GEODESICALLY TRACKING QUASI-GEODESIC PATHS FOR COXETER GROUPS

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Abstract. If Λ is the Cayley graph of a Gromov hyperbolic group, then it is a fundamental fact that quasi-geodesics in Λ are tracked by geodesics. Let \((W, S)\) be a finitely generated Coxeter system and Λ be the Cayley graph of \((W, S)\). For general Coxeter groups, not all quasi-geodesic rays in Λ are tracked by geodesics. In this paper we classify the Λ-quasi-geodesic rays that are tracked by geodesics. As corollaries we show that if \(W\) acts geometrically on a CAT(0) space \(X\), then CAT(0) geodesics in \(X\) are tracked by Cayley graph geodesics (taking the Cayley graph as equivariantly placed in \(X\)) and for any \(A \subset S\), the special subgroup \(\langle A \rangle\) is quasi-convex in \(X\). We also show that if \(g\) is an element of infinite order for \((W, S)\) then the subgroup \(\langle g \rangle\) is tracked by a Cayley geodesic in Λ (in analogy with the corresponding result for word hyperbolic groups).

1. Introduction

Suppose \(G\) is a group with finite generating set \(A\), and \(\Lambda = \Lambda(\Gamma, A)\) is the Cayley graph of \(\Gamma\) with respect to \(A\). If \(\Gamma\) is word hyperbolic then any quasi-geodesic in \(\Lambda\) is tracked by a geodesic (see [Sh]). The corresponding result for CAT(0) groups is not true. Our main goal in this paper is to classify the quasi-geodesics in the Cayley graph of a finitely generated Coxeter system that are tracked by geodesics. We define a “bracket number” for a Cayley path in terms of the wall crossings of the path and our main theorem is that a quasi-geodesic ray or line is tracked by a geodesic iff the bracket number of the ray or line is bounded. Our principal corollary to this theorem states that if \((W, S)\) is a finitely generated Coxeter system, and \(W\) acts geometrically on a CAT(0) space \(X\), then the CAT(0) geodesics of \(X\) are tracked by \((W, S)\) Cayley geodesics in \(X\). The corresponding result is not true, even for CAT(0) groups that embed as subgroups of finite index in Coxeter groups (see remark 6.5). If \(X\) is the Davis complex for \((W, S)\) or even if \(W\) acts as a reflection group on \(X\), the proof of the corollary is straightforward. Unfortunately, the reflection group argument has no analogue when \(W\) does not act as a reflection group on \(X\). The principal corollary directly implies that if \(A \subset S\) then the special subgroup \(\langle A \rangle\) is quasi-convex in \(X\).

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If a group $G$ acts geometrically on a CAT(0) space $X$ and one is interested in the asymptotic properties of $X$, it is a considerable advantage to know that CAT(0) geodesics in $X$ are tracked by Cayley geodesics. Clearly, the algebraic properties of $G$ are far more apparent in Cayley geodesics than in CAT(0) geodesics. This theme is highlighted in [MRT] where local connectivity of boundaries of right angled Coxeter groups are analyzed.

The work of B. Bowditch and G. Swarup (see [S]) imply that 1-ended word hyperbolic groups have locally connected boundary. One can easily see from our tracking results that any 1-ended hyperbolic Coxeter group has locally connected boundary.

2. Coxeter Preliminaries

We use M. Davis’ book [D] as a general Coxeter group reference for this section. A Coxeter system is a pair $(W,S)$ where $S$ is a generating set for the group $W$ and $W$ has presentation

$$(S : (s_is_j)^{m(i,j)} \text{ for all } s_i, s_j \in S)$$

where $m(i,j) \in \{1, 2, \ldots, \infty\}$, $m(i,j) = 1$ iff $i = j$ (so all generators are order 2) and $m(i,j) = m(j,i)$. If $m(i,j) = \infty$, the element $s_is_j$ is of infinite order (and the relation $(s_is_j)^\infty$ is left out of the presentation).

In the Cayley graph $\Lambda = \Lambda(W,S)$, for $w \in W$ and $s \in S$ there are formally two edges from $w$ to $ws = ws^{-1}$, one labeled by the letter $s$ and one labeled by the letter $s^{-1}$, but as every generator has order two, every edge is doubled. Ordinarily, it is just as easy to identify the letters $s$ and $s^{-1}$ and take a single edge from $w$ to $ws$ corresponding to the twin edges, and we will do so here. The Cayley graph is viewed as a 1-complex with unit length intervals for edges. An edge path in $\Lambda$ is a continuous map $\beta : [0,n] \rightarrow \Lambda$ such that $n \in \mathbb{Z}^+$ and for each non-negative integer $k < n$, $\beta$ maps the interval $[k, k+1]$ isometrically to an edge of $\Lambda$. Thus an edge path is determined by the sequence of its vertices, each adjacent to the previous, or alternatively, is determined by its starting point and the word of labels of its edges. Then $\Lambda$ is taken with edge path metric and the distance between vertices $v$ and $vw$ is the length of the shortest word representing $w$ (the length of a shortest path having label $w$). Similarly, for $\beta$ satisfying the corresponding condition, if $\beta : [0,\infty) \rightarrow \Lambda$, then $\beta$ is called a ray and, if $\beta : (-\infty, \infty) \rightarrow \Lambda$ then $\beta$ is called a line. An edge path, ray, or line $\beta$ is a geodesic if $\beta$ is an isometry.

A reflection in $W$ is a conjugate of an element of $S$. If $w \in W$ and $s \in S$, then the edge from $w$ to $ws$ is mapped to itself by the reflection $wsw^{-1}$, so that the vertices $w$ and $ws$ are interchanged, i.e., the edge is reflected across its midpoint. The set of those edges in $\Lambda$ which are mapped to themselves in this way by some particular reflection $r$ is called a wall of $\Lambda$. The walls of $\Lambda$ partition the edges of $\Lambda$ into disjoint sets. Notationally, we write a wall $Q$ as $[e]$ where $e$ is any edge of the wall $Q$ and we define $Q$ to be the union of the edges of $Q$ in $\Lambda$. An edge $e$ with label $t \in S$ belongs to a wall $Q$ corresponding to the reflection $wsw^{-1}$ iff a vertex of $e$ is $wq$ where
\(qtq^{-1} = s\). An edge path \(\beta\) containing more than one edge from a wall cannot be geodesic, for if \(e\) and \(d\) are edges in \(\beta\) (with \(e\) before \(d\), say) both in the wall given by a reflection \(r\), and \(\beta'\) is the segment of \(\beta\) from the terminal point of \(e\) to the initial point of \(d\), then \(r\beta'\) is a path from the initial point of \(e\) to the terminal point of \(d\) and replacing \((e, \beta', d)\) in \(\beta\) by \(r\beta'\) shortens \(\beta\). The converse of this observation is essentially the deletion condition for Coxeter groups; if \(\beta\) is not geodesic, then there are edges \(e\) and \(d\) in \(\beta\) belonging to the same wall. Replacing \((e, \beta', d)\) with \(r\beta'\) corresponds to deletion of the letters labeling edges \(e\) and \(d\) in the label of \(\beta\). Moreover, in this case, we can always take a \(e\) and \(d\) in a wall where the intervening segment \(\beta'\) in \(\beta\) is a geodesic which does not cross this wall. The closure of the complement of a wall in \(\Lambda\) has exactly two components (which are interchanged by the reflection) called the sides of the wall. Two walls are parallel if all edges of one are on the same side of the other. If two walls are not parallel, then they cross. The following theorem due to B. Brink and R. Howlett (see theorem 2.8 of [BrH]) is a fundamental result concerning the wall structure of \(\Lambda\).

**Theorem 2.1. (Parallel Wall theorem)** Suppose \((W, S)\) is a finitely generated Coxeter system and \(\Lambda\) is the Cayley graph of \(W\) with respect to \(S\). For each positive integer \(n\) there is a constant \(P(n)\) such that the following holds: given a wall \(Q\) and a point \(p\) in \(\Lambda\), if the distance from \(p\) to \(Q\) is at least \(P(n)\), then there exist \(n\) distinct pairwise parallel walls which separate \(Q\) from \(p\).

