1. Introduction

We will begin by recalling the definition of a manifold stratified pair. Suppose $X$ is a space and $B \subset X$. Then $B$ is forward tame in $X$ if there exist a neighborhood $U$ of $B$ in $X$ and a homotopy $h : U \times [0, 1] \to X$ such that $h_0$ is the inclusion of $U$ into $X$, $h_t$ restricts to the inclusion on $B$ for all $t \in [0, 1]$, $h_1(U) = B$, and $h_t(U - B) \subset X - B$ for each $t \in [0, 1)$. The homotopy link of $B$ in $X$ is defined by

$$\text{holink}(X, B) = \{ \lambda \in X^{[0, 1]} \mid \lambda(t) \in B \text{ if and only if } t = 0 \}. $$

A pair $(X, B)$, with $B$ a closed subset of $X$, is a manifold stratified pair with strata $B$ and $X - B$ if the following four conditions are satisfied:

1. $B$ is forward tame in $X$.
2. The holink evaluation map $q : \text{holink}(X, B) \to B$, defined by $\lambda \mapsto \lambda(0)$, is a fibration.
3. The fibers of $q : \text{holink}(X, B) \to B$ are compactly dominated.
4. $X$ is a locally compact, separable metric space, and $B$, $X - B$ are both topological manifolds (without boundary).

Manifold stratified pairs are a special case of the more general notion of manifold stratified spaces which may have more than two strata.

Given a manifold stratified pair $(X, B)$, we might ask when $B$ has a mapping cylinder neighborhood in $X$. Quinn has defined an obstruction $\sigma(X, B)$ to $B$ having a mapping cylinder neighborhood in $X$, and there exist manifold stratified pairs $(X, B)$ for which the obstruction $\sigma(X, B)$ does not vanish. In particular, if $W$ is an open manifold with one tame end, then the one-point compactification $W' = W \cup \{ w_\infty \}$ of $W$ provides an example of a manifold stratified pair $(W', w_\infty)$ such that the point at infinity, $w_\infty$, has a mapping cylinder neighborhood in $W'$ if and only if $W$ is the interior of a compact manifold with boundary.

Since their existence is obstructed, mapping cylinder neighborhoods are not the best geometric structure to require on neighborhoods of strata in a manifold stratified pair (or more generally in a manifold stratified space with $n$ strata, $n \geq 2$). Teardrop neighborhoods are a good alternative. If $p : M \to B \times \mathbb{R}$ is a continuous map, the teardrop of $p$, denoted by $M \cup_p B$, is the set $M \sqcup B$, equipped with the minimum topology so that

- the inclusion $M \hookrightarrow M \cup_p B$ is an open embedding and
- the function $c : M \cup_p B \to B \times (-\infty, \infty]$ which extends $p$ via the identity on $B = B \times \{ \infty \}$ is continuous.

We say that $B$ has a teardrop neighborhood in $X$ if there exist a neighborhood $N$ of $B$ in $X$ and a map $p : N - B \to B \times \mathbb{R}$, such that the natural map $(N - B) \cup_p B \to N$ is a homeomorphism.
Theorem 1.1 (Hughes-Taylor-Weinberger-Williams [HTWW00]). If \((X, B)\) is a manifold stratified pair, where \(B\) is a closed manifold and \(X - B\) is an \(n\)-manifold \((n \geq 5)\), then \(B\) has a neighborhood in \(X\) which is the teardrop of a manifold approximate fibration \(p : M \to B \times \mathbb{R}\).

Note that the above theorem of Hughes-Taylor-Weinberger-Williams only applies to the situation where \(B\) is a closed manifold. In order to obtain results for other manifolds without boundary \(B\), we will use the following theorem of Hughes.

Theorem 1.2 (Hughes [Hug04]). Let \(X\) be a manifold stratified space with finitely many strata \(\{X_i\}_{i \in I}\), and let \(B\) be a stratum of \(X\). Consider the one-point compactification \(X' = X \cup \{x_\infty\}\) to be a space with a stratification whose strata are \(\{X_i\}_{i \in I} \cup \{x_\infty\}\). Assume that

1. \(X'\) is a manifold stratified space, and
2. if \(Y\) is any stratum of \(X\) such that \(\text{cl}(B) \cap \text{cl}(Y) \neq \emptyset\), then \(\dim(Y) \geq 5\).

Then \(B\) has a neighborhood in \(X\) which is the teardrop of a manifold stratified approximate fibration.

In this paper, we will show how these teardrop neighborhoods relate to mapping cylinder neighborhoods. In particular, we will show:

1. if \(p : M \to B\) is a manifold approximate fibration, then its open mapping cylinder \(\text{cyl}(p)\) is homeomorphic (via the identity map) to the teardrop of \(p \times \text{id}_\mathbb{R} : M \times \mathbb{R} \to B \times \mathbb{R}\),
2. if \(B\) and \(M\) are manifolds without boundary with \(\dim(M) \geq 4\) and \(p : M \to B \times \mathbb{R}\) is a manifold approximate fibration, then its open mapping cylinder \(\text{cyl}(p)\) is homeomorphic to \((M \cup_p B) \times \mathbb{R}\), via a homeomorphism which restricts to the identity on \(B \times \mathbb{R}\),
3. if \((X, B)\) is a manifold stratified pair, where \(\dim(X - B) = n \geq 5\) and either
   - \(B\) is a closed manifold or
   - \(B\) is an open manifold such that the one-point compactification of \(X\), \(X' = X \cup \{x_\infty\}\), is a manifold stratified space with three strata, namely \(\{x_\infty\}, B, \) and \(X - B\),
   then \(B \times \mathbb{R}\) always has a mapping cylinder neighborhood in \(X \times \mathbb{R}\), even if \(B\) does not have a mapping cylinder neighborhood in \(X\), and
4. if \((X, B)\) is a manifold stratified pair, where \(\dim(X - B) = n \geq 5\) and either
   - \(B\) is a closed manifold or
   - \(B\) is an open manifold such that the one-point compactification of \(X\) is a manifold stratified space with three strata as above,
   then the obstruction to \(B\) having a mapping cylinder neighborhood in \(X\) can be reformulated in terms of splitting manifold approximate fibrations.

2. **Open Mapping Cylinder of** \(p : M \to B\) **Homeomorphic to the Teardrop of** \(p \times \text{id}_\mathbb{R} : M \times \mathbb{R} \to B \times \mathbb{R}\)

**Proposition 2.1.** Suppose \(p : M \to B\) is a proper map, where \(B\) is a locally compact and Hausdorff space. Then the open mapping cylinder of \(p\) is homeomorphic (via the identity map) to the teardrop of the map \(p \times \text{id}_\mathbb{R} : M \times \mathbb{R} \to B \times \mathbb{R}\).
Proof. The open mapping cylinder $\text{cyl}(p)$ of $p$ is defined to be the quotient space $M \times (-\infty, \infty) \sqcup B \sim$, where a point $(m, \infty) \in M \times \{\infty\}$ is identified with $p(m) \in B$. As a set, $\text{cyl}(p) = (M \times \mathbb{R}) \cup B$. Meanwhile, the teardrop $(M \times \mathbb{R}) \cup_{p \times \text{id}_\mathbb{R}} B$ of the map $p \times \text{id}_\mathbb{R}$ also is just $(M \times \mathbb{R}) \cup B$ as a set. Therefore, the identity is a function $id : \text{cyl}(p) \to (M \times \mathbb{R}) \cup_{p \times \text{id}_\mathbb{R}} B$ between the two spaces.

To see that the identity map is continuous, it suffices to show that the map $\text{cyl}(p) \to B \times (-\infty, \infty)$ given by $(m, t) \mapsto (p(m), t)$, $b \mapsto (b, \infty)$ is continuous. But this map is just the product of a map $\text{cyl}(p) \to B$ and a map $\text{cyl}(p) \to (-\infty, \infty)$, which are both easily seen to be continuous using the definition of the quotient topology on $\text{cyl}(p)$. Thus, the identity map is continuous.

It remains to show that the identity map is an open map as well when $p$ is proper and $B$ is locally compact and Hausdorff. It suffices to show that the identity map

$$id : \text{cyl}(p) \to M \times [-\infty, \infty) \cup_{p \times \text{id}_{[-\infty, \infty)}} B$$

is an open map, where $\text{cyl}(p) = M \times [-\infty, \infty) \sqcup B \sim$ is the closed mapping cylinder of $p$, since the subspace topologies on $\text{cyl}(p) \subset \text{cyl}(p)$ and $M \times \mathbb{R} \cup_{p \times \text{id}_\mathbb{R}} B \subset M \times [-\infty, \infty) \cup_{p \times \text{id}_{[-\infty, \infty)}} B$ agree with their usual topologies.

To this end, suppose $U$ is an open subset of $\text{cyl}(p)$. We wish to show that $U$ is open when viewed as a subset of $M \times [-\infty, \infty) \cup_{p \times \text{id}_{[-\infty, \infty)}} B$. Since the topologies on $\text{cyl}(p)$ and $M \times [-\infty, \infty) \cup_{p \times \text{id}_{[-\infty, \infty)}} B$ agree away from $B$, it suffices to show that for every $b \in U \cap B$, there exists an open neighborhood $W$ of $b$ in $M \times [-\infty, \infty) \cup_{p \times \text{id}_{[-\infty, \infty)}} B$ such that $b \in W \subset U$. We would like $W = c^{(-1)}(W')$, where $W'$ is an open subset of $B \times [-\infty, \infty]$ that contains $(b, \infty)$ and $c' : M \times [-\infty, \infty) \cup_{p \times \text{id}_{[-\infty, \infty)}} B \times \{\infty\} \to B \times [-\infty, \infty]$ is a collapse map. To guarantee that $W \subset U$, it would then suffice to choose $W'$ so that $f'(K) \cap W' = \emptyset$, where $f' = c' \circ id : \text{cyl}(p) \to B \times [-\infty, \infty]$ and $K = \text{cyl}(p) - U$, since if $w \in W = c^{(-1)}(W')$, $f'(w) = c'(w) \in W'$, so $f'(w) \notin f'(K)$, and hence $w \in \text{cyl}(p) - K = U$. Thus, we just need to find an open subset $W'$ of $B \times [-\infty, \infty]$ that contains $(b, \infty)$ and such that $f'(K) \cap W' = \emptyset$. We are guaranteed that such an open set $W'$ exists if $(b, \infty)$ is not in the closure of $f'(K)$.

