})$. Then $\phi([\langle A\rangle, \langle A\rangle]) = v^{-1}[\langle A'\rangle, \langle A'\rangle]v$, and so

$$vu[\langle A\rangle, \langle A\rangle]u^{-1}v^{-1} = [\langle A'\rangle, \langle A'\rangle].$$

The proof of the case $|\langle B\rangle| > |\langle B'\rangle|$ is the same as for the case $|\langle B\rangle| < |\langle B'\rangle|$ with the roles of B and B' reversed.

Suppose $|\langle B\rangle| > |\langle B'\rangle|$ and A is of type $\mathbf{C}_2$. By the Basic Matching Theorem, B is of type $\mathbf{C}_{2q+1}$ and B' is of type $\mathbf{B}_{2q+1}$ for some $q \geq 1$ and there is a monomorphism $\phi : \langle B'\rangle \to \langle B\rangle$ that restricts to conjugation on $[\langle B'\rangle, \langle B'\rangle]$ by an element of W. Let $\{x, y\}$ be the set of split ends of the C-diagram of $(\langle B'\rangle, B')$. Then $[\langle A\rangle, \langle A\rangle]$ is conjugate to $\langle xy\rangle$ by Lemma 5.14.
□

Lemma 6.11 *Let (W, S) be a Coxeter system with $A, B \subset S$ such that $\langle A\rangle$ is finite and irreducible. If $\langle A\rangle$ is conjugate to $\langle B\rangle$ in W and A is neither of type $\mathbf{A}_n$, for some n, nor of type $\mathbf{B}_5$, then $A = B$.*

Proof: Suppose $s \in S - A$, with $m(s, a) > 2$ for some $a \in A$, and suppose s is A-admissible. Then $K = A \cup \{s\}$ is irreducible. By Lemma 4.11, it suffices to show that $w_K s w_K = s$. This is clear if $\langle w_K\rangle$ is the center of $\langle K\rangle$. Suppose that $Z(\langle K\rangle) = 1$. Now K is not of type $\mathbf{A}_{n+1}$ nor of type $\mathbf{E}_6$, since A is not of type $\mathbf{A}_n$ nor of type $\mathbf{B}_5$. Hence K must be of type $\mathbf{B}_{2q+1}$ for some $q \geq 2$. Then A is of type $\mathbf{B}_{2q}$ and $w_K s w_K = s$. □

Theorem 6.12 *Let W be a finitely generated Coxeter group with two sets of Coxeter generators S and S'. Let A be a subbase of (W, S), and let $A' \subset S'$.*

If A' is irreducible and $[\langle A\rangle, \langle A\rangle]$ is conjugate to $[\langle A'\rangle, \langle A'\rangle]$ in W, then A' is unique up to conjugation in W; moreover, if A' is neither of type $\mathbf{A}_n$, for some n, nor of type $\mathbf{B}_5$, then A' is unique. If A is of type $\mathbf{C}_2$ and $A' = \{x, y\}$ and $[\langle A\rangle, \langle A\rangle]$ is conjugate to $\langle xy\rangle$ in W, then A' is unique.

Proof: Suppose A', A'_1 are irreducible subsets of S' and $[\langle A\rangle, \langle A\rangle]$ is conjugate to $[\langle A'\rangle, \langle A'\rangle]$ and to $[\langle A'_1\rangle, \langle A'_1\rangle]$. Then $[\langle A'\rangle, \langle A'\rangle]$ is conjugate to $[\langle A'_1\rangle, \langle A'_1\rangle]$. Hence A' is conjugate to A'_1 by Lemma 4.18. If A' is neither of type $\mathbf{A}_n$, for some n, nor of type $\mathbf{B}_5$, then A' is unique by Lemma 6.11.

Suppose A is of type $\mathbf{C}_2$ and $A' = \{x, y\} \subset S'$ and $[\langle A\rangle, \langle A\rangle]$ is conjugate to $\langle xy\rangle$ in W. Then $m(x, y) = 2$, since $[\langle A\rangle, \langle A\rangle]$ has order 2. Let B be a base of (W, S) containing A and let B' be the base of (W, S') that matches B. Suppose $|\langle B\rangle| = |\langle B'\rangle|$. Then B' contains a subbase A'_1 of type $\mathbf{C}_2$ such that $[\langle A\rangle, \langle A\rangle]$ is conjugate to $[\langle A'_1\rangle, \langle A'_1\rangle]$ in W by Theorem 6.10. Then xy is conjugate to an element of $\langle A'_1\rangle$. Hence A' is conjugate to A'_1 by Lemma 4.17, which is a contradiction, since A' and A'_1 have different types. Therefore $|\langle B\rangle| > |\langle B'\rangle|$ by Theorem 6.10.

Now by the Basic Matching Theorem, B is of type $\mathbf{C}_{2q+1}$ and B' is of type $\mathbf{B}_{2q+1}$ for some $q \geq 1$. Let $E' = \{u, v\}$ be the set of split ends of the C-diagram of $(\langle B'\rangle, B')$. Then $[\langle A\rangle, \langle A\rangle]$ is conjugate to $\langle uv\rangle$ by Lemma 5.14. Hence uv is conjugate to xy. Therefore E' is conjugate to A' by Lemma 4.17.

Suppose $s \in S' - E'$ is E'-admissible. Then either $m(s, u) = m(s, v) = 2$ or $m(s, u) = m(s, v) = 3$ by Theorem 5.15. Let K be the component of $E' \cup \{s\}$ containing s. Then $w_K s w_K = s$. Therefore $E' = A'$ by Lemma 4.11. Thus A' is unique. □

Theorem 6.13 (Edge Matching Theorem) *Let W be a finitely generated Coxeter group with two sets of Coxeter generators S and S'. Let $E = \{a, b\}$ be an edge of the P-diagram of (W, S) with $m(a, b) \geq 4$. Then there is a unique edge $E' = \{x, y\}$ of the P-diagram of (W, S') such that $[\langle E\rangle, \langle E\rangle]$ is conjugate in W to either $[\langle E'\rangle, \langle E'\rangle]$ or $\langle xy\rangle$.*

Proof: Assume first that E is a base of (W, S). Let E' be the base of (W, S') that matches E. Then $E' = \{x, y\}$ is an edge of the P-diagram Γ' of (W, S') such that $m(x, y) \geq 3$ by Theorem 4.19. To see that E' is unique, suppose $E'_1 = \{x_1, y_1\}$ is an edge of Γ' such that $[\langle E\rangle, \langle E\rangle]$ is conjugate to either

$[\langle E_1'\rangle, \langle E_1'\rangle]$ or $\langle x_1y_1\rangle$. Then $[\langle E'\rangle, \langle E'\rangle]$ is conjugate to either $[\langle E_1'\rangle, \langle E_1'\rangle]$ or $\langle x_1y_1\rangle$. Hence E' is conjugate to E_1' by Lemma 4.17 or Lemma 4.18. Therefore $E' = E_1'$ by Lemma 4.16.