For a path \(\beta\) in \(\Lambda\) and vertex \(t\) of \(\beta\) let the bracket number of \(t\) in \(\beta\) be the number of walls \(Q\) such that there is an edge of \(Q\) on either side of \(t\) in \(\beta\). Denote the bracket number of \(t\) in \(\beta\) as \(B(t, \beta)\). If \(\tau\) is a subpath of \(\beta\) the bracket number of \(\tau\) in \(\beta\) is the maximum of the numbers \(B(t, \beta)\) for all vertices \(t\) of \(\tau\). Denote this number \(B(\tau, \beta)\). Call \(B(\beta) \equiv B(\beta, \beta)\) the bracket number of \(\beta\).

### 3. Wall Computations

If \(\alpha\) is an edge path in the Cayley graph \(\Lambda\) having consecutive vertices \(a = v_0, v_1, \ldots, v_n = b\), then an \(L\)-approximation to \(\alpha\) is an edge path \(\beta\) in \(\Lambda\) connecting \(a\) and \(b\) of the form \(\beta = (\beta_1, \ldots, \beta_n)\) where each \(\beta_i\) is a geodesic connecting \(w_{i-1}\) to \(w_i\) for a sequence of vertices \(w_i\) each within \(L\) of the corresponding \(v_i\). The points \(w_i\) are called approximation points.

**Lemma 3.1.** Suppose \((W, S)\) is a finitely generated Coxeter system with Cayley graph \(\Lambda\). Then there is a function \(f\) such that, for any edge path \(\alpha\) in \(\Lambda\) and \(\beta\) an \(L\)-approximation of \(\alpha\), the bracket number \(B(\beta)\) is at most \(f(B(\alpha), L)\) (that is, \(B(\beta)\) is bounded by a constant depending only on \(B(\alpha)\) and \(L\) and is otherwise independent of the particular \(\alpha\)).

**Proof.** Let the consecutive vertices of \(\alpha\) be \(a = v_0, v_1, \ldots, v_n = b\), the approximation vertices of \(\beta\) be \(a = w_0, w_1, \ldots, w_m = b\) (so that \(d(w_1, v_i) \leq L\))
for all \( i \) and \( \beta_i \) be the geodesic subpath of \( \beta \) connecting \( w_{i-1} \) to \( w_i \), so \( \beta = (\beta_1, \ldots, \beta_m) \). If \( x \) is a vertex of \( \beta_i \) and \( B(x, \beta) \) is “large”, then (as each edge belongs to exactly one wall) there is a wall \( Q \) that brackets \( x \) on \( \beta \) that is “far” from \( x \) and hence far from \( v_i \). Hence it suffices to bound the distance between \( v_i \) and a wall \( Q \) that brackets \( x \) on \( \beta \). The Parallel Wall theorem implies this distance is large iff there is a large set \( Q \) of (mutually parallel) walls that separate \( Q \) from \( v_i \), so it suffices to bound the size of the set \( Q \) of walls that separate \( Q \) from \( v_i \). Say \( j < i < k \) such that \( e_j \) and \( e_k \) are edges of \( \beta_j \) and \( \beta_k \) respectively, and each of \( e_j, e_k \) belongs to the wall \( Q \). (See figure 1.)

![Figure 1](image-url)

A path \( \delta_i \), that begins at the end point of \( e_j \) follows \( \beta_j \) to \( w_j \) and then travels geodesically from \( w_j \) to \( v_j \) has length \( \leq 3L \). If \( \alpha_{j,i} \) is the subpath of \( \alpha \) from \( v_j \) to \( v_i \), then the path \( (\delta_j, \alpha_{j,i}) \) must cross each wall of \( Q \). Similarly define a path from \( e_k \) to \( v_i \) (which also crosses each wall of \( Q \)). Then at most \( 6L \) walls of \( Q \) do not bracket \( v_i \) on \( \alpha \). This bounds the size of \( Q \) by \( 6L + B(\alpha) \).

**Lemma 3.2.** Suppose \((W,S)\) is a Coxeter system and \( \alpha = (e_1, \ldots, e_n) \) is a geodesic edge path connecting vertices \( a \) and \( b \) in \( \Lambda(W,S) \) such that \( \alpha \) does not cross the wall \( Q \). If \( e_0 \) is an edge at \( a \) and \( e_{n+1} \) an edge at \( b \) such that \( e_0 \) and \( e_{n+1} \) belong to the wall \( Q \) then each vertex of \( \alpha \) is within \( P(1) \) of \( Q \) (where \( P \) is the function of theorem 2.1). In particular, if \( v \) is a vertex of \( \alpha \) and \( v' \) the reflection of \( v \) across \( Q \) then \( d(v, v') \leq 2P(1) + 1 \).

**Proof.** Otherwise, there is a wall \( Q' \) separating a vertex \( v \) of \( \alpha \) from \( Q \). Hence there is an edge of \( \alpha \) between \( a \) and \( v \) that belongs to \( Q' \) and an edge of \( \alpha \) between \( v \) and \( b \) that belongs to \( Q' \). This is impossible as \( \alpha \) is geodesic.

**Proposition 3.3.** Suppose \((W,S)\) is a Coxeter system and \( \alpha \) is an edge path of \( \Lambda(W,S) \) connecting \( a \) and \( b \). Then there is an \( L \)-approximation \( \beta \) to \( \alpha \) such that each vertex of \( \beta \) is on some geodesic connecting \( a \) and \( b \) and such that \( L \leq (2P(1) + 1)B(\alpha) \).
The segment of \((\alpha_i, \beta_i)\) between \(e\) and \(d\) is geodesic. Considering the reflection of this segment across the wall containing \(e\) and \(d\) (equivalently, delete \(e\) and \(d\) from \((\alpha_i, \beta_i))\). Then we see that \(v'_i\), the reflection of \(v_i\), is within \(2P(1)+1\) of \(v_i\) (lemma \(3.2\), and the distance from \(v'_i\) to \(a\) (respectively \(b\)) is less than that of \(v_i\) to \(a\) (respectively \(b\)). Hence \(\frac{1}{2}(d(a, v'_i) + d(v'_i, b) - d(a, b)) < N_i\) and a geodesic from \(a\) to \(v'_i\) followed by a geodesic from \(v'_i\) to \(b\) crosses at most \(N_i-1\) walls twice. Continuing as above at most \(N_i(\leq B(\alpha))\) such reflections are needed to move \(v_i\) to a point \(w_i\) on a geodesic between \(a\) and \(b\), and so \(d(w_i, v_i) \leq (2P(1) + 1)B(\alpha)\).

It remains to see that each vertex of a geodesic connecting \(w_i\) and \(w_{i+1}\) belongs to a geodesic connecting \(a\) and \(b\). Consider the edge path \((\delta_i, \beta_i, \gamma_i)\) where \(\delta_i\) is a geodesic connecting \(a\) to \(w_i\), \(\beta_i\) is a geodesic connecting \(w_i\) to \(w_{i+1}\) and \(\gamma_i\) is a geodesic connecting \(w_{i+1}\) to \(b\). The paths \(\delta_i\) and \(\gamma_i\) only cross walls crossed by some (equivalently any) geodesic connecting \(a\) to \(b\). If a vertex \(v\) of \(\beta_i\) is not on a geodesic connecting \(a\) and \(b\) then there is a wall \(R\) separating \(v\) from some (equivalently every) geodesic connecting \(a\) and \(b\). As \(R\) separates \(v\) from \(a\), and \(\delta_i\) does not cross \(R\), \(\beta_i\) must cross \(R\) between \(w_i\) and \(v\). Similarly \(\beta_i\) must cross \(R\) between \(v\) and \(w_{i+1}\). This is impossible as \(\beta_i\) is geodesic.