Since $b \in U \cap B \subset B$ and $B$ is locally compact and Hausdorff, there exists an open neighborhood $V$ of $b$ in $B$ such that $b \in V \subset cl(V) \subset U \cap B$ and $cl(V)$ (the closure of $V$) is compact. Let $K_1 = K \cap (p^{-1}(cl(V)) \times [-\infty, \infty) \cup cl(V)/\sim) \subset \text{cyl}(p)$. Since $p$ is proper, $p^{-1}(cl(V)) \times [-\infty, \infty) \cup cl(V)/\sim$ is compact. Since $K$ is closed, it follows that $K_1$ is a closed subset of a compact space, so $K_1$ is a compact subset of $\text{cyl}(p)$. Let $K_2 = K - (p^{-1}(V) \times [-\infty, \infty) \cup V/\sim) \subset \text{cyl}(p)$.

Then $K = K_1 \cup K_2$.

To show that $(b, \infty)$ is not in the closure of $f'(K)$, it suffices to show that $(b, \infty)$ is not in the closure of $f'(K_1)$ and also not in the closure of $f'(K_2)$. Since $K_1$ is compact and $f'$ is continuous, $f'(K_1)$ is a compact subspace of the Hausdorff space $B \times [-\infty, \infty]$, and hence $f'(K_1)$ is closed. Since $(b, \infty) \notin K_1$, $(b, \infty) \notin K_1$, and thus $(b, \infty) \notin f'(K_1) = cl(f'(K_1))$. As for $K_2$, note that $V \times [-\infty, \infty]$ is an open neighborhood of $(b, \infty)$ in $B \times [-\infty, \infty]$ such that $f'(K_2) \cap ([V \times [-\infty, \infty]) = \emptyset$. Since such an open neighborhood of $(b, \infty)$ exists, $(b, \infty) \notin cl(f'(K_2))$. Thus, $(b, \infty) \notin$
cl(f′(K)), so there exists an open neighborhood \( W' \) of \((b, \infty)\) in \( B \times [-\infty, \infty] \) such that \( f'(K) \cap W' = \emptyset \). Let \( W = c^{-1}(W') \). Then \( W \) is an open subset of \( M \times [-\infty, \infty] \cup_{p \times id, (-\infty, \infty)} B \times \{ \infty \} \) such that \((b, \infty) \in W \subset U\). This shows that the identity is an open map. □

Note that the previous proposition applies when \( p : M \to B \) is a manifold approximate fibration. We will use this fact to reinterpret the above result in terms of the commutativity of a certain diagram. For any fixed manifold without boundary \( B \) and integer \( n \geq 0 \), we will let \( \mathcal{MAF}^n(B) \) denote the set of equivalence classes of manifold approximate fibrations \( p : M \to B \), where \( M \) is an \( n \)-dimensional manifold without boundary. Two manifold approximate fibrations \( p_1 : M_1 \to B \) and \( p_2 : M_2 \to B \) lie in the same equivalence class if and only if they are controlled homeomorphic. Meanwhile, we will let \( \mathcal{SN}^n(B) \) denote the set of equivalence classes of \( n \)-dimensional stratified neighborhoods of \( B \); an element in \( \mathcal{SN}^n(B) \) is represented by a manifold stratified pair \((X, B)\), where \( \dim(X - B) = n \). \((X, B)\) and \((X', B)\) represent the same element in \( \mathcal{SN}^n(B) \) if and only if they are germ equivalent, i.e. if and only if there exist neighborhoods \( U \) and \( U' \) of \( B \) in \( X \) and \( X' \), respectively, and a homeomorphism \( \phi : U \to U' \) which restricts to the identity on \( B \). We can use the mapping cylinder construction to define a map

\[
\hat{\text{cyl}} : \mathcal{MAF}^{n-1}(B) \to \mathcal{SN}^n(B),
\]

which sends \([p : M \to B]\) to \([(\hat{\text{cyl}}(p), B)]\). Similarly, we can use the teardrop construction to define a map

\[
\text{TD} : \mathcal{MAF}^n(B \times \mathbb{R}) \to \mathcal{SN}^n(B)
\]

which sends \([q : M \to B \times \mathbb{R}]\) to \([(M \cup q B, B)]\). These two maps fit into the following diagram, which commutes by the main result of this section.

\[
\begin{array}{ccc}
\mathcal{MAF}^{n-1}(B) & \xrightarrow{x \times id} & \mathcal{MAF}^n(B \times \mathbb{R}) \\
\downarrow \hat{\text{cyl}} & & \downarrow \text{TD} \\
\mathcal{SN}^n(B) & & \\
\end{array}
\]

3. Open Mapping Cylinder of \( p : M \to B \times \mathbb{R} \) Homeomorphic to \((M \cup_p B) \times \mathbb{R}\)

In the last section, we showed that a certain diagram involving open mapping cylinders and teardrops of manifold approximate fibrations was commutative. In this section, we will show that the following similar diagram is also commutative.

\[
\begin{array}{ccc}
\mathcal{MAF}^n(B \times \mathbb{R}) & \xrightarrow{\text{TD}} & \mathcal{SN}^n(B) \\
\downarrow \hat{\text{cyl}} & & \downarrow x \times \mathbb{R} \\
\mathcal{SN}^n(B) & & \mathcal{SN}^{n+1}(B \times \mathbb{R}) \\
\end{array}
\]
**U-close maps.** We will begin with some constructions that will be used in the next subsection to prove that the above diagram is commutative. Suppose \( p : M \to B \) is a continuous function and that \( W \) is an open subset of \( B \) which is equipped with a continuous map \( q : W \to Z \times \mathbb{R} \) for some space \( Z \). Then we can form the teardrops \( p^{-1}(W) \cup_{qp} Z \) and \( W \cup_{q} Z \); we have collapse maps \( c : W \cup_{q} Z \to Z \times (-\infty, \infty] \) and \( d : p^{-1}(W) \cup_{qp} Z \to Z \times (-\infty, \infty] \), where \( d = c \circ \tilde{p} \). We will let \( M \cup_{qp} Z \) denote the space \( M \cup (p^{-1}(W) \cup_{qp} Z) \), and we will let \( B \cup_{q} Z \) denote the space \( B \cup (W \cup_{q} Z) \). Define a map \( \tilde{p} : M \cup_{qp} Z \to B \cup_{q} Z \) by \( \tilde{p}(m) = p(m) \) for \( m \in M \) and \( \tilde{p}(z) = z \) for \( z \in Z \).

**Lemma 3.1.** If \( p : M \to B \) is a continuous map between spaces, the extension \( \tilde{p} : M \cup_{qp} Z \to B \cup_{q} Z \) of \( p \) defined above is continuous.

**Proof.** Since \( \tilde{p} \) extends the continuous function \( p : M \to B \), it suffices to show that \( \tilde{p} \) is continuous when restricted to \( p^{-1}(W) \cup_{qp} Z \). To this end, it suffices to show that the composition \( c \circ \tilde{p} : p^{-1}(W) \cup_{qp} Z \to Z \times (-\infty, \infty] \) is continuous. But this follows immediately from the facts that \( c \circ \tilde{p} = d \) and that the teardrop topology on \( p^{-1}(W) \cup_{qp} Z \) is defined in such a way to guarantee that \( d \) is continuous. ∎

Suppose the space \( Z \) from above is metrizable. Then we will define an open cover \( \mathcal{U} \) of \( B \) as follows. First, choose a metric on the metrizable space \( Z \). For \( z \in Z \) and \( n \in \mathbb{N} \), let \( V_{z,n} = B(z, \frac{1}{n}) \) denote the open ball in \( Z \) which has radius \( \frac{1}{n} \) and is centered at \( z \). Note that \( V_{z,1} \supset V_{z,2} \supset V_{z,3} \supset \ldots \). Let

\[
U_{z,n} = q^{-1}(V_{z,n} \times (n-1, n+1)),
\]

which is an open subset of \( W \) (and \( B \)). The collection of all open subsets of this form covers \( q^{-1}(Z \times (0, \infty)) \subset \overline{W} \subset B \). The open cover \( \mathcal{U} \) of \( B \) consists of all of the open sets \( U_{z,n} \), where \( z \in Z \) and \( n \in \mathbb{N} \), along with \( B - q^{-1}(Z \times [-1, \infty)) \). We say that two maps \( f_{1}, f_{2} : X \to B \) are \( \mathcal{U} \)-close if for all \( x \in X \), there exists an open set \( U \) in \( \mathcal{U} \) such that \( f_{1}(x) \) and \( f_{2}(x) \) both lie in \( U \).

**Proposition 3.2.** Suppose that we have the following data:

- a continuous map \( p : M \to B \),
- an open subset \( W \) of \( B \) which is equipped with a map \( q : W \to Z \times \mathbb{R} \), where \( Z \) is a metrizable space,
- a continuous function \( h : B \cup_{q} Z \to B \cup_{q} Z \) which sends \( B \) to \( B \) and \( Z \) to \( Z \), and
- a function \( \tilde{h} : M \cup_{qp} Z \to M \cup_{qp} Z \) which sends \( M \) to \( M \) and \( Z \) to \( Z \) such that
  - \( \tilde{h} \) is continuous when restricted to \( M \),
  - \( \tilde{h}(z) = h(z) \) for all \( z \in Z \), and
  - \( \tilde{p} \circ \tilde{h} \) is \( \mathcal{U} \)-close to \( h \circ \tilde{p} \) when restricted to \( M \), where \( \mathcal{U} \) is the open cover of \( B \) defined above and \( \tilde{p} : M \cup_{qp} Z \to B \cup_{q} Z \) is the continuous map from [Lemma 3.1].

