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Reflection Independence in Even Coxeter Groups

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Abstract. If (W, S) is a Coxeter system, then an element of W is a reflection if it is conjugate to some element of S. To each Coxeter system there is an associated Coxeter diagram. A Coxeter system is called reflection preserving if every automorphism of W preserves reflections in this Coxeter system. As a direct application of our main theorem, we classify all reflection preserving even Coxeter systems. More generally, if (W, S) is an even Coxeter system, we give a combinatorial condition on the diagram for (W, S) that determines whether or not two even systems for W have the same set of reflections. If (W, S) is even and (W, S') is not even, then these systems do not have the same set of reflections. A Coxeter group is said to be reflection independent if any two Coxeter systems (W, S) and (W, S') have the same set of reflections. We classify all reflection independent even Coxeter groups.

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1. Introduction

A *Coxeter system* is a pair (W, S) where W is a group with *Coxeter presentation* $\langle S | (st)^{m_{st}} \forall s, t \in S \rangle$ where for all $s, t \in S$, $m_{st} \in \{1, 2, \dots, \infty\}$, $m_{st} = m_{ts}$ and $m_{st} = 1$ if and only if $s = t$. (The relator $(st)^\infty$ means that st has infinite order in W.)

The *Coxeter diagram* of the system (W, S) is the labeled graph $\mathcal{V}(W, S)$ with vertex set S, and an edge labeled m_{st} between distinct vertices s and t if and only if $m_{st} \neq \infty$.

If (W, S) is a Coxeter system, then the conjugates in W of elements of S are called the *reflections* of (W, S). An elementary quotient argument (see Lemma 2.4) shows that if (W, S) and (W, S') are Coxeter systems and S is a subset of the reflections of (W, S') then (W, S) and (W, S') have the same set of reflections. If α is an automorphism of W then $(W, \alpha(S))$ is another Coxeter system for W. These two systems have isomorphic diagrams but may not have the same set of reflections. If for any two Coxeter systems (W, S) and (W, S') the reflections of (W, S) and (W, S') are the same, then W is called *reflection independent*. A Coxeter system (W, S) is called *reflection preserving* if every automorphism of W preserves reflections in the system (W, S). Note that if W is reflection independent, then every Coxeter system (W, S) is reflection preserving.

A Coxeter group W is said to be *rigid* if given any two systems (W, S) and (W, S'), there is a (labeled) diagram isomorphism between the corresponding Coxeter diagrams. This is equivalent to the existence of an automorphism $\alpha: W \to W$ which satisfies $\alpha(S) = S'$. Hence, a rigid Coxeter group W is reflection independent iff some Coxeter system for W is reflection preserving. Radcliffe proves in his dissertation [1] that all *right-angled* Coxeter groups (those for which $m_{st} \in \{2, \infty\}$ for all $s \neq t$) are rigid. He goes on to prove that if a Coxeter diagram $\mathcal{V}$ for the Coxeter system (W, S) has all of its edges labeled by integers divisible by 4, then W is rigid. In [2] Bahls classifies the reflection independent Coxeter groups with all diagram labels divisible by 4. In [3], he proves that to any given Coxeter group there is at most one even system corresponding to that group. Other rigidity results can be found in [4] and [5].

A result of Tits (Lemma 3.1) implies that a finitely generated Coxeter group has only finitely many conjugacy classes of involutions. This can be used to show that for any Coxeter system, (W, S) there is an integer n such that for any automorphism α of W, α^n maps each element of S to a conjugate of itself (see [6] for this and other results along these lines).

In Theorem 5.1 below, we exhibit a combinatorial condition which (when combined with results from [3] and [7]) determines whether or not two even Coxeter systems have the same set of reflections. As a direct application, we will indicate how these results allow characterization of all even reflection preserving Coxeter systems, even reflection independent Coxeter groups, and even rigid Coxeter groups.

2. Basic Concepts

The notion of a Coxeter system (W, S) and its corresponding diagram were given in Section 1. There is a notion of a Coxeter graph (used, for example, in [8]) which differs from that of a Coxeter diagram. We will call a Coxeter group W *even* if there is a system (W, S) such that for all $s \neq t$, m_{st} is not odd.

Basic examples of Coxeter groups are furnished by the dihedral groups $D_n = \langle a, b | a^2, b^2, (ab)^n \rangle$. Such a group is the symmetry group of a regular n-gon.

The dihedral groups also show us that an even Coxeter group may have a Coxeter presentation for which not every m_{ij} is even: the dihedral group

$$D_6 = \langle a, b | a^2, b^2, (ab)^6 \rangle$$

is isomorphic to the group

$$\langle c, d, g | c^2, d^2, g^2, (cd)^3, (cg)^2, (dg)^2 \rangle$$

as can be easily verified by setting $g = (ab)^3, d = aba$ and $c = b$.

A word w in the letters of the generating set S is called a *geodesic* (or is said to be geodesic) if w is of minimal length among all words representing the same element of the group as w. The following result is usually called the *Deletion Condition*. It follows from an elementary van Kampen diagram argument or see, for example, [8].

PROPOSITION 2.1 (The Deletion Condition). *If (W, S) is a Coxeter system and the product $a = a_1 \cdots a_n$ is not a geodesic in S, then for some $1 \leqslant i < j \leqslant n$, $a = a_1 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_n$, i.e. a_i and a_j may be deleted.*

If (W, S) is a Coxeter system and $T \subseteq S$, then the subgroup W_T of W generated by T is a Coxeter group. The pair (W_T, T) is a Coxeter system and the Coxeter diagram for (W_T, T) is a full subdiagram of the diagram for (W, S) (see, for example, Chapter 5 of [8]). Hence, a Coxeter presentation for (W_T, T) can be read from the Coxeter presentation of (W, S).

If G is a group and $H \leqslant G$ is a subgroup, we say that a homomorphism $\phi: G \to H$ is a *retraction* if $\phi(h) = h$ for all $h \in H$; thus the inclusion of H into G is a right inverse to ϕ. The primary advantage to working in an even Coxeter group is the following.

PROPOSITION 2.2. *If (W, S) is an even Coxeter system and $V \subset S$ then there is a retraction $W \to W_V$ whose kernel is the normal closure of $S - V$.*

Proof. Define $N(S - V)$ to be the normal closure of $S - V$ in W. An elementary examination of presentations shows that the presentation for $W/N(S - V)$, obtained from the Coxeter presentation for W by adding the relations $t = 1$ for all $t \in S - V$, reduces to the obvious Coxeter presentation for W_V. □

If $(st)^{2n+1}$ is a relation in a Coxeter presentation, then s and t are conjugate, $s = (ts)^n t(st)^n$. Hence, in a general Coxeter diagram, vertices s and t are conjugate if there is an odd path between them. It is worth noting that in an even system, no two distinct generators $s \neq t \in S$ are conjugate to one another.

PROPOSITION 2.3. *If (W, S) is a Coxeter system and $V \subset S$ then $W/N(V)$ is isomorphic to the subgroup of W generated by the set of all $s \in S$ such that in the Coxeter diagram for (W, S), s is not connected to an element of V by an odd path.*

This fact allows us to prove the following, which will be key to the proof of the main theorem.