If \(\gamma\) is an edge path in \(\Lambda\) connecting the vertices \(a\) and \(b\), then each wall separating \(a\) and \(b\) is crossed an odd number of times by \(\gamma\) and each wall not separating \(a\) and \(b\) is crossed an even number of times by \(\gamma\). If \(\alpha\) is a geodesic connecting \(a\) and \(b\) then the walls separating \(a\) and \(b\) are the walls determined by the edges of \(\alpha\), so the walls separating \(a\) and \(b\) are in 1-1 correspondence with the edges of some (any) geodesic connecting \(a\) and \(b\). The following observations are straightforward.
Lemma 3.4. Suppose $\beta$ is an edge path in $\Lambda$ connecting the vertices $a$ and $b$ then the following are equivalent.

i) Each vertex of $\beta$ is on some geodesic connecting $a$ and $b$

ii) Each edge of $\beta$ belongs to a wall that separates $a$ from $b$.

iii) Each wall crossed by $\beta$ is crossed an odd number of times.

iv) For any vertices $c$ and $d$ of $\beta$ any wall separating $c$ and $d$ also separates $a$ and $b$.

The next result is a slightly more sophisticated version of lemma 3.2.

Lemma 3.5. Suppose $\alpha$ is a geodesic edge path in $\Lambda$ connecting the vertices $a$ and $b$, $v$ is a vertex of $\alpha$, and $a$ and $b$ are each within distance $A$ of $Q$ for some wall $Q$. Then $v$ is within distance $2A(2P(1) + 1) + P(1)$ of $Q$.

Proof. Let $a'$ (respectively $b'$) be a vertex of $Q$ within $A$ of $a$ (respectively $b$) and on the same side of $Q$ as $a$ (respectively $b$). Let $\beta$ (respectively $\gamma$) be a geodesic from $a'$ to $a$ (respectively $b$ to $b'$).

Case 1. The geodesic $\alpha$ does not cross $Q$.

In this case the path $\delta_0 = (\beta, \alpha, \gamma)$ does not cross $Q$. Since $|\beta| \leq A$ and $|\gamma| \leq A$, a sequence of at most $2A$ deletions (the first in the path $\delta_0$) will determine a geodesic connecting $a'$ to $b'$. As noted in section 2, if at some stage $\delta_i$ is not geodesic, then we can take edges $e_i$ and $d_i$ of $\delta_i$ in the same wall and such that $(\ast)$ the subpath of $\delta_i$ between $e_i$ and $d_i$ is geodesic and does not cross the wall. Replacing this subpath by its reflection in the wall and deleting $e_i$ and $d_i$ will give a shorter path $\delta_{i+1}$ from $a'$ to $b'$. Note that if $\delta_i$ does not cross $Q$, $e_i$ and $d_i$ are on the same side of $Q$, so the reflected subpath, being a geodesic by $(\ast)$, cannot cross $Q$, and then $\delta_{i+1}$ also does not cross $Q$. After $K \leq 2A$ such deletions, $\delta_K$ will be a geodesic connecting $a'$ and $b'$ which does not cross $Q$.

If $v$ is not between $e_1$ and $d_1$ then $v$ is a vertex of $\delta_1$. If $v$ is between $e_1$ and $d_1$, then $v_1$, the reflection of $v$ across the wall $[e_1] = [d_1]$, is within $2P(1) + 1$ of $v$, by lemma 3.2. (Note that the hypotheses of lemma 3.2 are satisfied since we require condition $(\ast)$.) In any case $\delta_1$ contains a vertex $v_1$ within $2P(1) + 1$ of $v$. If $e_2$ and $d_2$ are deleting edges of $\delta_1$ (satisfying $(\ast)$), then $\delta_2$, obtained from $\delta_1$ by deleting $e_2$ and $d_2$, contains a vertex $v_2$ within $2P(1) + 1$ of $v_1$, again by lemma 3.2, and so $v_2$ is within $2(2P(1) + 1)$ of $v$. Inductively then, $\delta_K$ contains a vertex $v_K$ within $K(2P(1) + 1)$ of $v$. By lemma 3.2, $v_K$ is within $P(1)$ of $Q$ so that $v$ is within $2A(2P(1) + 1) + P(1)$ of $Q$. This completes case 1.

Case 2. Suppose $\alpha$ crosses $Q$.

Say the edge $e$ of $\alpha$ between $v$ and $b$ belongs to $Q$. Repeat the case 1 argument with $\delta_0$ replaced by $(\beta, \alpha')$, where $\alpha'$ is the subsegment of $\alpha$ from $a$ to the initial point of $e$. Similarly if $e \in Q$ is an edge of $\alpha$ between $a$ and $v$. Note that in both case 2 scenarios, at most $A$ deletions are required to straighten to a geodesic, so the bound is reduced to $A(2(2P(1)+1)+P(1))$. $\square$
4. Tracking Quasi-geodesics

We are interested in quasi-geodesic edge paths in $\Lambda$. Recall that an edge path $\beta$ is a $(\lambda, \epsilon)$-quasi-geodesic if for each pair of integers $s$ and $t$ in the domain of $\beta$, $|s - t| \leq \lambda d(\beta(s), \beta(t)) + \epsilon$. If $\beta$ is a ray or line, we say $\beta$ is quasi-geodesic if it satisfies the corresponding condition for some $\lambda$ and $\epsilon$. If $\alpha$ and $\beta$ are edge paths, then $\beta$ is $K$-tracked by $\alpha$ if each vertex of $\beta$ is within $K$ of a vertex of $\alpha$. If $\alpha$ and $\beta$ are rays or lines, we say $\beta$ is tracked by $\alpha$ if they satisfy the corresponding condition for some $K$. If $\alpha$ is a $K$-approximation to $\beta$ then $\beta$ is $K$-tracked by $\alpha$, but in general there is no requirement that a tracking path $\alpha$ be piecewise geodesic nor even that the nearest points on $\alpha$ to vertices of $\beta$ occur in the same order. More generally, say that a set of vertices is $(K)$-tracked by $\alpha$ if every element of the set is within (some possibly unspecified bound) $K$ of a vertex of $\alpha$.

**Lemma 4.1.** For $i \in \{1, 2\}$ suppose $\beta_i$ is a $(\lambda_i, \epsilon_i)$-quasi-geodesic edge path in $\Lambda$, $\beta_1$ is $K$-tracked by $\beta_2$ and $\beta_1(0)$ is within $K$ of $\beta_2(0)$. Assume both $\beta_1$ and $\beta_2$ are lines, or both are rays, or both are finite length and the terminal points of $\beta_1$ and $\beta_2$ are within $K$ of one another. Then $\beta_2$ is $(\lambda_2(2K + 1) + \epsilon_2 + K)$-tracked by $\beta_1$.

**Proof.** Since each vertex of $\beta_1$ is within $K$ of a vertex of $\beta_2$, we may define an integer function $a$ such that for each integer $m$ (in the domain of $\beta_1$), $\beta_1(m)$ is within $K$ of $\beta_2(a(m))$. We take $a(0) = 0$ and if $\beta_i$ has $n_i$ edges then $a(n_1) = n_2$.

The first two sets of inequalities below follow from the definitions and triangle inequalities and the third set follows from the first two.

\[
\frac{|a(m + i) - a(m)| - \epsilon_2}{\lambda_2} - 2K \leq d(\beta_2(a(m + i)), \beta_2(a(m))) - 2K \\
\leq d(\beta_1(m + i), \beta_1(m)) \\
\leq d(\beta_2(a(m + i)), \beta_2(a(m))) + 2K \\
\leq |a(m + i) - a(m)| + 2K
\]

\[
\frac{i - \epsilon_1}{\lambda_1} \leq d(\beta_1(m + i), \beta_1(m)) \leq i
\]

(1) \[
\frac{i - \epsilon_1}{\lambda_1} - 2K \leq |a(m + i) - a(m)| \\
\leq \lambda_2(d(\beta_1(m + i), \beta_1(m)) + 2K) + \epsilon_2 \\
\leq (i + 2K)\lambda_2 + \epsilon_2
\]

The inequality $|a(i + 1) - a(i)| \leq \lambda_2(2K + 1) + \epsilon_2$ implies if $k$ is between $a(i)$ and $a(i + 1)$ for some $i$ then $\beta_2(k)$ is within $\lambda_2(2K + 1) + \epsilon_2 + K$ of $\beta_1(i)$. In the case $\beta_1$ and $\beta_2$ are finite, the condition that terminal points are within $K$ of one another (so that $a(n_1) = n_2$) implies that every integer
in the domain of $\beta_2$ is between $a(i)$ and $a(i+1)$ for some $i$ and this case is finished. If $\beta_1$ and $\beta_2$ are rays then $a(i)$ is non-negative and equation 1 (with $m = 0$) implies $a(i)$ is arbitrarily large for large $i$ and again every integer in the domain of $\beta_2$ is between $a(i)$ and $a(i+1)$ for some $i$. If $\beta_1$ and $\beta_2$ are bi-infinite, then the $a(i)$ may be positive or negative and (again by 1) for large $|i|$, $|a(i)|$ is large, and $\lim_{i\to+\infty} a(i) = \pm\infty$ and $\lim_{i\to-\infty} a(i) = \pm\infty$. It remains to see $\lim_{i\to-\infty} a(i) \neq \lim_{i\to+\infty} a(i)$. Equality is impossible, since otherwise, for every large positive integer $i$, $a(-i)$ would be between $a(j)$ and $a(j+1)$ for some (depending on $i$) large positive integer $j$. But equation 1 implies $a(j)$ and $a(j+1)$ are relatively close and $a(-i)$ and $a(j)$ are far apart. 