Then \( \tilde{h} : M \cup_{qp} Z \to M \cup_{qp} Z \) is continuous.

**Proof.** Since \( \tilde{h} \) is continuous when restricted to \( M \), it suffices to show that \( \tilde{h} \) is continuous when restricted to \( p^{-1}(W) \cup_{qp} Z \). Thus, it suffices to show that the composition

\[
d \circ \tilde{h} : p^{-1}(W) \cup_{qp} Z \to Z \times (-\infty, \infty]
\]

is continuous.
is continuous. To do this, we will show that if $A$ is any subset of $p^{-1}(W) \cup_{qp} Z$, then
\[ d(\tilde{h}(cl(A))) \subset cl(d(\tilde{h}(A))). \]

It suffices to show that if $z \in Z \cap cl(A \cap p^{-1}(W))$, then $d(\tilde{h}(z)) \in cl(d(\tilde{h}(A)))$. Let $V$ be any neighborhood of $d(\tilde{h}(z)) = (h(z), \infty)$ in $Z \times (-\infty, \infty]$; note that there exists an integer $N_0 > 0$ so that if $n > N_0$,
\[ B(h(z), \frac{1}{n}) \times (n, \infty] \]
is contained inside $V$. We wish to show that $d(\tilde{h}(A)) \cap V$ is nonempty.

For any integer $n > 0$, $\tilde{p}^{-1}(h^{-1}(c^{-1}(V_n)))$ is an open subset of $p^{-1}(W) \cup_{qp} Z$ which contains $z$ since $c$, $h$, and $\tilde{p}$ are all continuous maps. Since $z \in cl(A \cap p^{-1}(W))$, $\tilde{p}^{-1}(h^{-1}(c^{-1}(V_n))) \cap (A \cap p^{-1}(W))$ is nonempty for all $n$; let $x_n$ denote a point in this intersection. We claim that for sufficiently large $n > 0$, $d(\tilde{h}(x_n))$ is contained inside $V$. Note that $d(\tilde{h}(x_n)) = q(\tilde{p}(\tilde{h}(x_n)))$. Since $\tilde{p} \circ \tilde{h}$ is $U$-close to $h \circ \tilde{p}$ when restricted to $p^{-1}(W)$, for each integer $n > 1$, there exists $z_n \in Z$ and $m_n \in \mathbb{N}$ such that $\tilde{p}(\tilde{h}(x_n))$ and $h(\tilde{p}(x_n))$ both lie in $U_{z_n, m_n} = q^{-1}(B(z_n, \frac{1}{m_n}) \times (m_n - 1, m_n + 1))$. In other words, $c(\tilde{p}(\tilde{h}(x_n)))$ and $c(h(\tilde{p}(x_n)))$ both lie in
\[ B(z_n, \frac{1}{m_n}) \times (m_n - 1, m_n + 1) \]
. By construction of the $x_n$’s, we also know that $c(h(\tilde{p}(x_n)))$ lies in
\[ B(h(z), \frac{1}{n}) \times (n, \infty]. \]
In particular, this implies that $m_n + 1 > n$. Thus, we can choose an integer $N_1 > N_0 + 1$ so that $\frac{2}{m_n} + \frac{1}{n} < \frac{1}{N_0}$ for all $n > N_1$. If $n > N_1$, it then follows that $d(\tilde{h}(x_n)) = c(\tilde{p}(\tilde{h}(x_n)))$ lies in $B(h(z), \frac{1}{N_0}) \times (N_0, \infty] \subset V$. Thus, $d(\tilde{h}(x_n)) \in d(\tilde{h}(A)) \cap V$ for $n > N_1$. Since the intersection of any neighborhood $V$ of $d(\tilde{h}(z))$ in $Z \times (-\infty, \infty]$ with $d(\tilde{h}(A))$ is non-empty, $d(\tilde{h}(z)) \in cl(d(\tilde{h}(A)))$, as desired. □

**Proof of Commutativity.** We will now use the above results to show that if $p : M \to B \times \mathbb{R}$ is a manifold approximate fibration, where $B$ is a metrizable space and $M$ is a manifold without boundary of dimension at least 4, then $(M \cup_p B) \times \mathbb{R}$ is homeomorphic to $\text{cyl}(p)$, and ultimately that the diagram from the beginning of this section is commutative.

As sets, we see that $(M \cup_p B) \times \mathbb{R} = (M \cup B) \times \mathbb{R}$, whereas $\text{cyl}(p) = (M \times \mathbb{R}) \cup (B \times \mathbb{R})$. Even though these appear to be the same thing, something somewhat subtle is happening; the way that we are adding a copy of $B \times \mathbb{R}$ to $M \times \mathbb{R}$ differs in these two situations. In general, the identity will not be a homeomorphism between the two spaces. However, in the case when $p : M \to B \times \mathbb{R}$ is a manifold approximate fibration, we will still be able to show that the two spaces are homeomorphic. For example, consider the case when $B$ is a point, $M$ is the real line, and $p : M \to B \times \mathbb{R}$ is the identity. Then $(M \cup_p B) \times \mathbb{R}$ is clearly homeomorphic to $\text{cyl}(p)$, as seen in Figures 1 and 2 below.
Similarly, if \( B \) is a point, \( M = \mathbb{R} \times [0, 1] \), and \( p : M \to B \times \mathbb{R} = \mathbb{R} \) is projection onto the first factor, \( (M \cup_p B) \times \mathbb{R} \) is seen to be homeomorphic to \( \text{cyl}(p) \) in Figures 3 and 4.

\( \text{cyl}(p) \)

Remark 3.3. If \( p \) is not a manifold approximate fibration, the two spaces are not, in general, homeomorphic. Even if \( p : M \to B \times \mathbb{R} \) is an approximate fibration (but not a manifold approximate fibration), the spaces need not be homeomorphic. For example, let \( B \) be a point, and let \( M \) be the subset of \( \mathbb{R}^2 \) consisting of the \( x \)-axis along with the line segment from \((0, 0)\) to \((0, 1)\). Let \( p : M \to B \times \mathbb{R} = \mathbb{R} \) be the map which sends a point in \( M \) to its \( x \)-coordinate. Then \( p \) is an approximate fibration but \( (M \cup_p B) \times \mathbb{R} \) and \( \text{cyl}(p) \) are not homeomorphic, as we see in Figures 5 and 6 below.

\( \text{cyl}(p) \)

Suppose \( p : M \to B \times \mathbb{R} \) is a manifold approximate fibration. We can consider the space
\[
\overline{M} = (M \times \mathbb{R}) \sqcup (B \times \{\infty\} \times \mathbb{R}) \sqcup (B \times \mathbb{R} \times \{\infty\})
\]
with the minimal topology so that
\( (1) \) the inclusion of \( M \times \mathbb{R} \) into \( \overline{M} \) is an open embedding, and
(2) the collapse map
\[ \tau : M \to B \times (-\infty, \infty) \times (-\infty, \infty) \]

is continuous, where \( \tau(m, t) = (p(m), t) \) for \((m, t) \in M \times \mathbb{R}\), \( \tau(b, t) = (b(\infty, t) \) for \((b, t, \infty) \in B \times \{\infty\} \times \mathbb{R}\), and \( \tau(b, t, \infty) \) for \((b, t, \infty) \in B \times \mathbb{R} \times \{\infty\}\).

Note that \((M \cup_p B) \times \mathbb{R} = (M \cup_p B \times \{\infty\}) \times \mathbb{R} = (M \times \mathbb{R}) \sqcup B \times \{\infty\} \times \mathbb{R}\) is a subspace of \(\tilde{M}\), and \(\text{cyl}(\tilde{p}) = (M \times \mathbb{R}) \sqcup (B \times \mathbb{R}) = (M \times \mathbb{R}) \sqcup (B \times \mathbb{R} \times \{\infty\})\) is also a subspace of \(\tilde{M}\).

Instead of just adding \(B \times \{\infty\} \times \mathbb{R}\) and \(B \times \mathbb{R} \times \{\infty\}\) to \(M \times \mathbb{R}\), we could instead add the entire space \(B \times S^\infty(\mathbb{R}^2)\) to \(M \times \mathbb{R}\), where \(S^\infty(\mathbb{R}^2)\) is the sphere at infinity in \(\mathbb{R}^2\). Note that we can identify \(\{\infty\} \times \mathbb{R}\) and \(\mathbb{R} \times \{\infty\}\) with subspaces of \(S^\infty(\mathbb{R}^2)\), which we will denote by \(\mathbb{R}_1\) and \(\mathbb{R}_2\), respectively. To make talking about this sphere at infinity in \(\mathbb{R}^2\) easier, we will identify \(\mathbb{R}^2\) with the open cone \(\hat{c}(S^1)\) on \(S^1\), where \(c(S^1) = (S^1 \times [0, \infty)) / (\{0\} \times S^1)\). Then \(S^\infty(\mathbb{R}^2) = S^1\).