Assume now that E is a subbase of (W, S). Let B be a base of (W, S) containing E and let B' be the base of (W, S') that matches B. Assume first that if $|\langle B\rangle| > |\langle B'\rangle|$, then $m(a, b) > 4$. By Theorem 6.10, B' contains a subbase E' of (W, S') of the same type as E such that $[\langle E\rangle, \langle E\rangle]$ is conjugate to $[\langle E'\rangle, \langle E'\rangle]$. Suppose $E_1' = \{x_1, y_1\}$ is an edge of Γ' such that $[\langle E\rangle, \langle E\rangle]$ is conjugate to either $[\langle E_1'\rangle, \langle E_1'\rangle]$ or $\langle x_1y_1\rangle$. As in the previous case, E' is conjugate to E_1'. Therefore $E' = E_1'$ by Lemma 6.11.

Now assume that $|\langle B\rangle| > |\langle B'\rangle|$ and $m(a, b) = 4$. By Theorem 6.10, the base B' is of type $\mathbf{B}_{2q+1}$ for some $q \geq 1$ and $[\langle E\rangle, \langle E\rangle]$ is conjugate to $\langle xy\rangle$ where $E' = \{x, y\}$ is the set of split ends of the C-diagram of $(\langle B'\rangle, B')$. Suppose $E_1' = \{x_1, y_1\}$ is an edge of Γ' such that $[\langle E\rangle, \langle E\rangle]$ is conjugate to either $[\langle E_1'\rangle, \langle E_1'\rangle]$ or $\langle x_1y_1\rangle$. Then $\langle xy\rangle$ is conjugate to either $[\langle E_1'\rangle, \langle E_1'\rangle]$ or $\langle x_1y_1\rangle$. Hence E' is conjugate to E_1' by Lemma 4.17. Therefore $m(x_1, y_1) = 2$ and $[\langle E\rangle, \langle E\rangle]$ is conjugate to $\langle x_1y_1\rangle$. Hence $E' = E_1'$ by Theorem 6.12. □

7 Visual Graph of Groups Decompositions

Let (W, S) be a Coxeter system of finite rank. Suppose that $S_1, S_2 \subset S$, with $S = S_1 \cup S_2$, and $S_0 = S_1 \cap S_2$ are such that there is no defining relator of W (no edge of the P-diagram) between an element of $S_1 - S_0$ and $S_2 - S_0$. Then we can write W as a visual amalgamated product $W = \langle S_1\rangle *_{\langle S_0\rangle} \langle S_2\rangle$. We say that S_0 *separates* S if $S_1 - S_0 \neq \emptyset$ and $S_2 - S_0 \neq \emptyset$. The amalgamated product decomposition of W will be nontrivial if and only if S_0 separates S. If S_0 separates S, we call the triple (S_1, S_0, S_2) a *separation* of S. Note that S_0 separates S if and only if S_0 separates $\Gamma(W, S)$, that is, there are s_1, s_2 in $S - S_0$ such that every path in $\Gamma(W, S)$ from s_1 to s_2 must pass through S_0.

Let $\ell \in \langle S_0\rangle$ such that $\ell S_0 \ell^{-1} = S_0$. By Lemma 4.5, we have $S_0 = S_\bullet \cup (S_0 - S_\bullet)$ where $S_\bullet$ generates a finite group, each element of $S_\bullet$ commutes with each element of $S_0 - S_\bullet$, and ℓ is the longest element of $\langle S_\bullet\rangle$. The triple (S_1, ℓ, S_2) determines an elementary twist of (W, S) (or of its P-diagram) giving a new Coxeter generating set $S_* = S_1 \cup \ell S_2 \ell^{-1}$ of W.

In application, it is simpler to consider a more general kind of twisting. Suppose S_0 and $\bar{S}_0 \subset S_2$ generate conjugate subgroups of $\langle S_2\rangle$. Suppose

$d \in \langle S_2 \rangle$ is such that $d\bar{S}_0 d^{-1} = S_0$. Then $S_1 \cap dS_2 d^{-1} = S_0$, since

$$S_0 \subset S_1 \cap dS_2 d^{-1} \subset S_1 \cap \langle S_2 \rangle = S_0.$$

A *generalized twist* (or simply *twist*) of (W, S) in this situation gives a new Coxeter generating set $S_* = S_1 \cup dS_2 d^{-1}$ of W and a new visual amalgamated product decomposition $W = \langle S_1 \rangle *_{\langle S_0 \rangle} \langle dS_2 d^{-1} \rangle$

Elementary and generalized twists can be easily understood in terms of their effects on P-diagrams. The P-diagram of (W, S) is the union of the P-diagrams for $\langle S_1 \rangle$ and $\langle S_2 \rangle$ overlapping in the P-diagram for $\langle S_0 \rangle$. The P-diagram for (W, S_*) is obtained from the P-diagram of (W, S) by twisting the P-diagram of $\langle S_2 \rangle$, that is, removing the P-diagram for $\langle S_2 \rangle$, replacing it by the isomorphic P-diagram of $\langle dS_2 d^{-1} \rangle$, and attaching it to the P-diagram for $\langle S_1 \rangle$ along $S_0 = d\bar{S}_0 d^{-1}$. If $S_0 = \emptyset$, we call the twist *degenerate*. A degenerate twist does not change the isomorphism type of the P-diagram. This includes the case where $S_1 = S_0 = \emptyset$, $S_2 = S$, giving $S_* = dSd^{-1}$ the conjugation of S by an arbitrary $d \in W$. Any nondegenerate generalized twist of a Coxeter system (W, S) can be realized by a finite sequence of elementary twists.

Let Λ be a visual graph of groups decomposition of (W, S). Then the graph of Λ is a tree, since the abelianization of W is finite. The graph of groups decomposition Λ can be understood as a visual amalgamated product in many ways, e.g., by taking some of the vertex and edge groups to be generated by S_1, others to be generated by S_2, with the overlap being a single edge group of Λ. Hence we will also speak of twisting a visual graph of groups decomposition with respect to some such partitioning of the graph of groups and some conjugating element.

A graph of groups decomposition is said to be *reduced* if no edge group is equal to an incident vertex group. Suppose Λ is a reduced visual graph of groups decomposition of a Coxeter system (W, S) of finite rank. Suppose (for simplicity in this application) that no edge group of Λ is a proper subgroup of another edge group of Λ. Construct another visual graph of groups decomposition (though not reduced) $\tilde{\Lambda}$ as follows. The vertices of $\tilde{\Lambda}$ are of two distinct types, v-vertices and e-vertices. The v-vertices correspond to the vertices of Λ, and the e-vertices correspond to the distinct edge groups of Λ. An edge of $\tilde{\Lambda}$ will connect vertices p and q if p is a v-vertex and q is an e-vertex, and p corresponds to an endpoint of an edge of Λ with edge group corresponding to q. The vertex group of a v-vertex p of $\tilde{\Lambda}$ will be the vertex group for p in Λ. The vertex group of an e-vertex q of $\tilde{\Lambda}$ is the edge group of

Λ corresponding to q. Each edge of $\tilde{\Lambda}$, say from p to q, will have edge group equal to the edge group of Λ corresponding to the e-vertex q of that edge. The maps of edge groups into vertex groups in $\tilde{\Lambda}$ will be inclusion maps (as in visual decompositions generally, determined by which of the generators lie in each vertex and edge groups). Then by a series of reductions and expansions (inverse reductions) we can get from $\tilde{\Lambda}$ to Λ and we see that they are both visual graph of group decompositions of W (or by comparing the relations defining the fundamental groups of Λ and $\tilde{\Lambda}$). On the other hand, different reduced visual graphs of groups Λ and Λ_2 will correspond to the same $\tilde{\Lambda} = \tilde{\Lambda}_2$ provided they have the same vertex groups and edge groups, since the inclusion of edge groups into vertex groups determine the edges of $\tilde{\Lambda}$ when no edge group is a proper subgroup of another edge group. The point here is that $\tilde{\Lambda}$ provides a way of keeping track of which edge groups of Λ are equal and in which vertex groups without specifying what the subtree of edges of Λ with the same given edge group must look like, (in essence, without specifying the order of the vertex groups containing this edge group). We think of $\tilde{\Lambda}$ as a *flattened form* of Λ making uniform the relationship between vertex groups and different edge groups of Λ.