Proposition 4.2. Suppose $\beta$ is a quasi-geodesic edge path ray in $\Lambda$ and $\beta$ is tracked by a geodesic, then $\beta$ has bounded bracket number.

Proof. Assume that $\beta$ is a $(\lambda, \epsilon)$-quasi-geodesic. Suppose $\alpha$ is a geodesic such that each vertex of $\beta$ is within $L$ of a vertex of $\alpha$. For each integer $n \geq 0$, choose an integer $a(n)$ such that $d(\beta(n), \alpha(a(n))) \leq L$. We assume that $a(0) = 0$.

The next two sets of inequalities follow from the definitions and triangle inequalities, and the third follows from the first two.

$$a(n) - 2L \leq d(\beta(n), \beta(0)) \leq a(n) + 2L$$

$$\frac{n - \epsilon}{\lambda} \leq d(\beta(n), \beta(0)) \leq n$$

$$\frac{n - \epsilon}{\lambda} - 2L \leq a(n) \leq n + 2L$$

The proof now makes use of the following claim.

Claim 4.3. Suppose $K$ is an integer larger than $\lambda(4L + 1) + \epsilon$. Then for any integer $n$, $a(n + K) > a(n)$.

Proof. Note that if $m > \lambda(n + 4L) + \epsilon$ then $a(m) > n + 2L \geq a(n)$. So if $K > \lambda(4L + 1) + \epsilon$, and $a(n + K) \leq a(n)$, then there is a last integer $K_1 > \lambda(4L + 1) + \epsilon$ such that $a(n + K_1) \leq a(n)$. Then (see figure 3)

$$a(n + K_1 + 1) > a(n) \geq a(n + K_1)$$

Since $d(\beta(n + K_1), \beta(n + K_1 + 1)) = 1$ for all $n$, and $d(\beta(i), \alpha(a(i))) \leq L$ for all $i$, we have

$$d(\alpha(a(n + K_1)), \alpha(a(n + K_1 + 1))) \leq 2L + 1$$

But as $\alpha(a(n))$ is between $\alpha(a(n + K_1))$ and $\alpha(a(n + K_1 + 1))$ on the geodesic $\alpha$,

$$d(\alpha(a(n)), \alpha(a(n + K_1 + 1))) \leq 2L + 1$$

Then $d(\beta(n), \beta(n + K_1 + 1)) \leq 4L + 1$. But

$$d(\beta(n), \beta(n + K_1 + 1)) \geq \frac{1}{\lambda}(K_1 + 1 - \epsilon) > 4L + 1$$

the desired contradiction (so the claim is proved).
β \( n \) \beta(n) \beta(n + K_1) \beta(n + K_1 + 1)

≤ L \leq L \leq L

α \alpha(a(n + K_1)) \alpha(a(n)) \alpha(a(n + K_1 + 1))

Figure 3.

Now suppose \( v \equiv \beta(n) \) is a vertex of \( \beta \) with bracket number at least 
\( 2\lambda(4L + 1) + 2\epsilon + K \). Then there are \( K \) distinct walls, \( Q_1, \ldots, Q_K \) such that 
for each \( i \in \{1, \ldots, K\} \), there is an edge \( e_i \) of \( \beta \) preceding \( v \) and an edge \( d_i \) 
of \( \beta \) following \( v \) such that \( e_i \) and \( d_i \) belong to the wall \( Q_i \), the subpath of \( \beta \) 
between \( e_i \) and \( d_i \) does not cross \( Q_i \), \( e_i \) is not one of the \( \lambda(4L + 1) + \epsilon \) edges 
of \( \beta \) immediately preceding \( v \) and \( d_i \) is not one of the \( \lambda(4L + 1) + \epsilon \) edges of \( \beta \) 
immediately following \( v \). I.e. \( e_i = \beta([t_i, t_i + 1]) \) where \( t_i + 1 \leq n - \lambda(4L + 1) - \epsilon \) 
and \( d_i = \beta([u_i, u_i + 1]) \) where \( u_i \geq n + \lambda(4L + 1) + \epsilon \). (See figure 4.)

\[\begin{array}{c}
\beta \quad e_i \\
≤ L \quad \geq \lambda(4L + 1) + \epsilon \\
α \quad α(a(t_i + 1)) \quad α(a(n)) \quad α(a(u_i)) \\
\end{array}\]

Figure 4.

By claim 4.3, \( a(t_i + 1) < a(n) < a(u_i) \). Hence, by lemma 3.5, \( α(a(n)) \) is 
within \( 2L(2P(1) + 1) + P(1) \) of the wall \( Q_i \). For \( x \) a vertex of \( \Lambda \), let \( C(k) \) 
be the number of distinct walls that pass within \( k \) of \( x \). Note that \( C \) is 
independent of vertex in \( \Lambda \). Hence \( K \leq C(2L(2P(1) + 1) + P(1)) \), bounding 
the bracket number of a vertex of \( \beta \). \(\square\)
5. Proof of Main Theorem

In order to prove the main theorem, we need two results, one due to B. Brink and R. Howlett [BrH], and a second, due to R. P. Dilworth [Di].

**Theorem 5.1.** (Brink-Howlett) Suppose \((W, S)\) is a finitely generated Coxeter system, and \(\Lambda(W, S)\) is the Cayley graph of \(W\) with respect to \(S\). There is a bound \(F_{(W, S)}\) on the number of mutually crossing walls of \(\Lambda\).

Dilworth’s theorem requires several definitions. If \(A\) is a partially ordered set (a set with reflexive, antisymmetric and transitive binary relation \(\leq\) on \(A\)), then any two elements \(x\) and \(y\) are comparable if either \(x \leq y\) or \(y \leq x\). Otherwise they are in incomparable. A subset \(C\) of \(A\) is a chain when every pair of points in \(C\) is a comparable pair. A subset \(B\) of \(A\) is called an antichain when every pair of points in \(B\) is an incomparable pair. The number of points in a maximal antichain is called the width of \(A\).

**Theorem 5.2.** (Dilworth) If \(A\) is a partially ordered set of width \(w\), then \(A\) can be partitioned into \(w\) chains.

Suppose \(x\) and \(y\) are vertices of \(\Lambda(W, S)\) and \(W_{(x, y)}\) is the set of walls that separate \(x\) and \(y\). We partially order \(W_{(x, y)}\) by saying \(P \leq Q\) if either \(P = Q\), or \(P\) and \(Q\) are parallel and \(P\) separates \(x\) from \(Q\). Note that \(P\) and \(Q\) are parallel walls of \(W_{(x, y)}\), ifff they are comparable. Hence \(P\) and \(Q\) are incomparable ifft they cross. By proposition 5.1, the width of \(W_{(x, y)}\) is at most \(F_{(W, S)}\). Applying Dilworth’s theorem we have:

**Proposition 5.3.** Suppose \((W, S)\) is a finitely generated Coxeter system, and \(\Lambda(W, S)\) is the Cayley graph of \(W\) with respect to \(S\). For any vertices \(x\) and \(y\) of \(\Lambda\) the walls separating \(x\) and \(y\) can be partitioned into at most \(F_{(W, S)}\) chains (where any two walls in the same chain are parallel).