Consider the open subset \(B \times S^1 \times (1, \infty)\) of \(B \times \hat{c}(S^1)\). Let \(W = B \times S^1 \times (1, \infty)\) and \(Z = B \times S^1\). Then we have a map \(q = id : W \to Z \times (1, \infty)\), which we can use to form the teardrop \(W \cup_q Z\) after identifying \((1, \infty)\) with \(\mathbb{R}\); note that this space is homeomorphic to \(B \times S^1 \times (1, \infty)\). We will let \((B \times \hat{c}(S^1)) \cup_q Z\) denote the space \((B \times \hat{c}(S^1)) \cup (W \cup_q Z)\). Similarly, using the map \(q \circ (p \times \text{id}_B) : (p \times \text{id}_B)^{-1}(W) \to W\), we can form the teardrop \((p \times \text{id}_B)^{-1}(W) \cup_{q \circ (p \times \text{id}_B)} Z\), and we will let \((M \times \mathbb{R}) \cup_{q \circ (p \times \text{id}_B)} Z\) denote the space \((M \times \mathbb{R}) \sqcup (p \times \text{id}_B)^{-1}(W) \cup_{q \circ (p \times \text{id}_B)} Z\). Then \(M \cup_p B \times \mathbb{R}\) and \(\text{cyl}(\tilde{p})\) are also both subspaces of \((M \times \mathbb{R}) \cup_{q \circ (p \times \text{id}_B)} Z\). We will show that there is a homeomorphism \(\tilde{h} : (M \times \mathbb{R}) \cup_{q \circ (p \times \text{id}_B)} Z \to (M \times \mathbb{R}) \cup_{q \circ (p \times \text{id}_B)} Z\) which carries the subspace \((M \cup_p B) \times \mathbb{R}\) onto the subspace \(\text{cyl}(\tilde{p})\).

Let \(g_s : c(S^1) \cup_{id} S^1 \to \hat{c}(S^1) \cup_{id} S^1\) be the isotopy from the identity on \(c(S^1) = \hat{c}(S^1) \cup_{id} S^1\) which carries the subspace \(\mathbb{R}_1\) of \(S^1\) to the subspace \(\mathbb{R}_2\) of \(S^1\) which is obtained via rotation. Then an isotopy \(h_s : B \times \hat{c}(S^1) \cup_q Z \to (B \times \hat{c}(S^1)) \cup_q Z\) can be defined by \(h_s(b, x) = (b, g_s(x))\) for \((b, x) \in B \times \hat{c}(S^1)\) and \(h_s(b, y) = (b, g_s(y))\) for \((b, y) \in Z = B \times S^1\). This isotopy goes from \(h_0 = \text{id}\) to a homeomorphism \(h_1\) which sends the subspace \(B \times \mathbb{R}_1\) of \(B \times S^1\) to the subspace \(B \times \mathbb{R}_2\) of \(B \times S^1\). This isotopy restricts to give an isotopy \(H_s : B \times \hat{c}(S^1) \to B \times \hat{c}(S^1)\).

Note that we have a manifold approximate fibration \(p \times \text{id}_B : M \times \mathbb{R} \to B \times \mathbb{R}^2 = B \times \hat{c}(S^1)\). We will make use of Hughes’ Approximate Isotopy Covering Theorem, which we will now recall, to obtain an isotopy \(\tilde{H}_s : M \times \mathbb{R} \to M \times \mathbb{R}\) which is closely related to the isotopy \(H_s : B \times \hat{c}(S^1) \to B \times \hat{c}(S^1)\).

Theorem 3.4 (Approximate Isotopy Covering \cite{HR96}). Let \(p : M \to B\) be a manifold approximate fibration, where \(M\) is a manifold without boundary of dimension \(n \geq 5\). Let \(U\) be an open cover of \(B\), and let \(H_s : B \to B\) be an isotopy. Then there exists an isotopy \(\tilde{H}_s : M \to M\) such that \(p \circ \tilde{H}_s\) is \(U\)-close to \(H_s \circ p\) for each \(s \in I\).
First, we will define an open cover $\mathcal{U}$ of $B \times \hat{c}(S^1)$ similarly to how $\mathcal{U}$ was chosen in the previous section. Since we are assuming that $B$ is a metrizable space, $Z = B \times S^1$ is also a metrizable space, so we can choose a metric on $Z$. For $z \in Z$ and $n \in \mathbb{N}$, let $V_{z,n} = B(z, \frac{1}{n})$ denote the open ball in $Z$ which has radius $\frac{1}{n}$ and is centered at $z$. Let $U_{z,n} = V_{z,n} \times (n-1, n+1)$, which is an open subset of $W = B \times S^1 \times (0, \infty) \subset B \times \hat{c}(S^1)$. The open cover $\mathcal{U}$ of $B \times \hat{c}(S^1)$ consists of all the open sets $U_{z,n}$, where $z \in Z$ and $n \in \mathbb{N}$, along with the open set $B \times ((S^1 \times [0, 1])/(S^1 \times \{0\}))$. Since the dimension of $M$ is at least four, the Approximate Isotopy Covering Theorem implies that there exists an isotopy

$$\tilde{H}_s : M \times \mathbb{R} \to M \times \mathbb{R}$$

that covers $H_s : B \times \hat{c}(S^1) \to B \times \hat{c}(S^1)$ in the sense that $(p \times id_\mathbb{R}) \circ \tilde{H}_s$ is $\mathcal{U}$-close to $H_s \circ (p \times id_\mathbb{R})$ for each $s \in I$.

Recall that $H_s$ was defined to be the restriction of an isotopy

$$h_s : (B \times \hat{c}(S^1)) \cup q Z \to (B \times \hat{c}(S^1)) \cup q Z.$$

For each $s \in I$, we will extend $\tilde{H}_s$ to a function

$$\tilde{h}_s : (M \times \mathbb{R}) \cup_{q_0(p \times id_\mathbb{R})} Z \to (M \times \mathbb{R}) \cup_{q_0(p \times id_\mathbb{R})} Z,$$

where $\tilde{h}_s(z) = h_s(z)$ for $z \in Z$. We are particularly interested in the function

$$\tilde{h} := \tilde{h}_1 : (M \times \mathbb{R}) \cup_{q_0(p \times id_\mathbb{R})} Z \to (M \times \mathbb{R}) \cup_{q_0(p \times id_\mathbb{R})} Z$$

defined in this way. We know that $\tilde{h}$ is a homeomorphism when restricted to $M \times \mathbb{R}$, and we also know that $\tilde{h}$ is a homeomorphism $h := h_1$ when restricted to $Z$. We claim that $\tilde{h}$ is a homeomorphism on its entire domain. It is clear that $\tilde{h}$ is a bijection. To see that it is actually a homeomorphism, it suffices to show that $\tilde{h}$ and its inverse are continuous.

$$\begin{array}{ccc}
(M \times \mathbb{R}) \cup_{q_0(p \times id_\mathbb{R})} Z & \xrightarrow{\tilde{h}} & (M \times \mathbb{R}) \cup_{q_0(p \times id_\mathbb{R})} Z \\
| & | & | \\
(B \times \hat{c}(S^1)) \cup q Z & \xrightarrow{h} & (B \times \hat{c}(S^1)) \cup q Z
\end{array}$$

It follows from Proposition 3.2 that $\tilde{h}$ is continuous. To use Proposition 3.2 to conclude that $\tilde{h}^{-1} : (M \times \mathbb{R}) \cup_{q_0(p \times id_\mathbb{R})} Z$ is also continuous, we need to show that $(p \times id_\mathbb{R}) \circ \tilde{h}^{-1}$ is $\mathcal{U}$-close to $h^{-1} \circ (p \times id_\mathbb{R})$ when restricted to $M \times \mathbb{R}$. We know that $(p \times id_\mathbb{R}) \circ \tilde{h}_1$ is $\mathcal{U}$-close to $h_1 \circ (p \times id_\mathbb{R})$ when restricted to $M \times \mathbb{R}$; it follows that $(p \times id_\mathbb{R}) \circ \tilde{h}_1^{-1}$ is $h^{-1}(\mathcal{U})$-close to $h_1^{-1} \circ (p \times id_\mathbb{R})$, where $h^{-1}(\mathcal{U})$ is the open cover of $B \times \hat{c}(S^1)$ consisting of the open sets $h^{-1}(U)$, where $U$ is an element of the open cover $\mathcal{U}$ of $B \times \hat{c}(S^1)$. Since $h : B \times \hat{c}(S^1) \to B \times \hat{c}(S^1)$ is obtained via a rotation of $\hat{c}(S^1)$, $h^{-1}(\mathcal{U}) = \mathcal{U}$. Thus, $(p \times id_\mathbb{R}) \circ \tilde{h}^{-1}$ is $\mathcal{U}$-close to $h_1^{-1} \circ (p \times id_\mathbb{R})$ when restricted to $M \times \mathbb{R}$, so it follows from Proposition 3.2 that $\tilde{h}^{-1}$ is continuous. Thus, $\tilde{h} : (M \times \mathbb{R}) \cup_{q_0(p \times id_\mathbb{R})} Z \to (M \times \mathbb{R}) \cup_{q_0(p \times id_\mathbb{R})} Z$ is a homeomorphism which extends the homeomorphism $h = h_1$ on $Z$. By construction of our original isotopy, this homeomorphism maps the subspace $(M \cup q B) \times \mathbb{R}$ onto the subspace $\text{cyl}(p)$. Thus, we have just proven the following proposition.
Proposition 3.5. Suppose \( p : M \to B \times \mathbb{R} \) is a manifold approximate fibration, where \( B \) is metrizable and \( M \) is a manifold without boundary with \( \dim(M) \geq 4 \). Then \( (M \cup_p B) \times \mathbb{R} \) is homeomorphic to \( \text{cyl}(p) \).

Corollary 3.6. Suppose \( B \) is a metrizable space and \( n \geq 4 \). Then the following diagram is commutative:

\[
\begin{array}{ccc}
M \text{AF}^n(B \times \mathbb{R}) & \xrightarrow{\text{TD}} & SN^n(B) \\
& \searrow & \downarrow \times \mathbb{R} \\
& & SN^{n+1}(B \times \mathbb{R})
\end{array}
\]

Remark 3.7. Proposition 3.5 remains true if we instead just require that \( B \) is metrizable and \( p : M \to B \times \mathbb{R} \) is a map between spaces which has the approximate isotopy covering property.