LEMMA 2.4. *If (W, S) and (W, S') are Coxeter systems and S is a subset of the reflections of (W, S') then the reflections of (W, S) and (W, S') are the same.*

Proof. It suffices to show each element of S' is conjugate to an element of S. Let A' be the subset of S' consisting of elements of S' which are reflections of (W, S). If $s' \in S' - A'$ then, there is no odd path in the diagram for (W, S') from s' to an element in A'. Hence by Proposition 2.3, s' is not in the kernel of the quotient map from W to $W/N(A')$. But, $W/N(A') = W/N(S) = \{1\}$, the desired contradiction. □

We assume knowledge of van Kampen diagrams for the following proof.

LEMMA 2.5. *Suppose (W, S) is a Coxeter system, $u = s_1 \cdots s_n$ is geodesic in S and u is a reflection in (W, S). Then $n = 2m + 1$ for some integer m and $s_1 \cdots s_m = (s_{m+2} \cdots s_{2m+1})^{-1}$. (i.e. u is a conjugate of s_{m+1}.)*

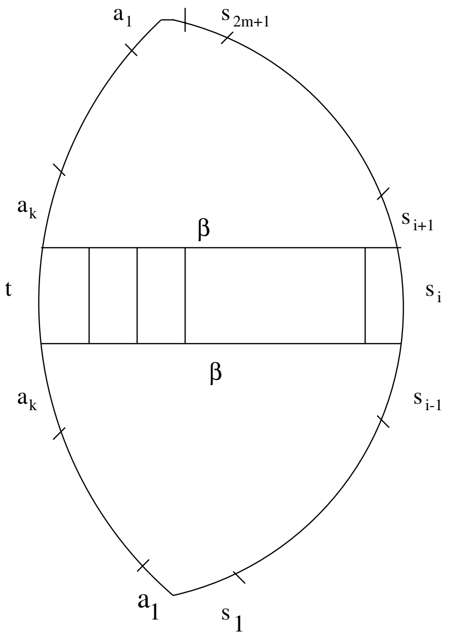

Figure 1. Lemma 2.5.

Proof. The Deletion Condition clearly implies that n is odd when u is a reflection. As u is a reflection, we can write u as $a_1 \cdots a_k t a_k \cdots a_1$ for t and all a_i in S. By the Deletion Condition we may assume that $a_1 \cdots a_k t$ (and so $t a_k \cdots a_1$) is geodesic. In a van Kampen diagram (Figure 1) for the trivial word $a_1 \cdots a_k t a_k \cdots a_1 s_{2m+1} \cdots s_1$, the string of 2-cells sharing opposite edges and beginning with the edge t must end at an edge labeled s_i. As $s_1 \cdots s_{i-1} = a_1 \cdots a_k \beta = s_{2m+1} \cdots s_{i+1}$ and u is geodesic, $i = m + 1$. □

Given a Coxeter diagram $\mathcal{V}$, we call a collection of vertices σ a *simplex* of $\mathcal{V}$ if the vertices in σ generate a complete subdiagram $\hat{\sigma}$, of $\mathcal{V}$. The *dimension* $\dim(\sigma)$ of the simplex σ is defined to be $|\sigma| - 1$. Clearly $\hat{\sigma}$ is simply the 1-skeleton of an abstract $\dim(\sigma)$-simplex generated by σ in the usual topological sense.

A simplex $\sigma \subseteq \mathcal{V}$ is said to be *maximal* if there is no simplex $\sigma' \subseteq \mathcal{V}$ such that $\sigma \subset \sigma'$, where the inclusion is strict. We say that σ is *spherical* if the subgroup W_σ generated by σ is finite, and in this case we say that W_σ is *spherical with respect to S* (or *with respect to* $\mathcal{V}$). We typically say simply that W_σ is spherical when the Coxeter presentation (or diagram) is clear.

A *maximal spherical simplex* is a spherical simplex that is not properly contained in another spherical simplex.

Consider a subgroup of W generated by C, a collection of vertices in $\mathcal{V}$ which is *not* a simplex. Then W_C is not finite, as there will be distinct elements $s, t \in C$ such that

$m_{st} = \infty$; thus every spherical subgroup is generated by a simplex. However, it need not be the case that every simplex is spherical; this we shall see later.

Throughout the remainder of this paper, 'diagram' will be taken to mean 'Coxeter diagram'.

3. Preliminary Lemmas

In this section we prove a result (Lemma 3.2) which will be of prime importance in proving our classification. This lemma will provide a way of relating the spherical simplices of one Coxeter diagram for a group W with the spherical simplices of another diagram for the same group.

The first piece in the puzzle is furnished by the following result of Tits, which is given as an exercise in [9].

LEMMA 3.1. *Let (W, S) be a Coxeter system. Then every finite subgroup of W is a subgroup of a conjugate of a spherical (with respect to S) subgroup.*

Consider two Coxeter systems (W, S) and (W, S') for W, and their respective diagrams $\mathcal{V}$ and $\mathcal{V}'$. The above lemma tells us that for any spherical simplex σ in $\mathcal{V}$ there is a spherical simplex σ' in $\mathcal{V}'$ and a group element w such that $W_\sigma \subseteq wW_{\sigma'}w^{-1}$. In case this containment is actually equality, we can say more.

LEMMA 3.2. *Suppose (W, S) and (W, S') are even Coxeter systems with corresponding diagrams $\mathcal{V}$ and $\mathcal{V}'$. Let σ_i (for $i \in \{1, 2, \ldots, n\}$) be simplices in $\mathcal{V}$ such that $\sigma = \cap\sigma_i \neq \emptyset$. Suppose that for each i, σ_i' is a simplex in $\mathcal{V}'$ such that for some $w_i \in W$, $W_{\sigma_i} = w_i W_{\sigma_i'} w_i^{-1}$. Let $\sigma' = \cap\sigma_i'$. Then there exists $w \in W$ such that $W_\sigma = wW_{\sigma'}w^{-1}$.*

Proof. We prove the lemma for the case $k = 2$, and an obvious induction proves the result in general.

From Theorem 2.7.4. in [10] (due to Kilmoyer) we have

$$W_\sigma = (w_1 W_{\sigma_1'} w_1^{-1}) \cap (w_2 W_{\sigma_2'} w_2^{-1}) = wW_{\bar{\sigma}}w^{-1} = vW_{\tilde{\sigma}}v^{-1},$$

where $w, v \in W$, $\bar{\sigma} \subset \sigma_1'$, and $\tilde{\sigma} \subset \sigma_2'$. Hence, $\bar{\sigma} = \tilde{\sigma} \subset \sigma'$ because (W, S') is even.

Similarly,

$$W_{\sigma'} = (w_1^{-1} W_{\sigma_1} w_1) \cap (w_2^{-1} W_{\sigma_2} w_2) = uW_{\hat{\sigma}}u^{-1}$$

for some $\hat{\sigma} \subset \sigma$.