Say a path is geodesic with respect to a set of walls if the path crosses each wall of the set either 0 or 1 times. The following lemma is clear.

**Lemma 5.4.** Suppose \(\alpha\) is an edge path in \(\Lambda\) and \(\alpha\) is geodesic with respect to the set of parallel walls \(Q\). If a subpath of \(\alpha\) is replaced by a geodesic edge path, then the resulting edge path is geodesic with respect to \(Q\).

**Theorem 5.5.** Suppose \((W, S)\) is a finitely generated Coxeter system with Cayley graph \(\Lambda\). Then there is a function \(f\) such that, for any \(\alpha\) a \((\lambda, \epsilon)\)-quasi-geodesic edge path from \(a\) to \(b\) in \(\Lambda\), there is a geodesic edge path \(\beta\) from \(a\) to \(b\) such that \(\alpha\) is \(K\)-tracked by \(\beta\) for \(K = f(B(\alpha), \lambda, \epsilon)\). That is, any \((\lambda, \epsilon)\)-quasi-geodesic \(\alpha\) is \(K\)-tracked by a geodesic \(\beta\) where the \(K\) depends only on the bracket number of \(\alpha\), \(B(\alpha)\), and on \(\lambda\) and \(\epsilon\), but is otherwise independent of the particular \(\alpha\).

**Proof.** Fix \(B_0\), \(\lambda\), and \(\epsilon\). Consider a \((\lambda, \epsilon)\)-quasi-geodesic \(\alpha\) from \(a\) to \(b\) with \(B(\alpha) \leq B_0\). By proposition 3.3, take \(\alpha'\) an \(L\)-approximation to \(\alpha\) with every vertex on some geodesic connecting \(a\) and \(b\), where \(L\) depends only
on \(B(\alpha) \leq B_0\). By proposition 3.1, \(B(\alpha') \leq B_1\) for a larger bound \(B_1\) still determined only from \(B_0\). Moreover, \(\alpha'\) is a \((\lambda', \epsilon')\)-quasi-geodesic for a \(\lambda'\) and \(\epsilon'\) determined only from \(L, \lambda\) and \(\epsilon\). More specifically, write \(\alpha'\) as \((\alpha'_1, \ldots, \alpha'_q)\) where the \(\alpha'_i\) are geodesics of length \(\leq 2L + 1\), with initial and terminal vertices of \(\alpha'_i\) within \(L\) of \(\alpha(i - 1)\) and \(\alpha(i)\) respectively. Then each interior vertex of \(\alpha'_i\) is within \(3L\) of \(\alpha(i - 1)\) and \(\alpha(i)\). Thus an \(\alpha'(i)\) and \(\alpha'(j)\) are within \(3L\) of vertices \(\alpha(l)\) and \(\alpha(m)\), respectively, such that vertices of \(\alpha\) between these have approximation points between \(\alpha'(i)\) and \(\alpha'(j)\), and hence

\[
|i - j| \leq (2L + 1)(|l - m| + 2) \\
\leq (2L + 1)(\lambda d(\alpha(l), \alpha(m)) + \epsilon + 2) \\
\leq (2L + 1)(\lambda(d(\alpha'(i), \alpha'(j)) + 6L) + \epsilon + 2) \\
= (2L + 1)\lambda d(\alpha'(i), \alpha'(j)) + (2L + 1)(\lambda(6L) + \epsilon + 2)
\]

from which \(\lambda'\) and \(\epsilon'\) can be read off.

A geodesic \(\beta\) that \(K\)-tracks \(\alpha'\) will \(K + L\)-track \(\alpha\). Thus it suffices to find, in terms of \(B_1, \lambda'\), and \(\epsilon'\), a \(K\) that works for those \((\lambda', \epsilon')\)-quasi-geodesics \(\alpha'\) with \(B(\alpha') \leq B_1\) and having every vertex on some geodesic between its endpoints, i.e., we may as well prove the theorem for \(\alpha\) having every vertex on some geodesic between its endpoints.

The proof is a double induction argument. For the “outside” induction, we show by induction on \(A\) with \(1 \leq A \leq F \equiv F_{(W,S)}\) that, for any positive integer \(B_0\) and constants \(\lambda\) and \(\epsilon\), there is a constant \(K_A\) such that, for any \((\lambda, \epsilon)\)-quasi-geodesic \(\alpha\) from \(a\) to \(b\), with \(B(\alpha) \leq B_0\) and every vertex on a geodesic between \(a\) and \(b\), if \(\mathcal{W}_{(a,b)}\), the set of walls separating \(a\) and \(b\), can be partitioned into \(A\) or fewer chains, then there is a geodesic edge path \(\beta\) from \(a\) to \(b\) such that \(\alpha\) is \(K_A\)-tracked by \(\beta\). By proposition 5.3, the set of walls separating any \(a\) and \(b\) can be partitioned into at most \(F\) chains, so \(K = K_F\) suffices for the theorem.

Note that if \(A = 1\) then all walls separating \(a\) and \(b\) are parallel. In this case, the walls separating \(a\) and \(b\) are ordered as \(Q_1, \ldots, Q_m\) where for \(i < j < k\), \(Q_j\) separates \(Q_i\) from \(Q_k\). Hence, there is a unique, geodesic edge path \(\beta\) connecting \(a\) and \(b\), and \(\beta\) crosses \(Q_1\), then \(Q_2\), etc. The path \(\alpha\) only crosses the walls separating \(a\) and \(b\) (see lemma 3.4) and, in this case, is a geodesic modulo backtracking. Eliminating backtracking on \(\alpha\) produces \(\beta\). Each vertex of \(\alpha\) is a vertex of \(\beta\) and the basis case is complete with \(K_1 = 0\).

Assume the (outside) induction statement is true for \(A < M\). For the “inside” induction, we show by induction on \(N\) with \(0 \leq N \leq M\) that, for any positive integer \(B_0\) and constants \(\lambda\) and \(\epsilon\), there is a constant \(K_{M,N}\) such that, for any \((\lambda, \epsilon)\)-quasi-geodesic \(\alpha\) from \(a\) to \(b\), with \(B(\alpha) \leq B_0\) and every vertex on a geodesic between \(a\) and \(b\), if \(\mathcal{W}_{(a,b)}\) can be partitioned into \(M\) chains \(Q_1, \ldots, Q_M\), such that \(\alpha\) is geodesic with respect to all but at most \(N\) of the \(Q_i\), then there is a geodesic edge path \(\beta\) from \(a\) to \(b\) such
that $\alpha$ is $K_{M,N}$-tracked by $\beta$. The induction step for the outside induction is completed then by taking $K_M = K_{M,M}$.

Fix a $B_0$, $\lambda$, and $\epsilon$. Consider a $(\lambda, \epsilon)$-quasi-geodesic $\alpha$ from $a$ to $b$, with $B(\alpha) \leq B_0$ and every vertex on a geodesic between $a$ and $b$, such that $M$ chains $Q_1, \ldots, Q_M$ partition the set of walls separating $a$ and $b$. Again by lemma 3.4, $\alpha$ only crosses walls separating $a$ and $b$, each an odd number of times, and each of which belongs to some $Q_i$. Suppose $\alpha$ is geodesic with respect to all but at most $N$ of these $Q_i$. If $N = 0$, then $\alpha$ is geodesic, so $\beta = \alpha$ with $K_{M,0} = 0$.