4. Vanishing of Mapping Cylinder Neighborhood Obstructions after Taking Product with \( \mathbb{R} \)

In this section, we will use the commutativity of the diagram from the end of the previous section and certain results about the image of the teardrop map to draw some conclusions about the vanishing of the mapping cylinder neighborhood obstruction for certain manifold stratified pairs of the form \( (X \times \mathbb{R}, B \times \mathbb{R}) \).

Recall the following result of Hughes-Taylor-Weinberger-Williams.

Theorem 4.1 ([HTWW00]). If \( (X, B) \) is a manifold stratified pair, where \( B \) is a closed manifold and \( X - B \) is an \( n \)-manifold \((n \geq 5)\), then \( B \) has a neighborhood in \( X \) which is the teardrop of a manifold approximate fibration \( p : M \to B \times \mathbb{R} \).

Note that this theorem implies that the vertical teardrop map in the diagram from Corollary 3.6 is surjective when \( B \) is a closed manifold and \( n \geq 5 \). (Hughes-Taylor-Weinberger-Williams also proved that this map is injective under these hypotheses on \( B \) and \( n \).) Combining this surjectivity result with the commutativity of the diagram in Corollary 3.6 we obtain the following proposition.

Corollary 4.2. Suppose \( (X, B) \) is a manifold stratified pair, where \( B \) is a closed manifold and \( X - B \) is a manifold of dimension \( n \geq 5 \). Then \( B \times \mathbb{R} \) has a neighborhood in \( X \times \mathbb{R} \) which is homeomorphic (rel \( B \times \mathbb{R} \)) to the mapping cylinder of a manifold approximate fibration over \( B \times \mathbb{R} \).

Proof. First note that \( (X, B) \) determines an element \( [(X, B)] \) in \( SN^n(B) \). Since the teardrop construction gives a surjection \( \text{TD} : M \text{AF}^n(B \times \mathbb{R}) \to SN^n(B) \), there exists a manifold approximate fibration \( p : M \to B \times \mathbb{R} \) so that

\[
[(M \cup_p B, B)] = [(X, B)]
\]

in \( SN^n(B) \). Commutativity of the diagram from Corollary 3.6 then implies that

\[
[(\text{cyl}(p), B \times \mathbb{R})] = [(X \times \mathbb{R}, B \times \mathbb{R})]
\]

in \( SN^{n+1}(B \times \mathbb{R}) \). It follows that \( B \times \mathbb{R} \) has a neighborhood in \( X \times \mathbb{R} \) which is homeomorphic to \( \text{cyl}(p) \) via a homeomorphism which restricts to the identity on \( B \times \mathbb{R} \). \( \square \)
Remark 4.3. This is analogous to the result of Guilbault from [Gui07] that if $M$ is a manifold of dimension at least six such that $M$ has finite homotopy type, then $M \times \mathbb{R}$ is always homeomorphic to the interior of a compact manifold with boundary, even if $M$ itself is not homeomorphic to the interior of a compact manifold with boundary.

We would like a result similar to [Corollary 4.2] for manifold stratified spaces $(X, B)$ when $B$ is not necessarily a closed manifold. For this, we recall the following result of Hughes.

Theorem 4.4 (Hughes [Hug04]). Let $X$ be a manifold stratified space with finitely many strata $\{X_i\}_{i \in I}$, and let $B$ be a stratum of $X$. Consider the one-point compactification $X' = X \cup \{x_\infty\}$ to be a space with a stratification whose strata are $\{X_i\}_{i \in I} \cup \{x_\infty\}$. Assume that

(1) $X'$ is a manifold stratified space, and

(2) if $Y$ is any stratum of $X$ such that cl$(B) \cap$ cl$(Y) \neq \emptyset$, then dim$(Y) \geq 5$.

Then $B$ has a neighborhood in $X$ which is the teardrop of a manifold stratified approximate fibration.

It follows from this theorem that if $(X, B)$ is a manifold stratified pair such that dim$(X - B) \geq 5$ and the one-point compactification of $X$, $X' = X \cup \{x_\infty\}$, is a manifold stratified pair with 3 strata, $\{x_\infty\}$, $B$, and $X - B$, then $B$ has a neighborhood in $X$ which is the teardrop of a manifold approximate fibration $p : M \to B \times \mathbb{R}$. For any fixed manifold without boundary $B$ and fixed integer $n \geq 0$, we will let $\hat{\mathcal{S}}^n(B)$ denote the subset of $\mathcal{S}^n(B)$ consisting of those elements that are represented by manifold stratified pairs $(X, B)$ such that the one point compactification $X'$ of $X$ is a manifold stratified space with 3 strata, $\{x_\infty\}$, $B$, and $X - B$. (Note that this implies that any ends of $B$ must be tame since $\{x_\infty\}$ must be tame in $B \cup \{x_\infty\} \subset X'$ and the subspace topology on $B \cup \{x_\infty\} \subset X'$ agrees with the topology on the one-point compactification $B'$ of $B$.) Hughes’ result implies that for any manifold without boundary $B$ and $n \geq 5$, the teardrop map $TD : MAF^n(B \times \mathbb{R}) \to \mathcal{S}^n(B)$ hits all of $\hat{\mathcal{S}}^n(B)$.

Corollary 4.5. Suppose $(X, B)$ is a manifold stratified pair such that $B$ is an open manifold with one tame end, the one-point compactification $X'$ of $X$ is a manifold stratified space with 3 strata, $\{x_\infty\}$, $B$, and $X - B$, and dim$(X - B) \geq 5$. Then $B \times \mathbb{R}$ has a neighborhood in $X \times \mathbb{R}$ which is homeomorphic (rel $B \times \mathbb{R}$) to the mapping cylinder of a manifold approximate fibration over $B \times \mathbb{R}$.

Proof. First note that $(X, B)$ determines an element $[(X, B)]$ in $\hat{\mathcal{S}}^n(B)$. By the result of Hughes, there exists a manifold approximate fibration $p : M \to B \times \mathbb{R}$ so that

$[(M \cup_p B, B)] = [(X, B)]$

in $\hat{\mathcal{S}}^n(B) \subset \mathcal{S}^n(B)$. Commutativity of the diagram from [Corollary 3.6] then implies that

$[(\text{cyl}(p), B \times \mathbb{R})] = [(X \times \mathbb{R}, B \times \mathbb{R})]$

in $\mathcal{S}^{n+1}(B \times \mathbb{R})$. It follows that $B \times \mathbb{R}$ has a neighborhood in $X \times \mathbb{R}$ which is homeomorphic to cyl$(p)$ via a homeomorphism which restricts to the identity on $B \times \mathbb{R}$. $\square$
5. The One-Point Compactification Condition

While we might hope that a result like Corollary 4.5 would be true without the condition on the one-point compactification of $X$, Proposition 5.1 and Corollary 5.2 show that this condition is not unexpected since for any manifold stratified pair $(X,B)$, where $B$ is an open manifold with one tame end, if $B$ has a neighborhood in $X$ which is the teardrop of a manifold approximate fibration or if $B$ has a mapping cylinder neighborhood in $X$, then the one-point compactification of $X$ must be a manifold stratified space with 3 strata.

Proposition 5.1. Suppose $B$ is an open manifold with one tame end (in the sense of Siebenmann) and $p : M \to B \times \mathbb{R}$ is a manifold approximate fibration. Then the one-point compactification $X' = X \cup \{x_\infty\}$ of $X = M \cup_p B$ is a manifold stratified space with 3 strata, $\{x_\infty\}$, $B$, and $M$.

Corollary 5.2. Suppose $(X,B)$ is a manifold stratified pair such that $B$ is an open manifold with one tame end (in the sense of Siebenmann). If $B$ has a mapping cylinder neighborhood in $X$, then the one-point compactification $X' = X \cup \{x_\infty\}$ of $X$ is a manifold stratified space with three strata, $\{x_\infty\}$, $B$, and $X - B$.

Proof. (Assuming Proposition 5.1) Since $B$ has a mapping cylinder neighborhood in $X$, there exists a closed neighborhood $N$ of $B$ in $X$ such that $N - B$ is a manifold with compact boundary and a proper map $q : \partial(N - B) \to B$ such that $\text{cyl}(q)$ is homeomorphic to $N$ via a homeomorphism which restricts to the identity on $B$ and $\partial(N - B) \times \{0\}$. By Theorem 5.11 in [Hug99a], $q$ is necessarily a manifold approximate fibration. By Proposition 2.1, the open mapping cylinder of $q$ is homeomorphic (relative to $B$) to the teardrop of the manifold approximate fibration $q \times id_\mathbb{R} : \partial(N - B) \times \mathbb{R} \to B \times \mathbb{R}$. By Proposition 5.1, the one-point compactification of the open mapping cylinder of $q$ is a manifold stratified space with three strata, $\{x_\infty\}$, $B$, and $\partial(N - B) \times \mathbb{R}$. Thus, the one-point compactification of the open mapping cylinder of $q$ in $X$ is a manifold stratified space with three strata, $\{x_\infty\}$, $B$, and $X - B$, since $[(X,B)] = [(\text{cyl}(q), B)]$ in $\text{SN}^n(B)$. □

The most difficult step in the proof of Proposition 5.1 is to prove that $\{x_\infty\}$ is forward tame in $M \cup \{x_\infty\} \subset X'$. The proof of this fact will follow immediately once we show that $\{x_\infty\}$ is stratified forward tame in $X'$, i.e. when we show that there exist a neighborhood $V'$ of $x_\infty$ in $X'$ and a nearly stratum-preserving homotopy $\tilde{F} : V' \times [0,1] \to X'$ such that $\tilde{F}$ restricts to the inclusion on $V' \times \{0\}$, $\tilde{F}(x_\infty, t) = x_\infty$ for all $t \in [0,1]$, $\tilde{F}(v, t) \in M \cup_p B$ for all $(v, t) \in (V' - \{x_\infty\}) \times [0,1)$, and $\tilde{F}(v, 1) = x_\infty$ for all $v \in V'$. If such a neighborhood $V'$ and nearly-stratum preserving homotopy $F$ exist, then $V = V' \cap (M \cup \{x_\infty\})$ is a neighborhood of infinity in $M \cup \{x_\infty\}$ and $F = \tilde{F}|_{V \times [0,1]} : V \times [0,1] \to M \cup \{x_\infty\} \subset X'$ is a nearly-stratum preserving homotopy which shows that $\{x_\infty\}$ is forward tame in $M \cup \{x_\infty\}$.