We have

$$W_\sigma \subseteq wW_{\sigma'}w^{-1} \subseteq wuW_\sigma u^{-1}w^{-1}.$$

The restriction of the retraction $W \to W_\sigma$ (with kernel $N(S - \sigma)$), applied to $wuW_\sigma u^{-1}w^{-1}$ is an injection and so an isomorphism to W_σ. Therefore, $W_\sigma = wuW_\sigma u^{-1}w^{-1}$ and so, $W_\sigma = wW_{\sigma'}w^{-1}$. □

4. The Structure of Even Diagrams for the Same Coxeter Group

Suppose we are given two even Coxeter presentations for a Coxeter group W and the diagrams corresponding to those presentations. We now know from the previous section how it is that intersections of simplices from one diagram relate to intersections of those in the other. We now expand upon these ideas.

The first fact of which we shall make use is the following:

LEMMA 4.1. *Suppose that $\mathcal{V}$ and $\mathcal{V}'$ are two even diagrams corresponding to the same Coxeter group W. If σ is a maximal spherical simplex in $\mathcal{V}$, then there exists a unique maximal spherical simplex σ' in $\mathcal{V}'$ such that $W_\sigma = wW_{\sigma'}w^{-1}$.*

Proof. This follows from Lemma 3.1 and an elementary quotient argument. Namely, Lemma 3.1 implies that W_σ is contained in $wW_{\sigma'}w^{-1}$ for some σ' in $\mathcal{V}'$ and then that $W_{\sigma'}$ is contained in $wW_{\sigma''}w^{-1}$ for some σ'' in $\mathcal{V}$. Because (W, S) is even, we may show that $\sigma = \sigma''$ by using the quotient map which identifies every element of σ to 1. Then the finiteness of W_σ implies that $W_\sigma = wW_{\sigma'}w^{-1}$. □

The spherical simplices of an even Coxeter diagram have an elementary decomposition structure. The following two results are classical, and elementary proofs can be found in the PhD qualifying paper [2], of the first author.

LEMMA 4.2. *Let $\mathcal{V}$ be an even Coxeter diagram. Then the subgroup generated by a spherical simplex σ is isomorphic to one of the following direct products*:

$$\prod_{i=1}^{m} D_{n_i} \quad \textit{or} \quad \prod_{i=1}^{m} D_{n_i} \times Z_2^r$$

where each n_i is even.

What can we say about two decompositions for the same such subgroup? The decompositions are almost as 'fine' as they can be, in that the factors are very nearly indecomposable. In fact, we have the following result (see [2]):

LEMMA 4.3. *Suppose the dihedral group D_n decomposes nontrivially as a direct product $D_n \cong H \times K$. Then $n = 2k$ where k is odd, $H \cong D_k$, and $K \cong Z_2$.*

This lemma allows us to consider any decomposition of the form $\prod D_{n_i}$ and create a 'finest' decomposition merely by splitting those D_{n_i} where $n_i = 2k_i$ for k_i odd, into the direct product of two groups: $D_{n_i} \cong D_{k_i} \times Z_2$.

Now consider two such 'finest' decompositions of a given group. These decompositions will be very similar to one another, by the Krull–Schmidt Theorem (see [11]).

THEOREM 4.4 (Krull–Schmidt). *Suppose that a group G satisfies both the ascending and descending chain conditions for normal subgroups (in particular, finite groups clearly satisfy both of these properties). Given any two decompositions of G,*

$$G \cong H_1 \times H_2 \times \cdots \times H_m$$

and

$$G \cong K_1 \times K_2 \times \cdots \times K_n$$

into indecomposable groups H_i and K_j, then $m = n$ and there is a permutation ϕ such that $H_i \cong K_{\phi(i)}$.

So suppose we are given two even spherical simplices σ_1 and σ_2 which generate isomorphic groups G_1 and G_2. Decomposing both groups into indecomposable factors according to the structure of the corresponding simplices, we have two isomorphic decompositions

$$\prod D_{n_i} \times \prod D_{m_j} \times (Z_2)^r$$

and

$$\prod D_{k_i} \times \prod D_{l_j} \times (Z_2)^s,$$

where all of the n_i and k_i are divisible by 4, and all of the m_j and l_j are odd. Because all of the factors are indecomposable and because the groups G_i are finite, the Krull–Schmidt Theorem states that there must be a bijection between the factors in these decompositions as given above. Thus the even dihedral groups in each decomposition match up with one another, as do the odd ones, and since by hypothesis both of the simplices that we began with are even, we see that each of the factors of the original decompositions of the form D_{2b_i} for b_i odd match up as well. The result is that both simplices σ_1 and σ_2 must have the same edge labels, and are therefore isomorphic as graphs.

This gives us the following result. (Recall that a simplex σ is a set of vertices in a diagram $\mathcal{V}$, whereas $\hat{\sigma}$ is the full subdiagram of $\mathcal{V}$ generated by σ.)

LEMMA 4.5. *If σ and σ' are even spherical simplices and $W_\sigma \cong W_{\sigma'}$, then $\hat{\sigma}$ and $\hat{\sigma}'$ are isomorphic as labeled diagrams.*

Now suppose that we are given two even diagrams $\mathcal{V}$ and $\mathcal{V}'$ for the same Coxeter group, and consider maximal spherical simplices σ_i in $\mathcal{V}$ and σ_i' in $\mathcal{V}'$ such that for each i there is an element $w_i \in W$ so that $W_{\sigma_i} = w_i W_{\sigma_i'} w_i^{-1}$. We know now that for every i, $\hat{\sigma}_i$ and $\hat{\sigma}_i'$ are isomorphic as labeled diagrams. If C_i (C_i') is the commutator subgroup of W_{σ_i} $(W_{\sigma_i'})$ then in W, $N(C_i) = N(C_i')$. A Coxeter diagram for the Coxeter group $W/N(C_i) = W/N(C_i')$ is obtained from $\mathcal{V}$ $(\mathcal{V}')$ by changing all edge labels in $\hat{\sigma}_i$ $(\hat{\sigma}_i')$ to 2. Doing the same for all C_i (C_i') produces diagrams for isomorphic right-angled Coxeter groups with the same unlabeled underlying graphs as $\mathcal{V}$ and $\mathcal{V}'$. As right-angled groups are rigid, we conclude that $\mathcal{V}$ and $\mathcal{V}'$ have isomorphic unlabeled graphs.

LEMMA 4.6. *If $\mathcal{V}$ and $\mathcal{V}'$ are even diagrams corresponding to the same Coxeter group W, then $\mathcal{V}$ and $\mathcal{V}'$ are isomorphic as unlabeled graphs.*

There is one more fact of which we take note. It will be necessary in the proof of the classification theorem given below. Note that this result holds for right-angled groups as well.

LEMMA 4.7. *Let $\mathcal{V}$ and $\mathcal{V}'$ be even diagrams for the same Coxeter group W. Then for any even integer n, the number of edges labeled n in $\mathcal{V}$ is the number of edges labeled n in $\mathcal{V}'$.*

Proof. Suppose that this result were not true, so that there exists a Coxeter group W with even diagrams $\mathcal{V}$ and $\mathcal{V}'$ for which there is an integer $k > 2$ and there are a different number of edges labeled k in $\mathcal{V}$ than there are in $\mathcal{V}'$. Let us furthermore assume that we have picked W, $\mathcal{V}$, $\mathcal{V}'$, and k so that the number of edges labeled k appearing in $\mathcal{V}$ is minimum over such witnesses to the falsity of the result.