Assume the (inside) induction statement is true for $N = H - 1$. Assume $\alpha$ is such that, taking $Q_i$ indexed conveniently, $\alpha$ is geodesic with respect to $Q_i$ for $H + 1 \leq i \leq M$. If $\alpha$ is also geodesic with respect to $Q_H$, then apply the induction hypothesis. Otherwise, write $\alpha$ as $(e_1, \ldots, e_n)$ with consecutive vertices $a \equiv a_1, \ldots, a_n \equiv b$. Let $i$ be the first integer such that $e_i$ is an edge of a wall of $Q_H$ where for some $j > i$, $e_j$ and $e_i$ are in the same wall $Q$. Take $j$ the largest integer such that $e_j \in Q$. Since $\alpha$ crosses $Q$ an odd number of times, the path $\alpha_{i,j} = (e_i, \ldots, e_{j-1})$ (from $a_i$ to $a_j$) crosses $Q$ an even number of times. A geodesic $\beta_{i,j}$ connecting $a_i$ to $a_j$ does not cross $Q$. All walls of $Q_H$ are parallel to one another, and $\beta_{i,j}$ begins and ends adjacent to (at endpoints of edges in), and on the same side of, $Q \in Q_H$. Suppose $Q'$ is a wall of $Q$ other than $Q$, with sides $S_1$ (containing $Q$) and $S_2$, so $\beta$ begins and ends in $S_1$. If $\beta_{i,j}$ crossed $Q'$ to side $S_2$, then as $\beta_{i,j}$ ends in side $S_1$, $\beta_{i,j}$ would cross $Q'$ a second time. But a geodesic cannot cross a wall twice and so $\beta_{i,j}$ does not cross a wall of $Q_H$. Hence $a_i$ and $a_j$ are not separated by a wall of $Q_H$. By proposition 3.3, take $\alpha'_{i,j}$ an $L$-approximation to $\alpha_{i,j}$ for $L$ determined from $B(\alpha_{i,j}) \leq B(\alpha) \leq B_0$, with $\alpha'_{i,j}$ a $(\lambda', \epsilon')$-quasi-geodesic edge path connecting $a_i$ to $a_j$ (for some $\lambda'$ and $\epsilon'$ determined from $L$ as above), $B(\alpha'_{i,j}) \leq B_1$ (for a $B_1$ determined from $L$ as above), and such that each vertex of $\alpha'_{i,j}$ is on a geodesic connecting $a_i$ to $a_j$. By lemma 3.4, each wall separating $a_i$ and $a_j$ also separates $a$ and $b$, and the walls separating $a_i$ and $a_j$ can be partitioned into fewer than $M$ chains. By (outside) induction, there is a geodesic $\beta_{i,j}$ connecting $a_i$ and $a_j$ which $K_{M-1}$-tracks $\alpha'_{i,j}$ and therefore $K_{M-1} + L$-tracks $\alpha_{i,j}$ (where the $K_{M-1}$ is determined using $B_1$, $\lambda'$, and $\epsilon'$). Replace $\alpha_{i,j}$ in $\alpha$ by $\beta_{i,j}$ to obtain a path $\alpha_1$.

Now the path $\alpha_1$ crosses $Q$ exactly once at $e_j$. The walls of $Q_H$ are ordered as $Q_1, Q_2, \ldots$ so that if $i < j$, then $Q_i$ separates $a$ from $Q_j$, and $Q_j$ separates $Q_i$ from $b$. A wall of $Q_H$ preceding $Q$ in this ordering is not crossed by $\alpha_1$ after $e_j$. Hence if $Q \subseteq Q_H$ is the set of walls of $Q_H$ preceding $Q$ and including $Q$, then $\alpha_1$ is geodesic with respect to $Q$ and (by lemma 5.4), $\alpha_1$ is geodesic with respect to each set $Q_i$ for $i > H$. Now suppose $e_k$ is the first edge of $\alpha_1$ such that $e_k$ is an edge of a wall $Q$ of $Q_H$, and for some $l > k$, $e_l \in Q$. Then $e_k$ follows $e_j$ on $\alpha_1$, and if we assume $e_l$ is the last edge of $\alpha_1$ in $Q$, then as above $(e_k, \ldots, e_{l-1})$ can be replaced by a geodesic.
close to \((e_k, \ldots, e_{l-1})\) with the same tracking constant \(K_{M-1} + L\). At each step, disjoint subpaths of the original \(\alpha\) are replaced.

Continuing, the resulting path \(\alpha_s K_{M-1} + L\)-tracks \(\alpha_s\) is geodesic with respect to \(Q_i\) for \(H \leq i \leq M\), and crosses the same walls as \(\alpha\). By lemma 3.4 every vertex of \(\alpha_s\) belongs to some geodesic connecting \(a\) to \(b\). We show \(\alpha_s\) is a \((\hat{\lambda}, \hat{\epsilon})\)-quasi-geodesic with \(B(\alpha_s) \leq B_2\) for appropriately bounded \(B_2\), \(\hat{\lambda}\) and \(\hat{\epsilon}\). Let \(\alpha_{ss}\) be a \(K_{M-1} + L\) approximation to \(\alpha\) obtained by replacing each \(\alpha_{ij}\) by a path \(\hat{\beta}_{ij}\) described as follows: If the consecutive vertices of \(\alpha_{ij}\) are \(w_0, \ldots, w_m\) then for \(1 \leq k \leq m - 1\) let \(\hat{w}_k\) be a vertex of the geodesic \(\beta_{ij}\) within \(K_{M-1} + L\) of \(w_k\). For \(0 \leq k \leq m - 1\) connect \(\hat{w}_k\) and \(\hat{w}_{k+1}\) with a geodesic subpath of \(\beta_{ij}\). Concatenate these geodesics to form \(\hat{\beta}_{ij}\). Then as above \(\alpha_{ss}\) is a \((\hat{\lambda}, \hat{\epsilon})\)-quasi-geodesic with \(B(\alpha_s) \leq B_2\) for appropriately bounded \(B_2\), \(\hat{\lambda}\) and \(\hat{\epsilon}\). Note that \(\beta_{ij}\) is obtained by eliminating backtracking in \(\hat{\beta}_{ij}\). Eliminating backtracking in a \((\lambda, \epsilon)\)-quasi-geodesic does not increase bracket number and the resulting path remains a \((\lambda, \epsilon)\)-quasi-geodesic. The path \(\alpha_s\) is obtained from \(\alpha_{ss}\) by eliminating backtracking so that \(\alpha_s\) is a \((\hat{\lambda}, \hat{\epsilon})\)-quasi-geodesic with bracket number \(\leq B_2\).

By induction hypothesis, there is a geodesic \(\beta\) that \(K_{M,H-1}\)-tracks \(\alpha_s\), and so \(\beta K_{M,H}\)-tracks \(\alpha\) for \(K_{M,H} = K_{M,H-1} + K_{M-1} + L\). Hence the inside induction step is established and the theorem follows.

Note that the bound \(F\) from proposition 5.3 on the number of chains needed to partition the set of walls separating two points \(a\) and \(b\) of \(\Lambda\) limits the total number of times the induction steps are carried out to arrive at a geodesic.

6. Consequences of the Main Theorem

Corollary 6.1. Suppose \((W, S)\) is a finitely generated Coxeter system, and \(\Lambda\) is the Cayley graph of \(W\) with respect to \(S\). Any infinite or bi-infinite \((\lambda, \epsilon)\)-quasi-geodesic edge path \(\alpha\) with bounded bracket number \(B\) is \(K'\)-tracked by an edge path geodesic where \(K'\) is a constant only depending on \(\lambda\), \(\epsilon\), \(B\) and \(S\).

Proof. The proof is a standard local finiteness argument in both the infinite and bi-infinite case. We give the bi-infinite case. Write \(\alpha\) as the edge path \((\ldots, e_{-1}, e_0, e_1, \ldots)\) in \(\Lambda\). Let \(v_i\) be the initial point of \(e_i\). By theorem 5.5, there is a \(\Lambda\)-geodesic \(\beta_n\) which \(K\)-tracks \(\alpha_n \equiv (e_{-n}, \ldots, e_n)\). Note that every vertex of \(\beta_n\) is within \(2K\) of a vertex of \(\alpha\). For each positive integer \(n\), some vertex \(x_n\) of \(\beta_n\) is within \(K\) of \(v_0\). Hence there is an infinite number of \(x_n\) that are equal. Of this infinite subcollection of \(x_n\), infinitely many have the same pair of edges one preceding and one following \(x_n\) on \(\beta_n\), of this infinite collection of \(x_n\) there is an infinite subcollection that have the same four edges - the two preceding and the two following \(x_n\) being exactly the same. Continuing, we have a bi-infinite geodesic \(\beta\) and each vertex of \(\beta\) is within
2K of a vertex of \( \alpha \). As \( \alpha \) is a \((\lambda, \epsilon)\)-quasi-geodesic, lemma 4.1 implies each point of \( \alpha \) is within \( \lambda(4K+1) + \epsilon + 2K \) of \( \beta \).