The following lemmas will be used to show that $\{x_\infty\}$ is stratified forward tame in the one-point compactification $X' = (M \cup_p B) \cup \{x_\infty\}$.

Lemma 5.3. If $B$ is an open manifold with a tame end (in the sense of Siebenmann), then the one-point compactification $Y' = Y \cup \{y_\infty\}$ of $Y = B \times [0, \infty)$ is a manifold stratified space.
Proof. Let $B' = B \cup \{b_\infty\}$ denote the one-point compactification of $B$. It follows from Proposition 10.13 in [HR96] that $(B', b_\infty)$ is a manifold stratified space with 2 strata. Meanwhile, it is clear that $[0, \infty]$ with the natural stratification is a manifold stratified space. Proposition 4.1(4) in [Hug02] then implies that $B' \times [0, \infty]$ is a manifold stratified space with the natural stratification. Let $Z = \{b_\infty\} \times [0, \infty] \cup B' \times \{\infty\}$. Then $Z$ is a compact union of strata of $B' \times [0, \infty]$, so Proposition 3.3 in [Hug02] implies that the quotient space $Q = (B' \times [0, \infty])/Z$ is a manifold stratified space.

We will define a function $\phi : Y' \to Q$, which sends $y \in Y = B \times [0, \infty)$ to itself and sends $y_\infty$ to the equivalence class of $(b_\infty, \infty)$ in $Q$. We claim that this function is a homeomorphism. It is easy to see that this function is a bijection. To see that it is continuous, first note that a subset $B$ of a manifold stratified space with the natural stratification. Let $Z$ from Proposition 10.13 in [HR96] that $(B, Z)$ is a manifold stratified space. Proposition 4.1(4) in [Hug02] then implies that $B$ is a manifold stratified space with the natural stratification. Let $Z = \{b_\infty\} \times [0, \infty] \cup B' \times \{\infty\}$. Then $Z$ is a compact union of strata of $B' \times [0, \infty]$, so Proposition 3.3 in [Hug02] implies that the quotient space $Q = (B' \times [0, \infty])/Z$ is a manifold stratified space.

Thus, $\phi$ is a homeomorphism. Therefore, $Y'$ is a manifold stratified space since $Q$ is.

Lemma 5.4. Suppose $\phi : V \times [0, 1] \to B \times [0, \infty)$ is a $W$-homotopy between maps $f_0, f_1 : V \to B \times [0, \infty)$ such that $f_0(v) \in B \times \{0\}$ if and only if $f_1(v) \in B \times \{0\}$, where $W$ is an open cover of $B \times [0, \infty)$ consisting of open sets of the form $W = W_1 \times W_2$, where $W_1$ is open in $B$ and $W_2$ is an interval in $[0, \infty)$ of the form $W_2 = (a, b)$, where $0 < a < b < a + 1$ or $W_2 = [0, b)$, where $0 < b < 1$. Then there exists a stratum-preserving $W$-homotopy $\phi^* : V \times [0, 1] \to B \times [0, \infty)$ from $f_0$ to $f_1$.

Proof. Let $\pi_1 : B \times [0, \infty) \to B$ and $\pi_2 : B \times [0, \infty)$ denote the projections onto the first and second factors, respectively. Define a function $\phi^* : V \times [0, 1] \to B \times [0, \infty)$ by setting $\pi_1 \phi^*(v, t) = \pi_1 \phi(v, t)$ and

$$\pi_2 \phi^*(v, t) = \begin{cases} \pi_2 \phi(v, t) & \text{if } \pi_2 \phi(v, 0) \geq 3 \\ \pi_2 \phi(v, t) \cdot (\pi_2 \phi(v, 0) - 2) + (\pi_2 \phi(v, 0)(1 - t) + \pi_2 \phi(v, 1)t) \cdot (3 - \pi_2 \phi(v, 0)) & \text{if } 2 \leq \pi_2 \phi(v, 0) \leq 3 \\ \pi_2 \phi(v, 0)(1 - t) + \pi_2 \phi(v, 1) \cdot t & \text{if } \pi_2 \phi(v, 0) \leq 2. \end{cases}$$

Although the definition of this new function appears complicated, what the formulas are accomplishing is straightforward. For any $v \in V$, the track $\pi_2(\phi((v) \times [0, 1]))$ of $v$ under $\pi_2 \circ \phi$ lies entirely within an interval of the form $W_2 = (a, b)$.
with $0 < a < b < a + 1$, or $W_2 = [0, b)$, where $0 < b < 1$. If that track begins in the interval $[3, \infty)$, the entire track remains unchanged, while if the original track begins in the interval $[0, 2]$, that track is replaced by the straight-line track between $\pi_2\phi(v, 0)$ and $\pi_2\phi(v, 1)$. Finally, if the original track begins in the interval $[2, 3]$, the track is replaced by a weighted average of the original track and the straight-line track.

Using the definition of $\phi^*$ given above, it is straightforward to verify that $\phi^*$ is a well-defined homotopy from $f_0$ to $f_1$. To verify that $\phi^*$ is actually a stratum-preserving homotopy, we must show that for each $v \in V$, $\pi_2\phi^*(v, 0) = 0$ if and only if $\pi_2\phi^*(v, t) = 0$ for all $t \in [0, 1]$ and that $\pi_2\phi^*(v, 0) > 0$ if and only if $\pi_2\phi^*(v, t) > 0$ for all $t \in [0, 1]$; these results follow directly from a case-by-case analysis of $\phi^*$ and from the given assumptions on $f_0, f_1$, and the open cover $W$ of $B \times [0, \infty)$.

Finally, to see that $\phi^*$ is a $W$-homotopy, we must show that for each $v \in V$, there exists an open set $W$ in $W$ such that $\phi^*((v) \times [0, 1])$ lies entirely within $W$. By our assumptions on $W$ and the definition of $\phi^*$, it suffices to show that for every $v \in V$, there exists an open interval $W_2 = (a, b)$ (with $0 < a < b < a + 1$) or $W_2 = [0, b)$ (with $0 < b < 1$) so that $\pi_2(\phi^*([v] \times [0, 1]))$ lies entirely within $W_2$. Since the original homotopy $\phi$ was a $W$-homotopy, for each $v \in V$, there exists such an interval $W_2$ so that $\pi_2(\phi([v] \times [0, 1]))$ lies entirely within $W_2$. A case-by-case analysis of $\pi_2 \circ \phi^*$ then shows that $\pi_2(\phi^*([v] \times [0, 1]))$ also lies within $W_2$.

Thus, any $W$-homotopy $\phi$ satisfying the conditions in the statement of the lemma can be replaced by a stratum-preserving $W$-homotopy.

**Lemma 5.5.** Suppose that $B$ is an open manifold, $E$ is a manifold stratified space with two strata, and $c : E \to B \times [0, \infty)$ is a stratum-preserving stratified approximate fibration. Given an open cover $\mathcal{U}$ of $B \times [0, \infty)$, there is a refinement $\mathcal{V}$ of $\mathcal{U}$ such that if $H : V \times [0, 1] \to B \times [0, \infty)$ is a stratum-preserving homotopy and $h : V \times \{0\} \to E$ is a map such that $c \circ h$ is $\mathcal{V}$-close to $H_{|V \times \{0\}}$ and such that $c(h(v, 0))$ lies in $B \times \{0\}$ if and only if $H(v, 0)$ lies in $B \times \{0\}$, then $h$ extends to a stratum-preserving map $\tilde{H} : V \times [0, 1] \to E$ such that $c \circ \tilde{H}$ is $\mathcal{U}$-close to $H$ and $c \circ \tilde{H}_{|V \times \{1\}}$ is $\mathcal{V}$-close to $H_{|V \times \{1\}}$.

**Proof.** (Compare with the proof of Lemma 1.1 in [CD77].) Let $W'$ be a twice star refinement of the open cover $\mathcal{U}$, and let $W$ be a refinement of $W'$ consisting of open sets of the form $W = W_1 \times W_2$, where $W_1$ is an open set in $B$ and $W_2$ is an interval of the form $W_2 = (a, b)$, where $0 < a < b < a + 1$, or $W_2 = [0, b)$, where $0 < b < 1$. The fact that such a refinement exists follows easily from the definition of the product topology on $B \times [0, \infty)$. Finally, let $\mathcal{V}$ be a star refinement of $\mathcal{W}$ such that any two maps into $B \times [0, \infty)$ which are $\mathcal{V}$-close are also $\mathcal{W}$-homotopic. The fact that such a refinement exists follows from [MS82].

Suppose $H : V \times [0, 1] \to B \times [0, \infty)$ and $h : V \times \{0\} \to E$ are maps such that $c \circ h$ is $\mathcal{V}$-close to $H_{|V \times \{0\}}$. By our choice of $\mathcal{V}$, this implies that there exists a $W$-homotopy $\phi : V \times [-1, 0] \to B \times [0, \infty)$ such that $\phi(v, -1) = c(h(v, 0))$ and $\phi(v, 0) = H(v, 0)$. Then by Lemma 5.4, there exists another $W$-homotopy $\phi^* : V \times [-1, 0] \to B \times [0, \infty)$ from $c \circ h$ to $H_{|V \times \{0\}}$, which is a stratum-preserving map.