Then consider an edge labeled k in $\mathcal{V}$ with end points in the maximal spherical simplex σ. There is a maximal spherical simplex $\sigma' \subseteq \mathcal{V}'$ such that $W_\sigma = wW_{\sigma'}w^{-1}$ for some $w \in W$. Hence, $\hat{\sigma}$ and $\hat{\sigma}'$ contain the same number of edges labeled k.

Let C (C') be the commutator subgroup of W_σ ($W_{\sigma'}$). In W, $N(C) = N(C')$. Diagrams for $W/N(C) = W/N(C')$ are obtained from $\mathcal{V}$ ($\mathcal{V}'$) by changing the labels of all edges in $\hat{\sigma}$ ($\hat{\sigma}'$) to 2. Since we have eliminated the same number of edges labeled k from both diagrams, the two new diagrams also have a different number of edges labeled k. But because we chose σ so that it contained an edge labeled k, we have eliminated at least one such edge, and have obtained a 'smaller' counterexample, whereas we assumed that we had a minimal counterexample. Thus the result is proved. □

Remark. Lemmas 4.5, 4.6, and 4.7 are used by Bahls in [3] to show that in fact two even diagrams for a Coxeter group are isomorphic as labeled graphs (See Theorem 5.2).

5. The Main Theorem

In this section we state our main theorem and indicate some of its consequences.

For a vertex x in a Coxeter diagram $\mathcal{V}$, let $st(x)$ denote the set containing x and all vertices adjacent to x by some edge. This is called the *star* of x. The *2-star* of x, written $st_2(x)$, is defined to be the set containing x and all vertices in $\mathcal{V}$ which are adjacent to x via an edge labeled 2. (Thus if $s \in st_2(x)$, s and x commute.) Define the *link* $lk(x)$ of the vertex x to be $st(x) - \{x\}$ and the *2-link* $lk_2(x)$ of the vertex x to be $st_2(x) - \{x\}$. For a given vertex $x \in \mathcal{V}$ (or $\mathcal{V}'$), denote by τ_x the (spherical) simplex which is formed by taking the intersection of all of the maximal spherical simplices containing x. Note that, in particular, τ_x satisfies the hypotheses of Lemma 3.2.

Our main theorem is:

THEOREM 5.1. *Suppose that $\mathcal{V}$ is an even Coxeter diagram corresponding to the system (W, S).*

(1) *If there are distinct vertices x and y in $\mathcal{V}$ such that $st(x) \subset st_2(y)$, then there is an automorphism α of W taking x to xy (which is not a reflection in (W, S)). (i.e. (W, S) and $(W, \alpha(S))$ have a different set of reflections.)*
(2) *If there are distinct vertices $x, y, z \in \mathcal{V}$ and an edge $[yz]$* in $\mathcal{V}$ *with label $n > 2$ such that $y, z \in \tau_x$ then there is an automorphism α of W taking x to $x(yz)^{\frac{n}{2}}$ (which is not a reflection in (W, S)).*
(3) *If no pair or triple of vertices, as described in (1)* and *(2)*, exists in $\mathcal{V}$ *then the reflections of (W, S) and (W, S') are the same for any other even system (W, S').*

This result characterizes all reflection preserving even Coxeter systems. If an even Coxeter group W is rigid, then our result determines if W is reflection independent.

For an even Coxeter system (W, S) and corresponding diagram $\mathcal{V}$, a simplex σ is spherical iff there are no consecutive edges $[xy]$ and $[yz]$ in $\hat{\sigma}$, both with labels > 2 (see Lemma 5.6 below). If one were interested in a more 'algorithmic' condition than that given in (2) of Theorem 5.1, then the following may suffice.

Remark. If x, y and z are distinct vertices in $\mathcal{V}$, and $[yz]$ has label $n > 2$, then $y, z \in \tau_x$ iff edges $[xy]$ and $[xz]$ exist and have label 2 and there is no vertex $w \in lk(x)$ such that one of the groups $W_{\{w,x,y\}}$, $W_{\{w,x,z\}}$ and $W_{\{w,y,z\}}$ is infinite (equivalently one of the subdiagrams of $\mathcal{V}$ generated by $\{w, x, y\}, \{w, x, z\}$ and $\{w, y, z\}$ is not a $\{2, 2, m\}$ triangle).

Remark. Following the results in [3] and [7], we can prove a great deal more concerning the uniqueness of representations of even Coxeter groups.

In [3], Bahls proves the following theorem:

THEOREM 5.2. *Any two even diagrams $\mathcal{V}$ and $\mathcal{V}'$ for the same Coxeter group W are isomorphic by a diagram isomorphism ψ which satisfies the following two conditions:*

(1) *If $\sigma \subseteq \mathcal{V}$ and $\sigma' \subseteq \mathcal{V}'$ are simplices generating conjugate spherical subgroups, then ψ takes the collection of vertices of σ to the collection of vertices of σ'.*
(2) *If $[xy] \subseteq \mathcal{V}$ and $[x'y'] \subseteq \mathcal{V}'$ are edges will labels greater than 2 such that the groups $\langle (xy)^2 \rangle$ and $\langle (x'y')^2 \rangle$ are conjugate, then $\{\psi(x), \psi(y)\} = \{x', y'\}$.*

As a consequence, there is at most one even diagram (up to isomorphism) for any given Coxeter group. Mihalik proves in [7] that there is a simple combinatorial condition which may be used to check whether or not an even Coxeter group is rigid. In particular, he proves the following result:

THEOREM 5.3. *Suppose that (W, S) is a finitely generated even Coxeter system with diagram $\mathcal{V}$. Then W has a noneven system if and only if there is an edge $[ab]$ of $\mathcal{V}$ with* *label* $2(2k+1)$ *for* $k>0$ *such that with the exception of* $[ab]$ *each edge containing a has label 2, and for each such edge* $[ac]$ *there is also an edge* $[bc]$ *with label 2.*

As a consequence of the proof of our result, it can be shown that if an even Coxeter group has a noneven presentation, then it is not reflection independent. Thus we have

THEOREM 5.4. *Suppose that* (W, S) *is an even Coxeter system. If* (W, S) *is reflection independent, then* W *is rigid.*

We can use this fact to determine whether or not a given even Coxeter system (W, S) represents a reflection independent group W. First we check to see if W is rigid, using Mihalik's criterion given above. If it is not rigid, it cannot be reflection independent. If W is rigid, then we may use the main theorem from this paper to determine whether or not the given system is reflection preserving. If (W, S) is reflection preserving, the fact that it is rigid ensures that it is also reflection independent. If (W, S) is not reflection preserving, it cannot be reflection independent either.

Therefore the results of this paper, in conjunction with those from [3] and [7], may be used to completely characterize those even Coxeter systems which are rigid, reflection preserving, or reflection independent.