The next result follows directly from proposition 4.2 and corollary 6.1.

**Corollary 6.2.** Suppose \((W, S)\) is a finitely generated Coxeter system, and \( \Lambda \) is the Cayley graph of \( W \) with respect to \( S \). Then a quasi-geodesic edge path ray in \( \Lambda \) is tracked by a geodesic iff it has bounded bracket number.

A metric space \((X, d)\) is a called a geodesic metric space if every pair of points are joined by a geodesic. It is proper if for any \( x \in X \), the ball of radius \( r \) about \( X \) is compact for all positive numbers \( r \). A group \( W \) acts geometrically on a space if the action is properly discontinuous, co-compact and by isometries.

Let \((X, d)\) be a proper complete geodesic metric space. If \( \Delta \ a \ b \ c \) is a geodesic triangle in \( X \), then consider \( \Delta \ a \ b \ c \) in the Euclidean plane \( \mathbb{E}^2 \) with the same side lengths, called a comparison triangle. We say \( X \) satisfies the CAT(0) inequality, and say \((X, d)\) is a CAT(0) space, if for any \( \Delta \ a \ b \ c \) in \( X \), and any two points \( p, q \) on \( \Delta \ a \ b \ c \), the corresponding points \( \overline{p}, \overline{q} \) the same distances from vertices on the sides in a comparison triangle satisfy

\[
d(p, q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q})
\]

If \((X, d)\) is a CAT(0) space, then the following basic properties hold:

1. The distance function \( d : X \times X \to \mathbb{R} \) is convex.
2. \( X \) has unique geodesic segments between points.
3. \( X \) is contractible.

For details, see [BH].

Suppose \((W, S)\) is a finitely generated Coxeter system, \( \Lambda \) is the Cayley graph of \( W \) with respect to \( S \), and \( W \) acts geometrically on a CAT(0) space \( X \). Fix a point \( x \in X \), and define \( \Lambda_x \subset X \) to have as vertices, the orbit \( Wx \), and as edges, CAT(0) geodesic paths connecting \( wx \) and \( wsx \), for \( w \in W \) and \( s \in S \). There is a proper \( W \)-equivariant map \( P_x : \Lambda \to \Lambda_x \) so that \( P_x \) maps the identity vertex of \( \Lambda \) to \( x \). This \( P_x \) is a quasi-isometry of \( \Lambda \) with edge path metric \( d_{\Lambda} \) into \((X, d)\), and we ordinarily think of \( \Lambda_x \) as being essentially a copy of \( \Lambda \) in \( X \) (though for some purposes we might require at least that \( x \) is not fixed by any generator). We consider rays and lines in \((X, d)\), and define tracking in \((X, d)\) with respect to the metric \( d \), all analogously to the definitions given earlier for Cayley graphs.

Intuitively, the next result says that when a Coxeter group acts geometrically on a CAT(0) space, CAT(0) geodesics are tracked by Cayley graph geodesics. This result generalizes the right angled version of the same result in [MRT].

**Corollary 6.3.** Suppose \((W, S)\) is a finitely generated Coxeter system, \( \Lambda \) is the Cayley graph of \( W \) with respect to \( S \), \( W \) acts geometrically on the proper CAT(0) space \( X \), and take \( x \in X \) with \( P_x : \Lambda \to \Lambda_x \) as above. Then
any CAT(0) geodesic ray in $X$ is tracked by (the image of) a Cayley graph geodesic in $\Lambda_x$.

Proof. For a given CAT(0) geodesic $\alpha$ we find a Cayley graph geodesic $\beta$ such that $P_x(\beta)$ tracks $\alpha$. It suffices to find $\lambda$, $\epsilon$, $K$ and $B$ such that any (finite) CAT(0) geodesic $\alpha$ is $K$-tracked by a Cayley $(\lambda, \epsilon)$-quasi-geodesic with bracket number $\leq B$. Since $W$ acts co-compactly on the CAT(0) space $X$, note that the above proof is valid even when $\alpha$ is of a fixed point of $W$ (in $\Lambda_x$) cannot be more than $B$-distance from $\Lambda_x$.

Figure 5.

Hence it suffices to bound the bracket number of such a $\beta$. If $v$ is a vertex of $\beta_i$ and $e$ and $d$ are edges of $\beta$ preceding and following $v$ respectively such that $e$ and $d$ belong to the same wall $Q$ of $\Lambda$, then $e$ is an edge of $\beta_i$ and $d$ is an edge of $\beta_i$ where $j \leq i \leq k$. The mid-points $e'$ of $P_x(e)$ and $d'$ of $P_x(d)$ are fixed (in $\Lambda_x$ and $X$) by the reflection $r_Q \in W$ for the wall $Q$. Hence the geodesic in $X$ connecting $d'$ and $e'$ is fixed by $r_Q$. Now, $d'$ (respectively $e'$) is within $D_X$ of $\alpha(j - 1)$ (respectively $\alpha(k + 1)$) and $P_x(v)$ is within $D_X$ of $\alpha(i)$. By the CAT(0) inequality for quadrilaterals (in particular for the quadrilateral determined by $d'$, $e'$, $\alpha(j - 1)$, and $\alpha(k + 1)$) it is within $D_X$ of a point of the $X$-geodesic connecting $d'$ to $e'$ and hence $\alpha(i)$ is within $D_X$ of a fixed point of $r_Q$. (See figure 5.)

Since the action of $W$ on $X$ is properly discontinuous, there is a bound $B$ on the number of reflections $r_Q$ such that $r_Q$ does not take the ball of radius $D_X$ centered at $\alpha(i) \in X$ (or any other point of $X$) off of itself. Hence there cannot be more than $B$ walls bracketing the vertex $v$ of $\beta$. □

Remark 6.4. Note that the above proof is valid even when $W$ does not act co-compactly on the CAT(0) space $X$, as long as the CAT(0) geodesic remains a bounded distance from $\Lambda_x$ for some $x$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure5}
\caption{Figure 5.}
\end{figure}
Remark 6.5. Unfortunately, this result does not hold for arbitrary finitely generated CAT(0) groups. Let $T$ be the Cayley graph of the free group $F_2 = \langle x, y \rangle$, a tree, and $R$ the Cayley graph of the infinite cyclic group $Z = \langle z \rangle$, a line, both CAT(0) spaces. Take $G = F_2 \times Z$ acting componentwise and geometrically on the CAT(0) space $X = T \times R$ with metric $d((w_1, h_1), (w_2, h_2)) = \sqrt{d_T(w_1, w_2)^2 + d_R(h_1, h_2)^2}$. Now with $a = xz$ and $b = yz$, $S = \{a, b, z\}$ is a generating set of $G$. Let $\Lambda$ be the Cayley graph with respect to this generating set and fix $* \in X$ to have components the identity vertices of the Cayley graphs $T$ and $R$, thus determining $\Lambda_*$. Let $\alpha$ be the CAT(0) geodesic ray from $*$ which passes through the points $v_n*$ where $v_0 = 1$, and for $n > 0$, $v_n = v_{n-1}2^n y^{-2^n}$, that is, a ray having a constant second component, contained in a horizontal copy of $T$, but following exponentially increasing long sequences of $x$ and $y^{-1}$ edges. A Cayley graph geodesic from $v_{n-1}*$ to $v_n*$ is labeled $a^2 b^{-2^n}$ but this path in $\Lambda_*$ increases to a second component value of $2^n$ at its midpoint, far from $\alpha$. Any Cayley graph geodesic approaching $\alpha$ within some bound arbitrarily far along must also diverge arbitrarily far from $\alpha$ far enough along.

We close this remark with the observation that the Coxeter group $W = (\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2) \times (\mathbb{Z}_2 \ast \mathbb{Z}_2)$ contains a subgroup of finite index isomorphic to $G \equiv F_2 \times Z$. So, while CAT(0) geodesics for $W$ are tracked by Cayley geodesics (determined by Coxeter generators), $G$ acts geometrically on a CAT(0) space where CAT(0) geodesics are not tracked by Cayley geodesics (for the obvious generators).