Define a map $G : V \times [-1, 1] \to B \times [0, \infty)$ which concatenates $\phi^*$ and $H$; namely,

$$
G(v, t) = \begin{cases} 
\phi^*(v, t) & \text{if } -1 \leq t \leq 0 \\
H(v, t) & \text{if } 0 \leq t \leq 1.
\end{cases}
$$
Then $G$ is a stratum-preserving homotopy, and we have the following stratified lifting problem:

$$
\begin{array}{ccc}
V \times \{-1\} & \xrightarrow{h} & E \\
\downarrow & & \downarrow c \\
V \times [-1, 1] & \xrightarrow{G} & B \times [0, \infty).
\end{array}
$$

Since $c$ is a stratified approximate fibration, there exists a stratum-preserving homotopy $\tilde{G}: V \times [-1, 1] \to E$ such that $\tilde{G}$ restricts to $h$ on $V \times \{-1\}$ and $c \circ \tilde{G}$ is $\mathcal{V}$-close to $G$.

We will reparametrize $\tilde{G}$ to obtain the desired stratum-preserving $U$-lift $\tilde{H} : V \times [0, 1] \to E$ of $H : V \times [0, 1] \to B \times [0, \infty)$. Let

$$
\tilde{H}(v, t) = \begin{cases} 
\tilde{G}(v, \frac{2t}{q(v)} - 1) & \text{if } 0 \leq t \leq \frac{q(v)}{2}, \\
\tilde{G}(v, 2t - q(v)) & \text{if } \frac{q(v)}{2} \leq t \leq q(v), \\
G(v, t) & \text{if } q(v) \leq t \leq 1.
\end{cases}
$$

where $q : V \to (0, 1)$ is a continuous function such that for each $v \in V$, the track $H(\{v\} \times [0, q(v)])$ lies within a single element of $\mathcal{V}$. Then for any $v \in V$, $\tilde{H}(v, -)$ traverses the entire homotopy $\phi^*$ quickly as $t$ varies from 0 to $\frac{q(v)}{2}$, and by the time $t$ reaches $q(v)$, it is traversing the original homotopy $H$ at regular speed. It follows from the fact that $G$ is a stratum-preserving map that $H$ is also.

We claim that $c \circ \tilde{H}$ is $U$-close to $H$. To see this, let $(v, t)$ be any element of $V \times [0, 1]$.

1. If $0 \leq t \leq \frac{q(v)}{2}$, then $c(\tilde{H}(v, t)) = c(\tilde{G}(v, \frac{2t}{q(v)} - 1))$, which is $\mathcal{V}$-close to $G(\{v\} \times [0, q(v)])$ by our choice of the lift $\tilde{G}$. But $G(v, \frac{2t}{q(v)} - 1) = \phi^*(v, \frac{2t}{q(v)} - 1)$ since $\frac{2t}{q(v)} - 1 \leq 0$, and $\phi^*(v, \frac{2t}{q(v)} - 1)$ is $\mathcal{W}$-close to $\phi^*(v, 0) = H(v, 0)$ since $\phi^*$ is a $\mathcal{W}$-homotopy. Finally, $H(v, 0)$ is $\mathcal{V}$-close to $H(v, t)$, since $q(v)$ was chosen in such a way so that $H(\{v\} \times [0, q(v)])$ is contained entirely within a single element of the open cover $\mathcal{V}$. Since $\mathcal{V}$ refines $\mathcal{W}$ and since $\mathcal{W}$ is a twice star refinement of $\mathcal{U}$, it follows that $c(\tilde{H}(v, t))$ is $\mathcal{U}$-close to $H(v, t)$ whenever $0 \leq t \leq \frac{q(v)}{2}$.

2. If $\frac{q(v)}{2} \leq t \leq q(v)$, then $c(\tilde{H}(v, t)) = c(\tilde{G}(v, 2t - q(v)))$, which is $\mathcal{V}$-close to $G(\{v\} \times [0, q(v)])$ by our choice of the lift $\tilde{G}$. But $G(v, 2t - q(v)) = H(v, 2t - q(v))$, which in turn is $\mathcal{V}$-close to $H(v, t)$ since $2t - q(v)$ and $t$ both lie within $[0, q(v)]$. Since $\mathcal{V}$ refines $\mathcal{W}$ and since $\mathcal{W}$ is a twice star refinement of $\mathcal{U}$, it follows that $c(\tilde{H}(v, t))$ is $\mathcal{U}$-close to $H(v, t)$ whenever $\frac{q(v)}{2} \leq t \leq q(v)$.

3. If $q(v) \leq t \leq 1$, then $c(\tilde{H}(v, t)) = c(\tilde{G}(v, t))$, which is $\mathcal{V}$-close to $G(v, t) = H(v, t)$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, it also follows that $c(\tilde{H}(v, t))$ is $\mathcal{U}$-close to $H(v, t)$ whenever $q(v) \leq t \leq 1$.

\[\Box\]

Lemma 5.6 (Stratum-Preserving, Open-Ended Homotopies). Suppose that $B$ is an open manifold, $E$ is a manifold stratified space with two strata, and $c : E \to B \times [0, \infty)$ is a stratum-preserving stratified approximate fibration, where $B \times [0, \infty)$ is given the natural stratification. Suppose that the following stratified lifting problem...
We can concatenate that there exists a stratum-preserving map \( \tilde{H} : V \times [0, 1) \to E \) such that \( \tilde{H} \) restricts to \( h \) on \( V \times \{0\} \) and \( c \circ \tilde{H} \) is \( \mathcal{U} \)-close to \( H \).

**Proof.** We will construct the desired stratum-preserving fibration, there exists a stratum-preserving lift \( \tilde{H} : V \times [0, 1) \to E \) by patching together stratum-preserving \( \mathcal{U} \)-lifts \( H_n : V \times [1 - \frac{1}{n}, 1 - \frac{1}{n+1}] \to E \) of the restrictions \( H_n = H_1 : V \times [1 - \frac{1}{n}, 1 - \frac{1}{n+1}] \to B \times [0, \infty) \) for each integer \( n \geq 1 \).

We begin by letting \( n = 1 \) and considering the following stratified lifting problem:

\[
\begin{array}{ccc}
V \times \{0\} & \xrightarrow{h} & E \\
\downarrow & & \downarrow c \\
V \times [0, 1) & \xrightarrow{H} & B \times [0, \infty),
\end{array}
\]

Let \( \mathcal{V} \) be the refinement of \( \mathcal{U} \) given by [Lemma 5.5](#). Since \( c \) is a stratified approximate fibration, there exists a stratum-preserving lift \( \tilde{H}_1 : V \times [0, \frac{1}{2}) \to E \) such that \( \tilde{H}_1 \) restricts to \( h \) on \( V \times \{0\} \) and such that \( c \circ \tilde{H}_1 \) is \( \mathcal{V} \)-close to \( H_1 \).

Now that we have a \( \mathcal{V} \)-lift of the restriction \( H_1 \) of \( H \) to \( V \times [0, \frac{1}{2}] \), we want to continue by finding an approximate lift \( \tilde{H}_2 \) of the restriction \( H_2 \) of \( H \) to \( V \times [\frac{1}{2}, \frac{2}{3}] \). In particular, we want the lift \( \tilde{H}_2 \) to extend the restriction \( h_2 : V \times \{\frac{1}{2}\} \to E \) of \( H_1 \) to \( V \times \{\frac{1}{2}\} \). Since \( c \circ h_2 \) and \( H_{1, V \times \{\frac{1}{2}\}} \) are \( \mathcal{V} \)-close maps into \( B \times [0, \infty) \), our choice of \( \mathcal{V} \) and [Lemma 5.5](#) (replacing the interval \( [0, 1] \) with the interval \( [\frac{1}{2}, \frac{2}{3}] \)) implies that there exists a stratum-preserving map \( \tilde{H}_2 : V \times [\frac{1}{2}, \frac{2}{3}] \to E \) such that

- \( \tilde{H}_2 \) restricts to \( h_2 \) on \( V \times \{\frac{1}{2}\} \),
- \( c \circ \tilde{H}_2 \) is \( \mathcal{V} \)-close to \( H_2 \) on all of \( V \times [\frac{1}{2}, \frac{2}{3}] \), and
- \( c \circ \tilde{H}_2 \) is \( \mathcal{V} \)-close to \( H_2 \) when restricted to \( V \times \{\frac{2}{3}\} \).

We can concatenate \( \tilde{H}_1 \) and \( \tilde{H}_2 \) to obtain a \( \mathcal{U} \)-lift of \( H \) restricted to \( [0, \frac{2}{3}] \) which is a \( \mathcal{V} \)-lift of \( H \) when restricted to \( V \times \frac{2}{3} \).

We can continue in this way to define analogous lifts \( \tilde{H}_n : V \times [1 - \frac{1}{n}, 1 - \frac{1}{n+1}] \to E \) for \( n \geq 3 \) so that \( \tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \ldots \), can all be concatenated to obtain a \( \mathcal{U} \)-lift \( \tilde{H} : V \times [0, 1) \to E \) of \( H : V \times [0, 1) \to B \times [0, \infty) \) which extends \( h \).

**Proposition 5.7.** If \( B \) is an open manifold with one tame end (or finitely many?) and \( p : M \to B \times \mathbb{R} \) is a manifold approximate fibration, then \( \{x_\infty\} \) is stratified forward tame in \( X' = (M \cup_p B) \cup \{x_\infty\} \) denotes the one-point compactification of the teardrop \( M \cup_p B \) of \( p \).

**Proof.** To show that \( \{x_\infty\} \) is stratified forward tame in \( X' = (M \cup_p B) \cup \{x_\infty\} \), we first recall from Proposition 7.1 in [Hug99](#) that the teardrop collapse map

\[
c : M \cup_p B \to B \times (-\infty, \infty)
\]
is a stratified approximate fibration, where $B \times (-\infty, \infty]$ is given the natural stratification with strata $B \times \mathbb{R}$ and $B \times \{0\}$. For ease of notation, we will identify $(-\infty, \infty)$ with $[0, \infty)$ via a homeomorphism which sends $+\infty$ to 0, and by an abuse of notation, we will identify the manifold approximate fibration $p : M \to B \times \mathbb{R}$ and the teardrop collapse map $c : M \cup_p B \to B \times (-\infty, \infty]$ with maps $p : M \to B \times (0, \infty)$ and $c : M \cup_p B \to B \times [0, \infty)$.