A particularly simple case is when the edge groups of Λ are all equal. Then $\tilde{\Lambda}$ has one v-vertex for each vertex p of Λ and one e-vertex q for the common edge group E with edges from q to p for each v-vertex p and with edge groups equal to E and inclusion maps into the vertex groups. The fundamental group of Λ is an amalgamated product of all the vertex groups of Λ identifying the copies of the edge group in each vertex group.

8 The Decomposition Matching Theorem

If U is a subgroup of W, write $U^* = \{wUw^{-1} : w \in W\}$ for the set of all subgroups conjugate to U in W. Write $U^* \preceq V^*$ if for some $w \in W$, $U \subseteq wVw^{-1}$ (independent of the representatives for the conjugacy classes). Clearly $\preceq$ is transitive and reflexive. Consider the conjugacy classes of a visual subgroup U and any subgroup V of a Coxeter system (W, S). If $U^* \preceq V^*$ and $V^* \preceq U^*$ then $U^* = V^*$, since if $U \subseteq wVw^{-1} \subseteq wzUz^{-1}w^{-1}$ then, since U is a visual subgroup, $wzUz^{-1}w^{-1} = U$ and U and V are conjugate by Lemma 4.3. Hence for the conjugacy classes of visual subgroups, $\preceq$ is a partial order. We say that $J \subseteq S$ is a c-minimal separating subset of generators if $\langle J\rangle^*$ is a $\preceq$-minimal element of the set of conjugacy classes of subgroups generated by separating subsets of S. Assuming there are separating subsets of S, there are finitely many since S is finite, and so there are c-minimal separating subsets of S.

Theorem 8.1 *Suppose (W, S) and (W, S') are two Coxeter systems for the same finitely generated Coxeter group. If (W, S) is complete, then (W, S') is complete; otherwise, for any given nontrivial splitting $A *_C B$ of W, there exist $S_0 \subseteq S$, $S'_0 \subseteq S'$, a visual graph of groups decomposition Λ for (W, S), and a visual graph of groups decomposition Λ' for (W, S') such that:*

1. *S_0 is a c-minimal separating subset of S, S'_0 is a c-minimal separating subset of S', with $\langle S_0\rangle^* = \langle S'_0\rangle^* \preceq C^*$;*
2. *the edge groups of Λ are conjugate to $\langle S_0\rangle$, the edge groups of Λ' are conjugate to $\langle S'_0\rangle$ (and hence are conjugate and conjugate to a subgroup of C); and*
3. *there is a 1-1 correspondence between the vertices of Λ and the vertices of Λ' such that each vertex group of Λ is conjugate to the corresponding vertex group of Λ'.*

Proof: If (W, S) is complete, then (W, S') is complete by Prop. 5.10. Suppose (W, S) is incomplete. Given a nontrivial splitting $W = A *_C B$, there is some visual splitting $W = A_1 *_{C_1} B_1$, with respect to S, with C_1 a subgroup of a conjugate of C by the visual decomposition theorem and Remark 1 in [9]. Consider the finite collection of conjugacy classes $\langle J\rangle^*$, partially ordered by $\preceq$, for subsets $J \subseteq S$ such that $\langle J\rangle^* \preceq C^*$ and there is a visual splitting $W = A_2 *_{\langle J\rangle} B_2$. Then there exists such a J with $\langle J\rangle^*$ minimal in this partial order.

Now starting with a splitting $W = A_2 *_{\langle J\rangle} B_2$ and working with respect to S', as above, there is a $J' \subseteq S'$ with $\langle J'\rangle^* \preceq \langle J\rangle^*$ and a visual splitting $W = A_3 *_{\langle J'\rangle} B_3$, with $\langle J'\rangle^*$ $\preceq$-minimal for such splitting S'-visual subgroups.

Working back again, from $W = A_3 *_{\langle J'\rangle} B_3$ and splitting visually with respect to S, there is a $J'' \subseteq S$ with $\langle J''\rangle^* \preceq \langle J'\rangle^*$ and an S-visual splitting over $\langle J''\rangle$ with $\langle J''\rangle^*$ $\preceq$-minimal. Now $\langle J''\rangle^* \preceq \langle J'\rangle^* \preceq \langle J\rangle^* \preceq C^*$ but J was taken so $\langle J\rangle^*$ was $\preceq$-minimal below C^* having an S-visual splitting over $\langle J\rangle$, hence $\langle J''\rangle^* = \langle J\rangle^*$ (but not necessarily $\langle J''\rangle = \langle J\rangle$), and so in fact $\langle J\rangle^* = \langle J'\rangle^*$ and (1) holds with $S_0 = J$ and $S'_0 = J'$.

Since S is finite and each vertex group of a reduced visual graph of groups decomposition of W is generated by a different subset of the generators, there is an obvious limit to the number of vertices in a reduced visual graph of groups decomposition of W, and in some sense, the more vertices, the finer the graph of groups decomposition. Take a reduced S-visual graph of groups decomposition Λ of W such that every edge group is conjugate to $\langle J \rangle$ and, among such, having a maximum number of vertices. By the visual decomposition theorem, take Λ' a reduced S'-visual graph of groups decomposition refining Λ, i.e., such that each vertex (edge) group of Λ' is a subgroup of a conjugate of a vertex (edge) group of Λ. Similarly, take Λ'' a reduced S-visual graph of groups decomposition of W refining Λ'. The edge groups of Λ'' are equal to conjugates of the edge groups of Λ by the c-minimality of S_0, and so are conjugate to the edge groups of Λ', and so (2) holds. We postpone the proof of (3) until after the proof of Lemma 8.3.

The following lemma characterizes the visual decomposition Λ.

Lemma 8.2 *Suppose (W, S) is a Coxeter system of finite rank and J is a c-minimal separating subset of S. Let $\mathcal{E}$ be the set of separating subsets of S that are conjugate to J in W. Let $\mathcal{V}$ be the set of all maximal subsets of S that are not separated by a set in $\mathcal{E}$. Suppose Λ is a reduced visual graph of groups decomposition of (W, S) having edge groups generated by conjugates of J (and hence elements of $\mathcal{E}$) and among such has a maximum number of vertices. Then all of the subgroups generated by sets in $\mathcal{V}$ are the vertex groups of Λ, and all of the subgroups generated by sets in $\mathcal{E}$ are the edge groups of Λ.*

Proof: All the visual conjugates of $\langle J \rangle$ are visual direct products $F \times G$ with conjugate finite factors F and the same factor G in common with all the visual conjugates of $\langle J \rangle$. If we split W by a separating visual conjugate of J, each of the other visual conjugates of J lies entirely in one of the factors of the free product with amalgamation, since the corresponding finite group F lies in one factor and the group G lies in each factor.