Finally, in order to highlight the distinction between reflection independence and reflection preservation, we indicate an example of a Coxeter group which is not reflection independent, but for which every Coxeter system is reflection preserving. Consider the group presented by

$$\langle a, x, y, z | a^2, x^2, y^2, z^2, (xy)^4, (yz)^3, (ay)^2, (az)^2 \rangle.$$

It can be shown using elementary techniques that this group is isomorphic to that presented by

$$\langle x', y', t' | x'^2, y'^2, t'^2, (x'y')^4, (y't')^6 \rangle$$

and that these are the only two means of presenting this group. The second of these presentations is seen to be reflection preserving by Theorem 5.1, and Lemma 3.1 may be utilized to show that the first presentation is also reflection preserving. However, as the group is not rigid, it cannot be reflection independent, by Theorem 5.4 above.

6. Structure Lemmas

In this section we prove part of the main theorem. We also use results from Section 4 to prove a collection of structural lemmas which are used in Section 7 to finish the proof of the main theorem.

By the Deletion Condition, all reflections have odd length. Hence the following Lemma verifies part (1) of our theorem.

LEMMA 6.1. *Suppose $\mathcal{V}$ is an even diagram corresponding to the Coxeter system (W, S). If there are vertices $x \neq y$ in $\mathcal{V}$ such that $st(x) \subseteq st_2(y)$, then there is an automorphism of W taking x to xy.*

Proof. It is easy to check that the map α: $W \to W$ extending the map $\alpha(x) = xy$ and $\alpha(s) = s$ for all $s \in S - \{x\}$ is a homomorphism. Since this map is clearly an involution, it is an automorphism. □

We will frequently need to determine whether or not a two-dimensional simplex is spherical. The following result can be found in [9], for example.

LEMMA 6.2. *Let σ be a two-dimensional simplex in a Coxeter diagram $\mathcal{V}$. Then σ is spherical if and only if the sum of the reciprocals of the edge labels of $\hat{\sigma}$ exceeds 1. In particular, the only even spherical two-dimensional simplices are those with labels $\{2, 2, n\}$ for n even.*

LEMMA 6.3. *Suppose $\mathcal{V}$ is an even diagram corresponding to the Coxeter system (W, S). If x, y, z are distinct vertices of $\mathcal{V}$, $[yz]$ is an edge of $\mathcal{V}$ with label $n > 2$ and $y, z \in \tau_x$ then there is an automorphism of W taking x to $x(yz)^{\frac{n}{2}}$.*

Proof. By Lemma 6.2, x commutes with both y and z. Suppose a is a vertex of $\mathcal{V} - \{x, y, z\}$ such that xa has finite order. Then a commutes with both y and z, as the edge $[xa]$ lies in a maximal spherical simplex (which must contain τ_x). Consider the map α of S into W which takes x to $x(yz)^{\frac{n}{2}}$ and each $s \in S - \{x\}$ to itself. To see that α extends to a homomorphism of W, it suffices to show that if $a \in S$ and $(xa)^m$ is a Coxeter relation of (W, S) (with $m \neq \infty$) then $(x(yz)^{\frac{n}{2}}a)^m$ is the trivial element of W. The relation x^2 becomes $(x(yz)^{\frac{n}{2}})^2$. As x has order 2 and commutes with the order 2 element $(yz)^{\frac{n}{2}}$, $(x(yz)^{\frac{n}{2}})^2 = 1$ in W. The relation $(xy)^2$ becomes $(x(yz)^{\frac{n}{2}}y)^2$. This element is trivial in W as the order 2 element x commutes with y and z and $(yz)^{\frac{n}{2}}y$ is a conjugate of y or z and hence has order 2. Similarly for the relation $(xz)^2$. If $a \in S - \{x, y, z\}$ and $(ax)^m$ is a Coxeter relation with $m \neq \infty$, then as a and x commute with y and z in W, $(ax(yz)^{\frac{n}{2}})^m = (ax)^m(yz)^{m\frac{n}{2}} = 1$ in W. Hence, α extends to a homomorphism of W. Clearly, α is an involution and hence an automorphism of W. □

By Lemma 2.5, $x(yz)^{\frac{n}{2}}$ is not a reflection of (W, S) and we have finished part (2) of our theorem.

If u is a word in the letters of a generating set for a group, then let $\bar{u}$ be the corresponding element of the group. The following result is of prime importance in finishing the proof of the main theorem.

LEMMA 6.4. *Suppose that we are given maximal spherical simplices σ_i and σ'_i of the even Coxeter systems (W, S) and (W, S'), respectively, for $i = 1, \ldots, k$ such that $W_{\sigma_i} = w_i W_{\sigma'_i} w_i^{-1}$. Define $\sigma = \cap \sigma_i$ and $\sigma' = \cap \sigma'_i$ and say $W_\sigma = wW_{\sigma'}w^{-1}$. Suppose that W_σ (and therefore $W_{\sigma'}$) is right-angled. Let x be an element of σ. If $x = a_1\bar{u}_1a_1^{-1} = a_2\bar{u}_2a_2^{-1}$ for geodesics u_i in the letters of σ' and $a_i \in W$, then $\bar{u}_1 = \bar{u}_2$.*

If we additionally assume that x is central in W_{σ_i}, then every letter of u_i is central in $W_{\sigma'_i}$.

Proof. As $W_{\sigma'}$ is right-angled and finite, and therefore Abelian, each letter of u_i occurs exactly once. Note that x is an element of the normal closure in W of the set of letters of u_1, and so each letter of u_2 must be a letter of u_1. (This may be shown by considering the quotient map which identifies to 1 all letters occurring in u_1.) Similarly each letter of u_1 appears in u_2. Thus $\bar{u}_1 = \bar{u}_2$.

Now assume that x is central in W_{σ_i}. Then $\bar{u}_1$ is central in $W_{\sigma'_i}$. We show that each letter of u_1 commutes with every letter in σ'_i. Consider a letter $s \in \sigma'_i$. If $s \in \sigma'$, we see s commutes with every letter of u_1 simply because $W_{\sigma'}$ is Abelian. If $s \in \sigma'_i - \sigma'$ and t is any letter in u_1, then by the fact that $s\bar{u}_1 = \bar{u}_1 s$, taking the retraction which maps from W onto $W_{\{s,t\}}$ shows that s and t must also commute. □

7. The Proof of the Main Theorem

Proof of Theorem 5.1. Suppose (W, S) and (W, S') are even with diagrams $\mathcal{V}$ and $\mathcal{V}'$, respectively. Then, by Lemma 2.4 and the results of the previous section, we need only prove that if there are no two (resp. three) distinct vertices x, y (resp. x, y, z) in $\mathcal{V}$ such that these vertices satisfy condition (1) (resp. (2)) of Theorem 5.1, then each element of S is conjugate to an element of S'.

Assume some $s \in S$ is not a reflection in (W, S'). By Lemma 3.2, $\tau_s \neq \{s\}$. We proceed with a least criminal argument based on the following:

Minimality condition: Choose $x \in S$ such that x is not a reflection of (W, S') and so that if $y \in S$ such that $|\tau_y| < |\tau_x|$, then y is a reflection of (W, S').

The argument splits naturally into two cases: When W_{τ_x} is Abelian and when it is not. We first consider the case W_{τ_x} Abelian, as the validity of this case will be used to establish the second case.

For the remainder of the proof we let $\{\sigma_j\}$ be the collection of maximal spherical simplices which contain x, and we let σ'_j denote the maximal spherical simplex of $\mathcal{V}'$ conjugate to σ_j, for each j. We also note that if $y \in \tau_x$, then $\tau_y \subseteq \tau_x$. This fact will be used without mention in the proofs that follow.