By Lemma 5.3, the one-point compactification $Y'$ of $Y = B \times [0, \infty)$ is a manifold stratified space with strata $\{y_{\infty}\}$, $B \times \{0\}$, and $B \times (0, \infty)$. In particular, this implies that the stratum $\{y_{\infty}\}$ is stratified forward tame $Y'$. Thus, there exists an open neighborhood $U'$ of $\{y_{\infty}\}$ in $Y'$ and a nearly stratum-preserving homotopy $G' : U' \times [0, 1] \to Y'$ such that $G'_0$ is the inclusion of $U'$ in $Y'$, $G'_t(\{y_{\infty}\}) = y_{\infty}$ for all $t$, $G'_t(U' - \{y_{\infty}\}) \subset Y' - \{y_{\infty}\} = B \times [0, \infty)$ for all $t \in [0, 1)$. If we let $U = U' - \{y_{\infty}\}$ and $G = G'_t|_{U \times [0, 1)}$, $U$ is a cocompact set in $B \times [0, \infty)$ and $G : U \times [0, 1) \to B \times [0, \infty)$ is a proper, stratum-preserving, open-ended homotopy. Let $V = c^{-1}(U)$. Then $V$ is cocompact in $M \cup_p B$, and we have the following stratified, open-ended lifting problem:

\[
\begin{array}{ccc}
V \times \{0\} & \longrightarrow & M \cup_p B \\
\downarrow & & \downarrow c \\
V \times [0, 1) & \longrightarrow & B \times [0, \infty),
\end{array}
\]

where $H = G \circ (c \times id|_{[0,1)})$ and the upper and left-hand maps are given by inclusion.

By Theorem 4.1 in [Cha76], we can choose an open cover $\mathcal{U}$ of $B \times [0, \infty)$ such that for any locally compact, separable metric space $X$ and any two maps $f, g : X \to B \times [0, \infty)$ which are $\mathcal{U}$-close, $f$ is proper if and only if $g$ is proper. We want to find a stratum-preserving map $\tilde{H} : V \times [0, 1) \to M \cup_p B$ extending the inclusion of $V = V \times \{0\}$ in $M \cup_p B$ so that $c \circ \tilde{H}$ is $\mathcal{U}$-close to $H$. Lemma 5.6 shows that we have a stratum-preserving homotopy $\tilde{H} : V \times [0, 1) \to M \cup_p B$ such that $\tilde{H}$ restricts to the inclusion on $V \times \{0\}$ and $c \circ \tilde{H}$ is $\mathcal{U}$-close to the proper map $H : V \times [0, 1) \to B$. By choice of our open cover $\mathcal{U}$, it follows that $c \circ \tilde{H}$ is also a proper map. It is then straightforward to verify that $\tilde{H}$ is proper, as well. Thus, we have found a cocompact neighborhood $V$ of $x_{\infty}$ in $X' = (M \cup_p B) \cup \{x_{\infty}\}$ and a stratum-preserving, proper open-ended homotopy $\tilde{H} : V \times [0, 1) \to M \cup_p B$ which extends the inclusion on $V \times \{0\}$.

Let $V' = V \cup \{x_{\infty}\}$; then $V'$ is a neighborhood of $x_{\infty}$ in $X' = (M \cup_p B) \cup \{x_{\infty}\}$ since $X' - V' = X - V$, which is compact. Let $\tilde{F} : V' \times [0, 1] \to X'$ denote the function that restricts to $\tilde{H}$ on $V \times [0, 1)$ and which sends $x_{\infty} \times t$ and $(v, 1)$ to $x_{\infty}$ for all $t \in [0, 1]$ and for all $v \in V$. Using the fact that $\tilde{H}$ is proper, it follows easily that $\tilde{F}$ is continuous; in fact, $\tilde{F} : V' \times [0, 1] \to X'$ is a homotopy which is stratum-preserving over $[0, 1)$. The existence of this neighborhood $V'$ of $x_{\infty}$ in $X'$ and this nearly stratum-preserving homotopy $\tilde{F}$ implies that $\{x_{\infty}\}$ is stratified forward tame in $X'$.

**Corollary 5.8.** If $B$ is an open manifold with one tame end (or finitely many?) and $p : M \to B \times \mathbb{R}$ is a manifold approximate fibration, then $\{x_{\infty}\}$ is forward
tame in $M \cup \{x_\infty\} \subset X'$, where $X' = (M \cup_p B) \cup \{x_\infty\}$ denotes the one-point compactification of the teardrop $M \cup_p B$ of $p$.

**Proof.** By [Proposition 5.7] there exists a neighborhood $V'$ of $x_\infty$ in $X'$ and a nearly stratum-preserving homotopy $\tilde{F} : V' \times [0, 1] \to X'$ such that $\tilde{F}_{\mid V' \times \{0\}}$ is the inclusion, $\tilde{F}(x_\infty, t) = x_\infty$ for all $t \in [0, 1]$, $\tilde{F}((V' - \{x_\infty\}) \times [0, 1]) \subset M \cup_p B$, and $\tilde{F}(v, 1) = x_\infty$ for all $v \in V'$. Restricting this neighborhood and nearly stratum-preserving homotopy to $V' \cap (M \cup \{x_\infty\})$ then implies that $\{x_\infty\}$ is forward tame in $M \cup \{x_\infty\} \subset X'$, as desired. \qed

**Proof.** (of [Proposition 5.1]) This proof follows easily from [Corollary 5.8] and the main result of [Hug99b] that $(M \cup_p B, B)$ is a manifold stratified pair. (give more details) \qed

6. Reformulating the Mapping Cylinder Neighborhood Obstruction

We have already seen that if $(X, B)$ is a manifold stratified pair such that $\dim(X - B) \geq 5$ and either

1. $B$ is a closed manifold, or
2. the one-point compactification $X'$ of $X$ is a manifold stratified space with 3 strata, $\{x_\infty\}, B$, and $X - B$

then there always exists a manifold approximate fibration $p : M \to B \times \mathbb{R}$ so that $[(M \cup_p B, B)] = [(X, B)]$. We will use this to determine a condition for when $B$ has a mapping cylinder neighborhood in $X$ for such manifold stratified pairs $(X, B)$.

**Definition 6.1.** (Compare with the definition of a split MAF in [BK02].) A manifold approximate fibration $p : M \to B \times \mathbb{R}$ is **splittable over** $B \times \{0\}$ if $p$ is controlled homeomorphic to a manifold approximate fibration $q : W \to B \times \mathbb{R}$ such that $q^{-1}(B \times \{0\}) \to B \times \{0\}$ is also a manifold approximate fibration and $q^{-1}(B \times \{0\})$ is a locally flat submanifold of $W$.

**Lemma 6.2.** If $p : M \to B \times \mathbb{R}$ is splittable over $B \times \{0\}$, then $p$ is controlled homeomorphic to $q \times id_{\mathbb{R}} : Z \times \mathbb{R} \to B \times \mathbb{R}$ for some manifold approximate fibration $q : Z \to B$.

**Theorem 6.3.** Suppose $(X, B)$ is a manifold stratified pair such that $\dim(X - B) = n \geq 5$ and either

1. $B$ is a closed manifold, or
2. the one-point compactification $X'$ of $X$ is a manifold stratified space with 3 strata, $\{x_\infty\}, B$, and $X - B$.

Then $B$ has a closed neighborhood $N$ such that $N - B$ is a manifold with boundary and $N$ is homeomorphic (relative to $B$ and $\partial(N - B)$) to the mapping cylinder of a proper map $q : \partial(N - B) \to B$ if and only if there exists a manifold approximate fibration $p : M \to B \times \mathbb{R}$ such that $[(X, B)] = [(M \cup_p B, B)]$ in $SN^n(B)$ such that $p$ is splittable over $B \times \{0\}$.

**Proof.** Suppose $B$ has a closed neighborhood $N$ which is homeomorphic to the mapping cylinder $\text{cyl}(q)$ for some proper map $q : \partial(N - B) \to B$. Then $(\text{cyl}(q), B)$ is a manifold stratified space such that $[(X, B)] = [(\text{cyl}(q), B)]$. By Theorem 5.11 in [Hug99a], $q$ is actually a manifold approximate fibration since $(\text{cyl}(q), B)$ is a
manifold stratified space. Let \( p = q \times \text{id}_\mathbb{R} : \partial (N - B) \times \mathbb{R} \rightarrow B \times \mathbb{R} \). Then \( p \) is clearly splittable over \( B \times \{0\} \), and \([ (X, B)] = [(\text{cyl}(q), B)] = \left\langle (\partial (N - B) \times \mathbb{R}) \cup q \times \text{id}_B, B \right\rangle \) by [Proposition 2.1].

Now suppose that there exists a manifold approximate fibration \( p : M \rightarrow B \times \mathbb{R} \) such that \([ (X, B)] = [(M \cup_p B, B)] \) and \( p \) is splittable over \( B \times \{0\} \). By [Lemma 6.2], \( p \) is controlled homeomorphic to \( q \times \text{id}_\mathbb{R} : Z \times \mathbb{R} \rightarrow B \times \mathbb{R} \) for some manifold approximate fibration over \( B \). Then \([ (M \cup_p B, B)] = \left\langle [(Z \times \mathbb{R}) \cup q \times \text{id}_B] \right\rangle \) (since the teardrop map is well-defined) and \([ (Z \times \mathbb{R}) \cup q \times \text{id}_B] = [(\text{cyl}(q), B)] \) by [Proposition 2.1]. Thus, \([ (X, B)] = [(\text{cyl}(q), B)] \), so \( B \) has a mapping cylinder neighborhood in \( X \). \( \square \)
References