Assume Λ is a reduced visual graph of groups with edge groups conjugate to $\langle J \rangle$ and among such having a maximal number of vertices. Note that each edge group, and hence each vertex group, contains the common subgroup G. The graph of Λ is a tree, since the abelianization of W is finite.

Suppose L is the set of generators of a vertex group V of Λ. We claim that L is not separated by a set in $\mathcal{E}$. On the contrary, suppose L is separated by a set K in $\mathcal{E}$, say x and y are in different components of the P-diagram of $\langle L - K \rangle$. We claim that $K \subseteq L$. On the contrary, suppose $K \not\subseteq L$. Then

$L \cap K$ does not separate S, by c-minimality of J, and so there is a path in the P-diagram of (W, S) from x to y that avoids $L \cap K$. Take a path from x to y which is in a union of as few vertex groups of Λ as possible. Let V' be a vertex group of Λ containing a generator in this path not in V. Then the path passes through some edge group E of V at some first point before V' and must pass back through E at some last point, since the graph of Λ is a tree. Neither of these points is a generator of G since these all lie in $L \cap K$. Hence these points are generators in the finite factor F of E. But the P-diagram of F is complete, and so there is a short circuit of the path going from the first to the last point in F avoiding V'. We conclude instead that the path hitting the fewest vertex groups of Λ is a path in $L - K$, contradicting the assumption that K separates L. Hence $K \subseteq L$ and there is a separation (L_1, K, L_2) of L. Each edge group of Λ incident to the vertex group $V = \langle L \rangle$ is contained in either the subgroup generated by L_1 or by L_2, and so we can split V into two vertices generated by L_1 and L_2, respectively, and joined by an edge group generated by K, with each component of the rest of Λ attached to one or the other of the new vertex groups by an edge group of Λ. Neither of the new vertex groups equals an incident edge group E, since the finite Coxeter groups E/G and $\langle K \rangle / G$ have the same rank. This gives a reduced visual graph of groups decomposition over separating conjugates of J with more vertex groups, contradicting the maximality of the number of vertices in Λ. Hence L cannot be separated by a set in $\mathcal{E}$ as claimed. Clearly, every subset of S that contains L properly is separated by the set of generators of some edge group of Λ that is incident to V. Therefore L is a maximal subset of S that is not separated by a set in $\mathcal{E}$, and so $L \in \mathcal{V}$.

Now suppose $L \in \mathcal{V}$. We claim that $\langle L \rangle$ is a vertex group of Λ. Every element of L is a generator of some vertex group of Λ. Suppose $L' \subseteq L$ is a maximal subset of L contained in some vertex group of Λ. If $L - L' \neq \emptyset$, say $x \in L - L'$, then L' and x are not both contained in a vertex group of Λ. Take vertex groups V and V' of Λ, with $x \in V$ and $L' \subseteq V'$, which are closest together in the graph of Λ. Let E be an edge group of the path between V and V'. Then E is generated by a visual conjugate K of J which separates the generators in $V - E$ from those in $V' - E$, and so $K \in \mathcal{E}$. Now $x \notin E$ otherwise x would also be in a vertex group closer to V' on the path between V and V'. Likewise, $L' \not\subseteq E$ or else L' would be contained in a vertex group closer to V on a path between V and V'. But then the P-diagram of $\langle L - K \rangle$ would have at least two components, one containing x and one containing some element of $L' - K$. This contradicts the assumption that $L \in \mathcal{V}$. Instead all of L must be contained in a vertex group V of Λ. As the set of generators in V is in $\mathcal{V}$, we have that $\langle L \rangle = V$.

Finally, suppose $K \in \mathcal{E}$. Then there is a separation (S_1, K, S_2) of S. Each $L \in \mathcal{V}$ generates a vertex group of Λ but is not separated by K by our previous argument, and so each $L \in \mathcal{V}$ is contained in either S_1 or S_2. Pick vertex groups V_1 and V_2 as close together in Λ as possible such that V_1 is generated by a subset of S_1 and V_2 is generated by a subset of S_2. Then V_1, and V_2 are adjacent since every vertex group in a path between these is generated by a subset of either S_1 or S_2. Now $V_1 \cap V_2$ is an edge group E of Λ which is generated by a subset of K but not by a proper subset of K by the c-minimality of J, and so $E = \langle K \rangle$. □

The next lemma explains the relationship between the visual decompositions Λ and Λ'' of (W, S).

Lemma 8.3 *Suppose Λ and Λ'' are reduced visual graph of groups decompositions of a Coxeter system (W, S) of finite rank. Suppose the edge groups of Λ are generated by conjugates of a c-minimal separating subset J of S, and, among visual decompositions with this same conjugacy class of edge groups, Λ has a maximum number of vertex groups. Suppose each vertex and edge group of Λ'' is a subgroup of a conjugate of a vertex or edge group of Λ, respectively. Then the vertex and edge groups of Λ are equal to the vertex and edge groups of Λ'', respectively, that is, $\tilde{\Lambda} = \tilde{\Lambda}''$.*

Proof: By the last lemma, the vertex groups of Λ are determined from the set of all separating sets of generators that are conjugate to J. Each edge group of Λ'' is generated by a separating subset of S and is contained in a conjugate of a $\langle J \rangle$, and so, by the c-minimality of J, must be a conjugate of $\langle J \rangle$ and an edge group also of Λ.

Let T be the Bass-Serre tree with standard transversal T_*, corresponding to the graph of groups Λ, i.e., the vertices of T are the cosets of each vertex group of Λ and T_* consists of the cosets of each vertex group that contain the identity. A vertex group G of Λ'' stabilizes a vertex V of T, since G is a subgroup of a conjugate of a vertex group of Λ. But each generator of G also stabilizes a vertex of T_* and the geodesic path from that vertex of T_* to V. Hence G also stabilizes the vertex of T_* nearest to V. Thus each vertex group of Λ'' is actually a subgroup of a vertex group of Λ.

As the vertex groups of Λ are proper subgroups, Λ'' has at least two vertices, and each vertex group of Λ'' contains an edge group of Λ'', which is a conjugate of $\langle J\rangle$, as a proper subgroup. Hence no vertex group of Λ'' is contained in an edge group of Λ, since all the visual conjugates of $\langle J\rangle$ have the same rank. Consequently, each vertex group of Λ'' can be contained in only one vertex group of Λ, otherwise a vertex group of Λ'' would be contained in the intersection of vertex groups for two different vertices of Λ and so would be contained in each edge group for edges of Λ in the geodesic path between these vertices, which is not the case.