Case 1. W_{τ_x} is Abelian.

Let τ' be the (right-angled) simplex in $\mathcal{V}'$ such that $W_{\tau_x} = wW_{\tau'}w^{-1}$ for some $w \in W$.

Consider the product $z' = z'_1 \cdots z'_l$ with $z'_i \in \tau'$ such that $x = wz'w^{-1}$. Note that this product must have length at least 2 (or x would be a reflection in $\mathcal{V}'$).

CLAIM 7.1. *There is some $i \in \{1, \ldots, l\}$ such that $\tau_{z'_i} = \tau'$.*

Proof. Suppose this were not the case. Then $\tau_{z'_i}$ is a proper subset of τ', and there is a proper subset τ_i of τ_x such that W_{τ_i} is conjugate to $W_{\tau_{z'_i}}$ (by Lemma 3.2). The definition of τ_x implies that $x \notin \tau_i$. By our minimality condition, some element of τ_i is conjugate to z'_i. That is, for every z'_i appearing in z', there is a generator $z_i \in \tau_x$ such that $z_i = w_i z'_i w_i^{-1}$ for some element $w_i \in W$. Since $W_{\tau_x} = wW_{\tau'}w^{-1}$, we have

$z_i = wu_i'w^{-1}$ for some $u_i' \in W_{\tau'}$. By Lemma 6.4 $u_i' = z_i'$ and so $z_i = wz_i'w^{-1}$ for each z_i' appearing in z'.

Thus we have

$$x = wz'w^{-1} = wz_1' \cdots z_l'w^{-1} = ww^{-1}z_1w \cdots w^{-1}z_lww^{-1} = z_1 \cdots z_l$$

which cannot be. Thus it must be that some z_i' appearing in z' satisfies $\tau_{z_i'} = \tau'$. □

Without loss we may assume that z_1' satisfies the conclusion of Claim 7.1.

CLAIM 7.2. *A Coxeter diagram for $W/N(x) = W/N(z')$ is obtained from $\mathcal{V}'$ by removing the vertex z_1' (and all incident edges) and changing all edge labels greater than 2 between a vertex of $lk(z_1')$ and a point of $\{z_2', \ldots, z_l'\}$ to the label 2.*

Proof. Consider the presentation of $W/N(x) = W/N(z')$ obtained by adding the relation $z_1' = z_2' \cdots z_l'$ to the relations of P' ($\equiv$ the presentation defined by $\mathcal{V}'$). Let $\overline{P}'$ be the equivalent presentation for $W/N(z')$ obtained by removing the generator z_1' and the relation $z_1' = z_2' \cdots z_l'$ and replacing each occurrence of z_1' in a relation by the word $z_2' \cdots z_l'$. Let R' be the set of relations of P' that do not contain the letter z_1'. The relation $(z_1')^2$ of P' is changed to $(z_2' \cdots z_l')^2$, which is easily seen to be a consequence of the relations of R' (as the z_j' commute and have order 2). If $z_i' \in \tau' - \{z_1'\}$ then the relation $(z_1'z_i')^2$ of P' becomes $(z_2' \cdots z_l'z_i')^2$ in $\overline{P}'$, which is also a consequence of the relations of R'.

Suppose $a \in S' - \tau'$, and $(z_1'a)^n$ is a relation of P'. As $\tau_{z_1'} = \tau'$, a maximal spherical simplex of $\mathcal{V}'$ containing the edge $[z_1'a]$ also contains τ'. Hence, if $n > 2$, then $(az_i')^2 \in R'$ for all $i \in \{2, \ldots, l\}$. This implies that $(z_2' \cdots z_l'a)^2$ (and therefore $(z_2' \cdots z_l'a)^n$) is a consequence of R'.

If instead, $n = 2$, then again $(z_i'a)^{m_i} \in R'$ for all $i \in \{2, \ldots, l\}$ and at most one m_i is larger than 2. If all $m_i = 2$, then as above $(z_1'a)^2$ is a consequence of R'. If some $m_i \neq 2$ (say for simplicity $m_2 > 2$) then $(z_2'a)^2$ is a consequence of $R' \cup \{(z_2' \cdots z_l'a)^2\}$. As $(z_2' \cdots z_l'a)^2$ is a consequence of $R' \cup \{(z_2'a)^2\}$ we are finished. □

The following is clear.

CLAIM 7.3. *A diagram for $W/N(x)$ is obtained from $\mathcal{V}$ by removing x (and all incident edges).*

Now for each element or subset A of W, let $\overline{A}$ be the image of A under the quotient map of W to $W/N(x)$. Claim 7.2 implies that $\{\bar{z}_2', \ldots, \bar{z}_l'\}$ is central in $W_{\bar{\sigma}_i'}$ for all i. As $\bar{z}_2'$ is conjugate to some product of $\bar{z}_j$ in $W/N(x)$ (where the elements $\bar{z}_i$ are defined as in Claim 7.1), each of these $\bar{z}_j$ is central in $W_{\bar{\sigma}_i}$ for all i, by Lemma 6.4. If $\bar{z}_j$ is central in $W_{\bar{\sigma}_i}$ then by Claim 7.3, z_j is central in W_{σ_i}, but no z_j is central in all W_{σ_i} as $st(x) \not\subset st_2(z_j)$. This is the desired contradiction and Case 1 is finished.

Case 2. W_{τ_x} is not Abelian.

The only case to consider is when τ_x contains an edge $[xy]$ with label $n > 2$. By our hypotheses all other labels on edges with both vertices in τ_x are 2.

Say $\mathcal{V}$-simplex W_{τ_x} is conjugate to $W_{\tau'}$ for τ' a $\mathcal{V}'$-simplex. Then by Lemma 4.5, $\hat{\tau}_x$ and $\hat{\tau}'$ are isomorphic as labeled diagrams. Let $[x'y']$ be the edge of τ' with label n. Let

$Z = \tau_x - \{x, y\}$ and $Z' = \tau' - \{x', y'\}$. Then Z and Z' are central in W_{τ_x} and $W_{\tau'}$, respectively. As $\tau_z \subset \tau_x$ for all $z \in Z$ and since by hypothesis, not both x and y are in τ_z for any $z \in Z$, τ_z is Abelian for all $z \in Z$. Assume $Z = \{z_1, \ldots, z_k\}$. By Case 1, we may assume that each $z_i \in Z$ is conjugate to $z_i' \in Z'$. Because x does not commute with y, x must be conjugate to an element of order 2 in $W_{\tau'}$ which is not central in $W_{\tau'}$. Thus x is one of the elements $wq'z'w^{-1}$, where z' is a (possibly trivial) element of $W_{Z'}$ and q' is in the set

$$Q = \{x', y', x'y'x', y'x'y', \ldots, (x'y')^{n/2-1}x', (y'x')^{n/2-1}y'\}.$$

Note that the element $(x'y')^{n/2} = (y'x')^{n/2}$ is central in $W_{\tau'}$ and so x cannot be conjugate to an element of the form $(x'y')^{n/2}z'$.