The following result answers a question posed by K. Ruane.

Corollary 6.6. Suppose $(W, S)$ is a finitely generated Coxeter group with Cayley graph $\Lambda$, acting geometrically on the CAT(0) space $X$, and take an $x \in X$, and $P_x : \Lambda \rightarrow \Lambda_x$, as before, mapping $\Lambda$ quasi-isometrically and $W$-equivariantly into $X$. Then for each subset $A \subset S$, (the image of) the subgroup $\langle A \rangle$ is quasi-convex in $X$.

Proof. Let $K$ be the tracking constant from corollary 6.3. Suppose $a_1, a_2 \in \langle A \rangle$ and $\alpha$ is a CAT(0) geodesic in $X$ from $P_x(a_1)$ to $P_x(a_2)$. Let $\beta$ be a $\Lambda_x$, edge path geodesic which $K$-tracks $\alpha$. I.e. there is a $\Lambda(W, S)$ geodesic $\beta'$, from $a_1$ to $a_2$ such that $P_x(\beta') = \beta$. Since $a_i \in \langle A \rangle$, the edge labels of $\beta'$ are all in $A$. This means all vertices of $\beta'$ are in $\langle A \rangle$, and so the image of $\alpha$ is within $K$ of $P_x(\langle A \rangle)$. \qed

The next result says that elements of infinite order in a Coxeter group are tracked by geodesics in the standard Cayley graph.

Corollary 6.7. Suppose $(W, S)$ is a finitely generated Coxeter system and $g \in W$ is an element of infinite order. Then in the Cayley graph $\Lambda(W, S)$ the elements $\{\ldots, g^{-2}, g^{-1}, 1, g, g^2, \ldots\}$ are tracked by a Cayley graph geodesic.

Proof. We know by G. Moussong [Mo], all finitely generated Coxeter groups are CAT(0). Let $X$ be any CAT(0) space such that $W$ acts geometrically
on $X$. We write $|g| = \inf\{d(x, gx) : x \in X\}$ and define the min set of $g$ by $\text{min}(g) = \{x \in X : d(x, gx) = |g|\}$, those points of $X$ moved a minimum amount by $g$. The min set of $g$ contains a geodesic line $l$ that is invariant under the action of $g$. Let $x$ be any point in $X$ and $\Lambda_x$ the copy of $\Lambda(W, S)$ in $X$ at $x$. Let $\alpha$ be an $S$-geodesic for $g$. Observe that the edge path line $l_g$ in $\Lambda_x$ determined by positive and negative iterates of $\alpha$ at $x$ is a bounded distance from $l$. The proof of corollary 6.3 shows that $l_g$ is a quasi-geodesic with bounded bracket number and so by corollary 6.1 is tracked by a Cayley graph geodesic.

One of the fundamental asymptotic results for word hyperbolic groups is that 1-ended word hyperbolic groups have locally connected boundary. This result follows from a long program of results by several authors, notably B. Bowditch, and concluded by G. Swarup [S]. To give a feeling for the reach of our results, we outline an elementary proof of this fact for Coxeter groups.

**Corollary 6.8.** If $W$ is a 1-ended word hyperbolic Coxeter group then the boundary of $W$ is locally connected.

**Proof.** We use an elementary form of a construction of a “filter” in [MRT] (where a partial classification of right angled Coxeter groups with locally connected boundaries is produced). Suppose $W$ acts geometrically on the CAT(0) space $X$, with base point $x$. Let $\Lambda_x$ be the copy of the Cayley graph of $(W, S)$ at $x$ in $X$ with proper $W$-equivariant map $P_x : \Lambda(W, S) \to \Lambda_x$. Suppose $r$ and $s$ are “close” geodesic rays in $X$, with $r(0) = s(0) =$
Choose $\Lambda$ (edge path) geodesics $r'$ and $s'$ at $*$ (the identity vertex of $\Lambda(W,S)$), such that $P_2(r')$ and $P_2(s')$ $K$-track $r$ and $s$ respectively. Since $r$ and $s$ are close in $\partial X$, we may assume that $r'$ and $s'$ have long initial segments with “close” terminal points. For simplicity we assume these initial segments agree. If $y$ is the last vertex of this common initial segment, say the edge of $r'$ following $y$ has label $a_1$ and the edge of $s'$ following $y$ has label $b_1$. The presentation diagram $\Gamma(W,S)$ of $(W,S)$ has vertex set $S$ and an edge labeled $m(i,j)$ between distinct vertices $s_i, s_j$ if $m(i,j) \neq \infty$. Since $W$ is 1-ended no subset $A$ of $S$ with $\langle A \rangle$ a finite group separates $\Gamma$ (see corollary 16 of [MT]). The set $B$ of $S$-elements that label edges at $y$ with end points closer to $*$ than $y$ is to $*$ generates a finite subgroup of $W$ (see lemma 4.7.2 of [D]). The set of vertices of $\Gamma$ corresponding to $B$ does not separate $\Gamma$ and $B$ does not contain $a_1$ or $b_1$. Hence there is an edge path in $\Gamma$ from $a_1$ to $b_1$ avoiding $B$. Let the consecutive vertices of this path be $a_1 = v_1, v_2, \ldots, v_n = b_1$. If $q(i,i+1)$ is the (finite) order of $v_iv_{i+1}$ then the relation $(v_iv_{i+1})^{q(i,i+1)}$ determines a loop at $y \in \Lambda$. By way of the deletion condition, it is an easy exercise to see that for any subset $C$ of $S$ and geodesic $\alpha$ connecting vertices $v_1, v_2$ of $\Lambda$ there is a unique closest vertex $x$ of $v_2(\langle C \rangle)$ to $v_1$, and for any geodesic $\beta$ at $x$ in the letters of $C$, $\langle \alpha, \beta \rangle$ is geodesic in $\Lambda$. Since the edges labeled $v_i$ and $v_{i+1}$ at $y$ extend the geodesic from $*$ to $y$, it must be that $y$ is the closest point of $y(v_i, v_{i+1})$ to $*$, and the two half loops at $y$ making up this loop extend the Cayley geodesic from $*$ to $y$. Consider the subgraph $F_1$ of $\Lambda$ determined by the edge paths $r'$, $s'$ and the edge loops for each $v_iv_{i+1}$ (see figure 6). Each $v_i$ determines an edge of $F_1$ (with label $v_i$) beginning at $y$. At the end point of this edge there are two edges of $F_1$ that extend a Cayley geodesic from $*$ to $y$. Build a set of loops as with $a_1$ and $b_1$ for each of these pairs of edges. Then $F_2$ is $F_1$ union all new loops. Continuing we build a 1-ended subgraph $F = \cup_{i=1}^{\infty} F_i$ of $\Lambda$ such that for each vertex $v$ of $F$, not on the common overlap of $r'$ and $s'$, there is a Cayley geodesic from $*$ to $v$ in $F$ which passes through $y$. We claim that $L$, the limit set of $P_x(F)$ is a “small” connected set containing $r$ and $s$ (and so $\partial X$ is locally connected). Certainly, $r$ and $s$ are in $L$. Since $F$ is 1-ended and $P_x$ is proper, $L$ is connected. If $v$ is a vertex of $F$, then there is a Cayley geodesic $\alpha_v$ from $*$ to $v$ (which passes through $y$ for all but finitely many $v$). If $z \in L$ then let $z_1, z_2, \ldots$ be a sequence of vertices of $F$ such that $P_x(z_i)$ converges to $z$. The CAT(0) geodesic from $x$ to $P_x(z_i)$ is $K$- tracked by a Cayley geodesic $\beta_i$ in $\Lambda_x$. As $W$ is word hyperbolic the Cayley geodesics $P_x(\alpha_v)$ and $\beta_i$ (with the same end points) must $\delta$-fellow travel (for a fixed constant $\delta$). In particular each $\beta_i$ must pass “close” to $P_x(y)$ and so $z$ is close to both $r$ and $s$ in $\partial X \equiv \partial W$. 

References


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