Summarizing, for each vertex U of Λ'', there exists a unique vertex $f(U)$ of Λ such that the vertex group $\Lambda''(U)$ of Λ'' at U is a subgroup of the vertex group $\Lambda(f(U))$ of Λ at $f(U)$. We claim that for each vertex V of Λ, the vertex group $\Lambda(V)$ is generated by the vertex groups of Λ'' for vertices in $f^{-1}(V)$. In particular, there will be at least one vertex of Λ'' in $f^{-1}(V)$, and so at least as many vertices in Λ'' as in Λ. But Λ has a maximal number of vertices for visual reduced graph of groups decompositions of (W, S) with edge groups that are conjugates of $\langle J\rangle$, so Λ'', which also satisfies these conditions, has no more vertices than Λ. Hence Λ and Λ'' have the same number of vertices; moreover, for each vertex V of Λ, we conclude that $f^{-1}(V)$ is a unique vertex of Λ'', and the vertex groups of these vertices in Λ and Λ'' must be equal. Hence the vertex and edge groups of Λ are the same as the vertex and edge groups of Λ'', respectively, and so $\tilde{\Lambda} = \tilde{\Lambda}''$.

To establish the claim that each vertex group $\Lambda(V)$ of Λ is generated by the vertex groups of Λ'' that it contains, we will show that each edge group of Λ for edges incident to V is contained in a vertex group of Λ'' which is contained in $\Lambda(V)$. A generator of $\Lambda(V)$ which is not contained in any edge group incident to V is an element of only that vertex group of Λ, but is also an element of some vertex group of Λ'' and that vertex group of Λ'' can only be contained in $\Lambda(V)$. Thus we will get that each generator of $\Lambda(V)$ is in a vertex group of Λ'' which is contained in $\Lambda(V)$.

Consider then an edge group C of an edge incident to V in Λ. Delete the edges $E_1, \dots E_n$ of the underlying tree of Λ that are incident to V with the edge group C, leaving a connected component T_0 containing V, and connected components $T_1, \dots, T_n$ with T_i containing the vertex V_i of E_i opposite V for each i. Then $W = A *_C B$ where A is the group generated by the vertex groups of the tree T_0 and B is the group generated by the vertex groups of the forest $T_1, \dots, T_n$. Neither A nor B equals C as Λ is reduced. Each vertex group of Λ'' is contained in a unique vertex group of Λ and so is contained in either A or B but not in both, since the intersection of A and B is the edge group C. There is at least one vertex group of Λ'' in each of A and B. Hence there are adjacent vertices of Λ'' having vertex groups one in A and one in B, whose intersection is the edge group of Λ'' for the edge between these vertices. But the intersection of these vertex groups of Λ'' is also contained in C. Since the edge groups of Λ and Λ'' are visual subgroups conjugate to $\langle J\rangle$, we have that C is the edge group of Λ'' for the edge between these vertices of Λ''. Hence C is contained in a vertex group of Λ'' contained in A. If this vertex group of Λ'' is contained in a vertex group in A other than V, then C would be contained in the edge groups in a geodesic path between between V and this other vertex in T_0. But the edge groups for edges incident to V in T_0 are different conjugates of $\langle J\rangle$ than C, since we deleted all edges incident to V having C as edge group. As C cannot be contained in a different conjugate of $\langle J\rangle$, instead the vertex group of Λ'' that is in A and contains C is actually contained in the vertex group $\Lambda(V)$ of V in Λ. This completes the analysis of the claim and so completes the proof of the lemma. □

We now finish the proof of Theorem 8.1. By Lemma 8.3, the vertex groups of Λ'' are in fact equal to the vertex groups of Λ. Finally we compare Λ and Λ'' with Λ'. Each vertex group G of Λ'' is a subgroup of a conjugate of a vertex group G' of Λ' which is in turn a subgroup of a conjugate of a vertex group H of Λ. But G is a vertex group of Λ and cannot be contained in a conjugate of another vertex group of Λ (since again Λ is reduced). Hence $G = H$, $G^* \preceq G'^* \preceq G^*$ so $G^* = G'^*$, and each vertex group of Λ is conjugate to a vertex group of Λ'. On the other hand, if H' is a vertex group of Λ' then H' is a subgroup of a conjugate of a vertex group H of Λ. But H is also a vertex group of Λ'', is contained in a conjugate of a vertex group G' of Λ', so $G' = H'$ is conjugate to H. Hence the vertex groups of Λ' correspond to conjugate vertex groups of Λ, as required for (3). This completes the proof of Theorem 8.1. □

Lemma 8.4 *Suppose Λ is a reduced visual graph of groups decomposition for a Coxeter system (W, S) of finite rank such that the edge groups of Λ are conjugates. Then Λ can be twisted resulting in a new generating set S_* for W and a reduced visual graph of groups decomposition Ψ such that the edge groups of Ψ are all equal.*

Proof: If not, take an example of a Λ for (W, S) and an edge group E_1, having a minimum number of edges labelled by groups different from E_1, which cannot be twisted to a Ψ with equal edge groups. Then some vertex group V contains incident edge groups E_1 and E_2 with $E_1 \neq E_2$. Let W_1 be the group generated by the vertex groups of Λ that are joined to V by a geodesic path in the underlying tree of Λ terminating in an edge incident to V labelled by E_1 and let W_2 be the group generated by the rest of the vertex groups of Λ. Then we have a free product decomposition $W = W_1 *_{E_1} W_2$ with $E_2 \subset V \subset W_2$. Let $d \in W$ be of minimal length, with respect to S, such that $E_1 = dE_2d^{-1}$. By considering the normal form for d with respect to the amalgamated product $W_1 *_{E_1} W_2$, we deduce that $d \in W_2$.

Suppose $E_1 = \langle S_0 \rangle$, $E_2 = \langle \bar{S}_0 \rangle$, $W_1 = \langle S_1 \rangle$, and $W_2 = \langle S_2 \rangle$ for S_0, $\bar{S}_0$, S_1, $S_2 \subset S$. By Lemma 4.3, we have $d\bar{S}_0d^{-1} = S_0$. Then twist the visual decomposition $\langle S_1 \rangle *_{\langle S_0 \rangle} \langle S_2 \rangle$ by conjugating the generators S_2 by d, giving $S_* = S_1 \cup dS_2d^{-1}$, and corresponding Λ_* where we conjugate each vertex and edge group of Λ with generators in S_2. An edge labelled E_1 cannot be in the twisted part of Λ since the generators S_0 would have to be contained in each vertex and edge group in a geodesic path between such an edge and an edge with label E_1 incident at V. Hence all the edges having label E_1 originally still have label E_1 in Λ_*. The edge incident to V labelled E_2 in the original Λ is conjugated by d to $dE_2d^{-1} = E_1$ and so we have at least one more edge labelled by E_1, and hence at least one fewer edge labelled by a group different from E_1. Thus Λ_* contradicts the minimality of number of edges labelled by groups different from a particular edge group assumed for Λ. Instead, twisting to reduce the number of edges labelled by a group different from a chosen edge group must eventually transform a given Λ to a graph of groups Ψ having all the same edge groups. $\square$

Lemma 8.5 *Suppose Ψ is a reduced graph of groups decomposition for a Coxeter system (W,S) of finite rank such that all of the edge groups of Ψ are equal. Suppose Ψ_0' is a similar decomposition for (W,S') such that each vertex group is conjugate to a vertex group of Ψ and the equal edge groups of Ψ_0' are conjugate to the edge groups of Ψ. Then by a sequence of twists applied to Ψ_0' there results a new set of generators S_*' and corresponding visual graph of groups Ψ' such that the vertex groups of Ψ' are equal to those of Ψ and the edge groups of Ψ' are all equal and equal to the edge groups of Ψ, and hence $\tilde{\Psi} = \tilde{\Psi}'$.*