If z' is trivial, we are done, since each of the elements in Q is a reflection. Suppose that this is not the case. If x is $wq'z'w^{-1}$ where q' is of the form $(x'y')^k x'$, then because each of the elements z_i' commutes with both x' and y', we may write x as

$$w(x'y')^k x'z'w^{-1} = w(x'y')^{k/2}(x'z')(y'x')^{k/2}w^{-1}$$

when k is even and

$$w(x'y')^k x'z'w^{-1} = w(x'y')^{\frac{k-1}{2}}x'(y'z')x'(y'x')^{\frac{k-1}{2}}w^{-1}$$

when k is odd. A completely analogous argument shows that y is conjugate to $x'z''$ or $y'z''$ for some $z'' \in W_{Z'}$. An examination of the quotient of $W_{\tau_x} = wW_{\tau'}w^{-1}$ by its center shows that x and y are not conjugate to $x'z'$ and $x'z''$, respectively or $y'z'$ and $y'z''$, respectively. Hence, we may assume that x is conjugate to $x'z'$ and y to $y'z''$. In particular $N(x) = N(x'z')$.

On one hand, a diagram for the group $W/N(x)$ is found by merely removing the vertex x and all incident edges from the diagram for (W, S). On the other hand, we will produce a diagram for the isomorphic group $W/N(x'z')$ by considering the effect of adding the relation $x' = z'^{-1} = z' = z_1' \cdots z_s'$ to the presentation P' for the system (W, S') and by Tietze transformations changing this presentation to a Coxeter presentation. (Recall that a *Tietze transformation* is an operation on a group presentation which results in a second presentation, for an isomorphic group. Precisely, if one is given the presentation $\langle S|R\rangle$, one may add a generator s while simultaneously adding the relation $s = w(S)$ expressing s as a word in the original generators, or one may add a relation $r = 1$ which follows as a consequence of the original relations R. Corresponding to each of these transformations is an 'inverse' transformation, removing a generator or removing a relation.)

First alter this presentation for $W/N(x'z')$ by removing the generator x' and the relation $x' = z_1' \cdots z_s'$ and replace every occurrence of x' in the remaining relations with the product z'. Call the resulting presentation $\overline{P}'$.

Let R' be the set of relations of P' that do not contain the letter x'. The relation $(x')^2$ has been changed to $(z')^2$, which is a consequence of R' (since all of the z_i' have order 2 and commute with one another in W). The relation $(x'z_j')^2$ is changed to $(z'z_j')^2$ in $\overline{P}'$, which is also a consequence of R'.

Next consider the relation $(x'y')^n$. As every z_i' commutes with y' in W, the relation $(z'y')^n$ is a consequence of R' (in fact, $z'y')^2$ follows).

If there is a relation of the form $(x'a')^p$ (for a' not in τ') in P', a different approach must be used. We will show that $p = 2$, and the corresponding relator $(z'a')^2$ in $\overline{P}'$, can be removed from $\overline{P}'$ as a consequence of other relators, but in doing so a Coxeter relation of the form $(z_i'a')^k$ (for $k > 2$) must also be changed to $(z_i'a')^2$.

CLAIM 7.4. *If $a' \in S' - \tau'$ and $x'a'$ has order p, then $p = 2$.*

Proof. If $p > 2$, then a maximal spherical simplex β' containing a' and x' does not contain y', and $x' \in \beta' \cap \tau' \subset \{x'\} \cup Z'$. Let β be a maximal simplex in $\mathcal{V}$, conjugate to β'. Now β does not contain x, for otherwise, β would equal some σ_i and W_{τ_x} contains more elements than $W_{\beta'\cap\tau'}$ contrary to Lemma 3.2. So $\beta \cap \tau_x \subset \{y\} \cup Z$.

Now for each $t \in \beta \cap \tau_x$, W_{τ_t} is Abelian and so by Case 1, t is conjugate to an element of $\beta' \cap \tau'$. As no two elements of $\beta \cap \tau_x$ are conjugate and $|\beta \cap \tau_x| = |\beta' \cap \tau'|$ some element of $\beta \cap \tau_x$ is conjugate to x' (i.e. x' is conjugate to y or some z_i). If x' were conjugate to $z_i \in Z$, then as z_i is conjugate to z_i', x' is conjugate to z_i', which is impossible.

Hence, we must have x' conjugate to y. But this is also impossible as y is conjugate to $y'z''$ and clearly x' is not conjugate to $y'z''$, as if this were the case, the quotient map which identifies y and all z_i' with 1 would also identify x' to 1, a contradiction. □

CLAIM 7.5. *If $a' \in (S' - \tau') \cap st(x')$ then $a'y'$ has order 2.*

Proof. If not, there is a maximal spherical simplex β' containing a' and x' (but not y'). Now $x' \in \beta' \cap \tau'$ and β' is conjugate to a maximal simplex β in $\mathcal{V}$ which does not contain x. Since $W_{\beta\cap\tau_x}$ is Abelian and conjugate to $W_{\beta'\cap\tau'}$, x' is conjugate to an element of $\tau_x - \{x\} = \{y\} \cup Z$. If x' were conjugate to $z_i \in Z$, then x' is conjugate to z_i' which is impossible. If x' were conjugate to y, then x' would be conjugate to $y'z''$ which it is clearly not, as in the proof of Claim 7.4. This completes the proof of our claim. □

CLAIM 7.6. *A diagram for $W/N(x'z')$ is obtained from $\mathcal{V}'$ by removing the vertex x' (and all incident edges) and changing some labels > 2 on edges between vertices of $(S' - \tau') \cap st(x')$ and Z' to the label 2.*

Proof. If $a' \in (S' - \tau') \cap st(x')$ then by Claims 7.4 and 7.5, there exists a maximal spherical simplex β' in $\mathcal{V}'$, containing a', x' and y'. Now, $W_{\beta'}$ is conjugate to W_β for β a maximal spherical simplex in $\mathcal{V}$. Observe that β must contain x (otherwise $x \notin N(\beta \cup Z) = N(\beta' \cup Z')$, but $x = x'z'$) and therefore τ_x. This implies β' contains τ'. As $W_{\beta'}$ is finite, at most one edge $[a'z_i']$ has label > 2. Our relation $(z'a')^2$ is a consequence of R' if $z_i'a'$ has order 2 for all i. Otherwise we assume $(a'z_1')^k$ is a relation with $k > 2$. The relations $(z'a')^2$, $(a'z_1')^k$, $(z_i'z_j')^2$ for all $i \neq j$, $(z_i')^2$ for all i, and $(a'z_i')^2$ for $i > 1$ are equivalent to (i.e. have the same normal closure in the free group on S'

as) the relations $(z_i'z_j')^2$ for all $i \neq j$, $(z_i')^2$ for all i, and $(a'z_i')^2$ for all i. Making the corresponding replacements in our presentation for $W/N(z'x')$ gives a Coxeter presentation of $W/N(z'x')$ with diagram as described in the statement of our claim. $\square$

If v is a vertex of a diagram $\mathcal{V}$ then we define the *spoke of v in* $\mathcal{V}$ (written $sp(v, \mathcal{V})$) to be the subdiagram of $\mathcal{V}$ consisting of all labeled edges containing v.