Proof: Let $\tilde{T}$ be the Bass-Serre tree for $\tilde{\Psi}$. Then each vertex group V' of Ψ_0' stabilizes a v-vertex of $\tilde{T}$, but stabilizes at most one v-vertex since V'

cannot be a subgroup of a conjugate of an edge group of Ψ'_0, and the same is true for any Ψ'_* resulting by twists conjugating vertex groups and preserving the same edge groups from Ψ'_0. Let T_0 be the spanning tree for the v-vertices of $\tilde{T}$ that are stabilized by a vertex group of such a Ψ'_* and take Ψ'_* so that T_0 has a minimal number of vertices. The smallest T_0 can be is one v-vertex for each vertex group of Ψ'_* plus one e-vertex, corresponding to the common edge group of $\tilde{\Psi}$, connected to each of the v-vertices of T_0. In this case, conjugating Ψ'_* carries T_0 to the standard transversal T_1 of $\tilde{T}$ and so takes Ψ'_* to a Ψ' having the same vertex and edge groups as Ψ.

Suppose instead that T_0 has more than one e-vertex. Suppose further that some v-vertex wV of T_0, for V a vertex group of Ψ, stabilized by a vertex group $V' = wVw^{-1}$ of Ψ'_* has more than one edge of T_0 incident at that vertex. Let E be the common edge group of Ψ so there are e-vertices uE and vE adjacent to wV, the edge group E' of Ψ'_* is $uEu^{-1} = vEv^{-1}$, and $uv^{-1} \in V'$. Twist Ψ'_* to Ψ'_{**} by conjugating each vertex group of Ψ'_* stabilizing a v-vertex of T_0 on the vE side of wV by the element uv^{-1}. Then $uv^{-1}E'vu^{-1} = uEu^{-1} = E'$, and so edge groups have not changed.

If V'_2 is a vertex group of Ψ'_* stabilizing a v-vertex w_2V_2 on the vE side of wV, then $uv^{-1}V'_2vu^{-1}$ stabilizes $uv^{-1}w_2V_2$. If p is a geodesic path from wV to w_2V_2 in T_0, then translating p by uv^{-1} results in a path from $uv^{-1}wV = wV$ to $uv^{-1}w_2V_2$. Since the first edge in p is vE, the first edge in the translated path is uE. We conclude that the spanning tree for the v-vertices stabilized by Ψ'_{**} consists of the part of T_0 on the vE side of wV translated by uv^{-1} together with the rest of T_0. Since the e-vertex vE is carried to the e-vertex uE in the new spanning tree, there are fewer vertices in the new spanning tree, contradicting the minimality of T_0 for Ψ'_*.

Finally suppose that all the v-vertices of T_0 that are stabilized by a vertex group of Ψ'_* are the leaves of T_0 (the end points of T_0), and that T_0 has at least two e-vertices. Then T_0 has a v-vertex that is not stabilized by a vertex group of Ψ'_*. Let $\tilde{T}'$ be the Bass-Serre tree of $\tilde{\Psi}'_*$. Let $\tilde{T}'_*$ be the result of replacing, equivariantly with respect to the action of W, each translate of the standard transversal T'_1 of $\tilde{T}'$ by a copy of T_0 so that T_0 is attached by identifying each vertex of T_0 stabilized by a vertex group V' of Ψ'_* with the vertex V' of T'_1 (this vertex remains labelled V'). In particular, the e-vertices of $\tilde{T}'$ (those that are labelled by cosets of the edge group of Ψ'_*) are replaced by copies of the level one core of T_0 (the tree T_0 minus its leaves and their adjoining edges). Then W acts on the tree $\tilde{T}'_*$ translating the vertices labelled by cosets of vertex groups of Ψ'_* in the same way as in $\tilde{\Psi}'_*$.

Define a map $\tau : \tilde{T}'_* \to \tilde{T}$ by mapping the vertex V' of the attached T_0 to the vertex wV of T_0 when a vertex group V' of Ψ'_* stabilizes the vertex wV in T_0, and by mapping the translates of T_0 in $\tilde{T}'_*$ isomorphically to corresponding translates of T_0 in $\tilde{T}$ so as to make τ respect the action of W. Then τ is locally injective, since the cosets $uv'E'$ of the edge group E' in a given v-vertex uV' of $\tilde{T}'$, which is also a vertex of $\tilde{T}'_*$, correspond to the cosets $uwvE$ of the edge group E in the v-vertex $\tau(uV') = uwV$ of $\tilde{T}$ under the correspondence $v' = wvw^{-1}$, since $E' = wEw^{-1}$ and $V' = wVw^{-1}$. Hence τ is injective, since τ is a map of trees. But T_0 has an interior v-vertex $U = tV$, corresponding to a vertex group V of Ψ, which is not stabilized by a vertex group of Ψ'_*. Now $U = \tau(U')$ where U' is an interior v-vertex of the attached T_0 in $\tilde{T}'_*$. Let $w^{-1}V'$ be the v-vertex of $\tilde{T}'$ stabilized by V. Then $w^{-1}V'w = V$, and so $V' = wVw^{-1}$. Hence the vertex group V' of Ψ'_* stabilizes the vertex wV in T_0, and so $\tau(tw^{-1}V') = tw^{-1}wV = tV = U$. As $tw^{-1}V' \neq U'$ in $\tilde{T}'_*$, we have a contradiction to τ being injective. $\square$

Applying these lemmas to the result of the last theorem we have the following conclusion.

Theorem 8.6 (The Decomposition Matching Theorem) *Suppose (W, S) and (W, S') are Coxeter systems for the same finitely generated Coxeter group and W has a nontrivial splitting as $A *_C B$. Then there are sequences of twists applied to (W, S) and (W, S') giving rise to Coxeter systems (W, S_*) and (W, S'_*), respectively, such that there exists a nontrivial reduced visual graph of groups decomposition Ψ of (W, S_*) and a nontrivial reduced visual graph of groups decomposition Ψ' of (W, S'_*) having the same graphs and the same vertex and edge groups and all edge groups equal and a subgroup of a conjugate of C.*

Proof: Take Λ and Λ' from Theorem 8.1, twist Λ to get a visual decomposition Ψ of (W, S_*) with one edge group and twist Λ' to get a visual decomposition Ψ'_0 with one edge group. Then twist Ψ'_0 to a visual decomposition Ψ'_1 of (W, S'_*) having the same vertex and edge groups as Ψ, so $\tilde{\Psi} = \tilde{\Psi}'$. Now Ψ and Ψ'_1 only differ by expansions and contractions rearranging the edge group attachments to vertex groups. So there is a visual decomposition Ψ' of (W, S'_*) with the same graph of groups structure as Ψ. $\square$

9 The Simplex Matching Theorem

The next lemma is known to experts. For a proof see Paris [11].

Lemma 9.1 *Let W be a finitely generated Coxeter group with two complete Coxeter systems (W, S) and (W, S'). Let*

$$(W, S) = (W_0, S_0) \times (W_1, S_1) \times \cdots \times (W_k, S_k)$$

with (W_0, S_0) finite and (W_i, S_i) infinite and irreducible for each $i = 1, \ldots, k$. Suppose

$$(W, S') = (W'_0, S'_0) \times (W'_1, S'_1) \times \cdots \times (W'_\ell, S'_\ell)$$

with (W'_0, S'_0) finite and (W'_j, S'_j) infinite and irreducible for each $j = 1, \ldots, \ell$. Then $W_0 = W'_0$. Let $Z = Z(W_0)$. Then $k = \ell$ and after reindexing we have $ZW_i = ZW'_i$ for each $i = 1, \ldots, k$.

Lemma 9.2 *Let W be a finitely generated Coxeter group with two Coxeter systems (W, S) and (W, S'). Let $S_1 \subset S$ and $S'_1 \subset S'$ and suppose that $W_1 = \langle S_1 \rangle = \langle S'_1 \rangle$. If the basic subgroups of (W, S) isomorphically match basic subgroups of (W, S'), then the basic subgroups of (W_1, S_1) isomorphically match basic subgroups of (W_1, S'_1).*

Proof: On the contrary, suppose $(\langle B_1 \rangle, B_1)$ is a basic subgroup of (W_1, S_1) that matches with a nonisomorphic basic subgroup $(\langle B'_1 \rangle, B'_1)$ of (W_1, S'_1). Without loss of generality, we may assume that $|\langle B_1 \rangle| > |\langle B'_1 \rangle|$. Then either B_1 is of type $\mathbf{C}_{2q+1}$ and B'_1 is of type $\mathbf{B}_{2q+1}$ for some $q \geq 1$ or B_1 is of type $\mathbf{D}_2(4q+2)$ and B'_1 is of type $\mathbf{D}_2(2q+1)$ for some $q \geq 1$. Let B be the base of (W, S) containing B_1 and let B' be the base of (W, S') matching B. Then B is not of type $\mathbf{A}_5$ and B_1 is not of type $\mathbf{C}_2$. By Theorem 6.10, there is a $B''_1 \subset B'$ such that $(\langle B''_1 \rangle, B''_1)$ is a finite irreducible subgroup of (W, S') and $[\langle B_1 \rangle, \langle B_1 \rangle]$ is conjugate to $[\langle B''_1 \rangle, \langle B''_1 \rangle]$ in W; moreover $(\langle B_1 \rangle, B_1) \cong (\langle B''_1 \rangle, B''_1)$, since $(\langle B \rangle, B) \cong (\langle B' \rangle, B')$. Now B'_1 is conjugate to B''_1 in W by Theorem 6.12. Therefore $|\langle B_1 \rangle| = |\langle B''_1 \rangle| = |\langle B'_1 \rangle|$ which is a contradiction. □

Theorem 9.3 (The Simplex Matching Theorem) *Let (W, S) and (W, S') be finite Coxeter systems with isomorphic matching basic subgroups. Then (W, S) and (W, S') have the same number of visual subgroups of each complete system isomorphism type. In particular, $|S| = |S'|$.*

Proof: The proof is by induction on $|S|$. This is clear if $|S| = 1$, so assume $|S| > 1$ and the theorem is true for all Coxeter systems with fewer generators than $|S|$. Assume first that (W, S) is complete. Then (W, S') is complete by Prop. 5.10. Let

$$(W, S) = (W_1, S_1) \times \cdots \times (W_n, S_n)$$

be the factorization of (W, S) into irreducible factors and suppose (W_i, S_i) is finite if and only if $i \leq k$. Let

$$(W, S') = (W_1', S_1') \times \cdots \times (W_m', S_m')$$

be the factorization of (W, S') into irreducible factors and suppose (W_i, S_i') is finite if and only if $i \leq \ell$. By Lemma 9.1, we have

$$(W_1, S_1) \times \cdots \times (W_k, S_k) = (W_1', S_1') \times \cdots \times (W_\ell', S_\ell').$$

By the Matching Theorem for systems of a finite Coxeter group, we can reindex so that W_i is noncyclic if and only if $i \leq p$ and W_j' is noncyclic if and only if $j \leq p$ and $[W_i, W_i] = [W_i', W_i']$ for each $i \leq p$. By hypothesis, $(W_i, S_i) \cong (W_i', S_i')$ for each $i \leq p$. As the remaining finite factors have order 2, we have $k = \ell$ and $(W_i, S_i) \cong (W_i, S_i')$ for $p < i \leq k$.

By quotienting out the finite normal subgroup $(W_1, S_1) \times \cdots \times (W_k, S_k)$, we may assume that W_i and W_j' are infinite for each i and j. By Lemma 9.1, we have that $m = n$ and after reindexing $W_i = W_i'$ for each i. Hence we may assume that W is infinite and (W, S) and (W, S') are irreducible. By Theorem 2.4, we have that $(W, S) \cong (W, S')$. Thus in general $(W, S) \cong (W, S')$ when (W, S) is complete.

Now assume (W, S) is incomplete. Then there are a, b in S such that $m(a, b) = \infty$. Hence

$$W = \langle S - \{a\} \rangle *_{\langle S - \{a,b\} \rangle} \langle S - \{b\} \rangle$$

is a nontrivial visual amalgamated decomposition. By the Decomposition Matching Theorem, Theorem 8.6, there exist four nontrivial reduced visual graph of group decomposition of W, visual with respect to different sets of generators, a Λ with respect to S, a Λ' with respect to S', a Ψ with respect to another set of Coxeter generators R of W, and a Ψ' with respect to another set of Coxeter generators R' of W such that (1) the edge groups of Λ and Λ' are all conjugate and conjugate to a subgroup of $\langle S - \{a, b\} \rangle$; (2) there is a

1-1 correspondence between the vertices of Λ and the vertices of Λ' such that each vertex group of Λ is conjugate to the corresponding vertex group of Λ'; (3) Ψ is a twisted form of Λ having all edge groups equal and conjugate to the edge groups of Λ, and having vertices in a 1-1 correspondence with those of Λ such that each vertex group of Ψ is conjugate to the corresponding vertex group of Λ, and Ψ' is similarly a twisted form of Λ'; (4) Ψ' is the same graph of groups as Ψ and differs from Ψ only in being a visual graph of groups decomposition of W with respect to a different set of Coxeter generators.

The Coxeter systems (W, R) and (W, S) are twist equivalent and so have the same number of visual subgroups of each complete system isomorphism type. Moreover (W, R) and (W, S) have isomorphic matching basic subgroups. Likewise the Coxeter systems (W, R') and (W, S') have the same number of visual subgroups of each complete system isomorphism type, and (W, R') and (W, S') have isomorphic matching basic subgroups.