CLAIM 7.7. *The diagrams $sp(x, \mathcal{V})$ and $sp(x', \mathcal{V}')$ are isomorphic as labeled diagrams.*

Proof. By Claim 7.4, all edges in $sp(x', \mathcal{V}')$ have label 2 except $[x'y']$. If $a \in lk(x) - \{y\}$, then any maximal spherical simplex containing $[ax]$ contains y and so the edge label of $[ax]$ is 2. So, all edges of $sp(x, \mathcal{V})$ have label 2 except $[xy]$. It suffices to show $sp(x, \mathcal{V})$ and $sp(x', \mathcal{V}')$ have the same number of edges. Let $\overline{\mathcal{V}}$ and $\overline{\mathcal{V}}'$ be the subdiagrams of $\mathcal{V}$ and $\mathcal{V}'$, respectively for $W/N(Z) = W/N(Z')$ obtained by removing the vertices of Z and Z' from $\mathcal{V}$ and $\mathcal{V}'$, respectively. By Lemma 4.6, $\overline{\mathcal{V}}$ and $\overline{\mathcal{V}}'$ have the same number of edges. Under the quotient map from W to $W/N(Z)$, the image of x is conjugate to the image of x' and so diagrams for $W/N(Z \cup \{x\}) = W/N(Z' \cup \{x'\})$ can be obtained by removing $sp(x, \overline{\mathcal{V}})$ from $\overline{\mathcal{V}}$ and $sp(x', \overline{\mathcal{V}}')$ from $\overline{\mathcal{V}}'$. As the resulting diagrams have the same numbers of edges (by Lemma 4.6), $sp(x, \overline{\mathcal{V}})$ and $sp(x', \overline{\mathcal{V}}')$ have the same number of edges. As the number of edges in $sp(x, \mathcal{V})$ (and $sp(x', \mathcal{V}')$) is $|Z|$ plus the number of edges in $sp(x, \overline{\mathcal{V}})$ (and $|Z'|$ plus the number of edges in $sp(x', \overline{\mathcal{V}}')$, respectively), we are finished. $\square$

If $a \in st(x)$ then there is a maximal spherical simplex containing $[ax]$ and therefore τ_x. Hence az_i has finite order for all i. As $st(x) \not\subset st_2(z_i)$ there exists $a \in st(x)$ such that $[az_i]$ has label $k > 2$. Choose a maximal spherical simplex β containing $[ax]$ (and therefore τ_x). If β' is the maximal spherical simplex of $\mathcal{V}'$ such that W_β is conjugate to $W_{\beta'}$ then $\tau' \subset \beta'$. As $[az_i] \subset \beta$, z_i is not central in W_β.

CLAIM 7.8. *The element z_i' is not central in $W_{\beta'}$.*

Proof. Recall, if d is a geodesic in a Coxeter system (W, S), then $\bar{d}$ is the corresponding element in W. Say $wz_iw^{-1} = z_i'$ for $w \in W$. If $vW_\beta v^{-1} = W_{\beta'}$, then say $vz_iv^{-1} = \bar{q}' \in W_{\beta'}$ for some geodesic q' in $(W_{\beta'}, \beta')$. Now, $w^{-1}z_i'w = v^{-1}\bar{q}'v$ and so $vw^{-1}z_i'wv^{-1} = \bar{q}'$. By Lemma 2.5, we may assume the geodesic q' is written as $q_1't'(q_1')^{-1}$, where $t' \in \beta'$ and q_1' is geodesic in β'. As (W, S') is even, $t' = z_i'$ and if z_i' is central in $W_{\beta'}$, then $z_i' = \bar{q}' = vz_iv^{-1}$. But this implies z_i is central in W_β, the desired contradiction. $\square$

By Claim 7.8 there is a spherical triangle $[x'z_i'e']$ in $\mathcal{V}'$ with label $k > 2$ on $[z_i'e']$ and the other two labels equal to 2. As $sp(x)$ has one edge labeled n and the rest labeled by 2, the Coxeter diagram for $W/N(x)$ obtained from $\mathcal{V}$ by removing the vertex x has one fewer edge with label > 2 than $\mathcal{V}$. Now consider the effect on $\mathcal{V}'$ when the relation $x'z'$ is added to the Coxeter presentation with generating set S' and a Coxeter presentation is formed as in Claim 7.6. Certainly the vertex x' is removed and so the resulting diagram for $W/N(x) = W/N(x'z')$ has at least one fewer edge with label > 2 than $\mathcal{V}'$. But by Claim 7.6, the edge $[e'z_i']$ of $\mathcal{V}'$ with label $k > 2$ is replaced in the diagram of $W/N(x'z')$ by an edge labeled 2. Hence, the diagram for $W/N(x'z')$ has at least two fewer edges with label > 2 than $\mathcal{V}'$. Now comparing the number of edges with label > 2 in the two diagrams for $W/N(x) = W/N(x'z')$ contradicts Lemma 4.7.

This completes the non-Abelian case and our proof of Theorem 5.1.

As an immediate corollary we have □

COROLLARY 7.9. *If the Coxeter system (W, S) is right-angled, then W is reflection independent if and only if every vertex in $\mathcal{V}(W, S)$ is the intersection of the maximal simplices containing it.*

Proof. Note that maximal simplices of $\mathcal{V}$ are maximal spherical simplices, since in a right-angled Coxeter group every simplex is spherical.

By Lemma 3.2, if $\tau_x = \{x\}$ for every x then our group is reflection independent.

If for some vertex x there is some other vertex y in τ_x then $st(x) \subseteq st_2(y)$, since every edge has label 2. Thus our group is not reflection independent. □

Two vertices of a Coxeter diagram are in the same *odd component* iff there is an edge path with odd labels between them.

PROPOSITION 7.10. *Suppose (W, S) and (W, S') are Coxeter systems with diagrams* $\mathcal{V}$ and $\mathcal{V}'$. *If (W, S) and (W, S') have the same set of reflections, then there is a bijection ψ, from the set of odd components of $\mathcal{V}$ to the set of odd components of $\mathcal{V}'$ such that for any odd component C of $\mathcal{V}$, W_C is isomorphic to $W_{\psi(C)}$.*

Proof. As noted in Section 2, distinct elements x and y of S are conjugate iff they are in the same odd component of $\mathcal{V}$. Suppose C is an odd component of $\mathcal{V}$, $x \in C$ and x is conjugate to x' an element of the odd component C' of $\mathcal{V}'$. Then each element of C is conjugate to each element of C' and so $N(C) = N(C')$ (in W). This defines a bijection between the odd components of $\mathcal{V}$ and $\mathcal{V}'$. Observe that there are retractions of W to W_{S-C} and $W_{S'-C'}$ with kernel $N(C) = N(C')$. I.e. W_{S-C} is isomorphic to $W_{S'-C'}$. In fact, if D is a union of odd components of $\mathcal{V}$ and D' is the corresponding union of odd components of $\mathcal{V}'$, Then W_{S-D} is isomorphic to $W_{S'-D'}$. If we take $D = S - C$ for C an odd component of W, then we have W_C isomorphic to $W_{C'}$. □

This proposition implies that if (W, S) is an even Coxeter system and (W, S') is not even, then (W, S) and (W, S') do not have the same set of reflections.

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