Let $\{(W_i, R_i)\}_{i=1}^k$ be the Coxeter systems of the vertex groups of Ψ, and let (W_0, R_0) be the Coxeter system of the edge group of Ψ. Then $k \geq 2$, $R = \cup_{i=1}^k R_i$, and $\cap_{i=1}^k R_i = R_0$, and $R_i - R_0 \neq \emptyset$ for each $i > 0$, and $m(a, b) = \infty$ for each a in $R_i - R_0$ and b in $R_j - R_0$ with $i \neq j$. Let $\{(W_i', R_i')\}_{i=1}^k$ be the Coxeter systems of the vertex groups of Ψ' indexed so that $W_i' = W_i$ for each i, and let (W_0, R_0') be the Coxeter system of the edge group of Ψ'. Then $W_0' = W_0$, $R' = \cup_{i=1}^k R_i'$, and $\cap_{i=1}^k R_i' = R_0'$, and $R_i' - R_0 \neq \emptyset$ for each $i > 0$, and $m(a', b') = \infty$ for each a' in $R_i' - R_0'$ and b' in $R_j' - R_0'$ with $i \neq j$. Moreover (W_i, R_i) and (W_i, R_i') have isomorphic matching basic subgroups for each i by Lemma 9.2.

Let $\mathcal{C}$ be a complete system isomorphism type and let $\mathcal{C}(S)$ be the number of visual subgroups of (W, S) of isomorphism type $\mathcal{C}$. By the induction hypothesis, $\mathcal{C}(R_i) = \mathcal{C}(R_i')$ for each i. Observe that

$$\begin{aligned}
\mathcal{C}(S) &= \mathcal{C}(R) \\
&= \sum_{i=1}^k \mathcal{C}(R_i) - (k-1)\mathcal{C}(R_0) \\
&= \sum_{i=1}^k \mathcal{C}(R_i') - (k-1)\mathcal{C}(R_0') \\
&= \mathcal{C}(R') \quad = \quad \mathcal{C}(S'),
\end{aligned}$$

which completes the induction. □

10 The Maximum Rank of a Coxeter Group

In this section we describe an algorithm for constructing a Coxeter system of maximum rank for a finitely generated Coxeter group. Let (W, S) be a Coxeter system of finite rank. We say that (W, S) can be *blown up along a base* B if (W, S) and B satisfy the hypothesis of either Theorem 5.6 or 5.9. If (W, S) can be blown up along a base B, then we can blow up (W, S) to a Coxeter system (W, S') as in the statement of Theorem 5.6 or 5.9 such that $|S'| = |S| + 1$, the base B matches a base B' of (W, S') with $|\langle B\rangle| > |\langle B'\rangle|$, and each other base C of (W, S) is also a base of (W, S'). We say that (W, S') is obtained by *blowing up* (W, S) *along the base* B.

By the process of blowing up along a base, we can effectively construct a sequence $S = S^{(0)}, S^{(1)}, \ldots, S^{(\ell)}$ of Coxeter generators of W such that $(W, S^{(i+1)})$ is obtained by blowing up $(W, S^{(i)})$ along a base for each $i = 0, \ldots, \ell - 1$ and $(W, S^{(\ell)})$ cannot be blown up along a base. The sequence terminates since the sum of the orders of the basic subgroups decreases at each step of the sequence. By the next theorem, the system $(W, S^{(\ell)})$ has maximum rank over all Coxeter systems for W.

Theorem 10.1 (The Maximum Rank Theorem) *Let (W, S) be a Coxeter system of finite rank. Then the following are equivalent:*

1. *We have $|S| \geq |S'|$ for every set of Coxeter generators S' of W.*
2. *Each base B of (W, S) matches a base B' of (W, S') with $|\langle B\rangle| \leq |\langle B'\rangle|$ for every set of Coxeter generators S' of W.*
3. *The system (W, S) cannot be blown up along a base.*

Proof: Suppose that $|S| \geq |S'|$ for every set of Coxeter generators S' of W and on the contrary, a base B of (W, S) matches a base B' of (W, S') with $|\langle B\rangle| > |\langle B'\rangle|$. By the Basic Matching Theorem either B is of type $\mathbf{C}_{2q+1}$ and B' is of type $\mathbf{B}_{2q+1}$ for some $q \geq 1$ or B is of type $\mathbf{D}_2(4q+2)$ and B' is of type $\mathbf{D}_2(2q+1)$ for some $q \geq 1$. By Theorems 5.7 and 5.8, we have that (W, S) and B satisfy the hypothesis of Theorem 5.6 or 5.9. Therefore (W, S) can be blown up along B to obtain a system (W, S') with $|S'| = |S| + 1$ contrary to the maximality of $|S|$. Therefore (1) implies (2).

Suppose that each base B of (W, S) matches a base B' of (W, S') with $|\langle B\rangle| \leq |\langle B'\rangle|$ for every set of Coxeter generators S' of W. If $|\langle B\rangle| = |\langle B'\rangle|$

for every base B of (W,S), then (W,S) and (W,S') have isomorphic matching basic subgroups by the Basic Matching Theorem, and so $|S| = |S'|$ by the Simplex Matching Theorem 9.3.

Suppose a base B of (W,S) matches a base B' of (W,S') with $|\langle B\rangle| < |\langle B'\rangle|$. By the Basic Matching Theorem either B' is of type $\mathbf{C}_{2q+1}$ and B is of type $\mathbf{B}_{2q+1}$ for some $q \geq 1$ or B' is of type $\mathbf{D}_2(4q+2)$ and B is of type $\mathbf{D}_2(2q+1)$ for some $q \geq 1$. By Theorems 5.7 and 5.8, we have that (W,S') and B' satisfy the hypothesis of Theorem 5.6 or 5.9, and so (W,S') can be blown up along B' to obtain a system (W,S'') with $|S''| = |S'| + 1$ such that B' matches a base B'' of (W,S'') with $|\langle B'\rangle| > |\langle B''\rangle| = |\langle B\rangle|$ and for each other base C' of (W,S') the base C' is a base of (W,S'').

If (W,S) and (W,S'') do not have isomorphic matching basic subgroups, we can blow up (W,S'') along a base. Continuing in this way, we obtain a sequence of Coxeter generators $S' = S^{(1)},\ldots,S^{(\ell)}$ of W such that $(W,S^{(i+1)})$ is obtained by blowing up $(W,S^{(i)})$ along a base for each $i = 1,\ldots,\ell-1$ and (W,S) and $(W,S^{(\ell)})$ have isomorphic matching basic subgroups. In particular, $|S^{(i+1)}| = |S^{(i)}| + 1$ for each $i = 1,\ldots,\ell-1$. By the Simplex Matching Theorem, $|S| = |S^{(\ell)}|$, and so $|S| > |S'|$. Thus (2) implies (1).

Finally (2) and (3) are equivalent by the Basic Matching Theorem and Theorems 5.7 and 5.8. □

We end our paper with the following theorem that says that any two Coxeter systems of maximum rank for a finitely generated Coxeter group have a lot in common.

Theorem 10.2 (The Maximum Rank Simplex Matching Theorem) *Suppose W is a finitely generated Coxeter group and S and S' are Coxeter generators of W of maximum rank. Then (W,S) and (W,S') have the same number of visual subgroups of each complete system isomorphism type.*

Proof: By the Maximum Rank Theorem, each base B of (W,S) matches a base B' of (W,S') with $|\langle B\rangle| = |\langle B'\rangle|$. Therefore (W,S) and (W,S') have isomorphic matching basic subgroups by the Basic Matching Theorem. Hence (W,S) and (W,S') have the same number of visual subgroups of each complete system isomorphism type by the Simplex Matching Theorem. □

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