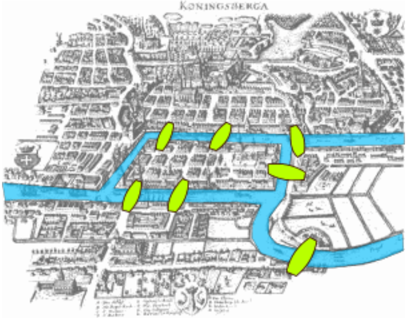
Graph Theory II

Nashville Math Club

October 20, 2020

The seven bridges of Konigsberg



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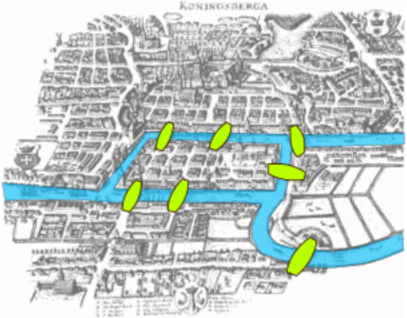

Question

Can we walk through all seven bridges of Konigsberg exactly once?

The handshake lemma

Lemma

The sum of the degrees of all the vertices in a graph is equal to two times the number of edges in the graph.

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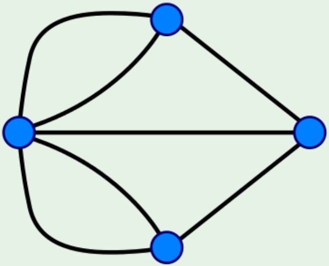

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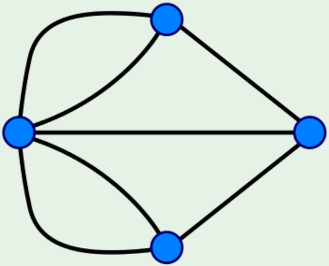

$$5 + 3 + 3 + 3 = 14 = 2 \times 7.$$

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Each landmass is a vertex and each bridge is an edge.
Notice that whenever one enters a vertex by an edge they must leave the vertex by a different edge.
Therefore, each vertex (except for the starting and finishing ones) must have an even degree.
The walk is only possible if exactly zero or two vertices have odd degree.

Euler paths and circuits

Definition

- An **Euler path** in a graph or multigraph is a path which uses each edge exactly once.

Euler paths and circuits

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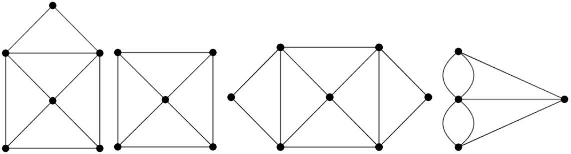
- An **Euler path** in a graph or multigraph is a path which uses each edge exactly once.
- An **Euler circuit** is an Euler path which starts and stops at the same vertex.

Euler paths and circuits



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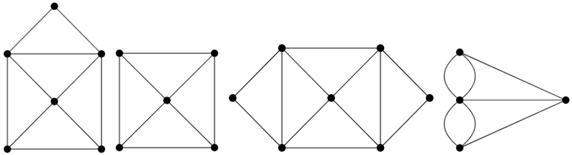

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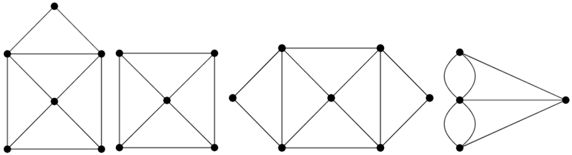
- *Which of these graphs have Euler paths or circuits?*

Euler paths and circuits



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- *Is there a connection between the degrees of each vertex and your answer above?*

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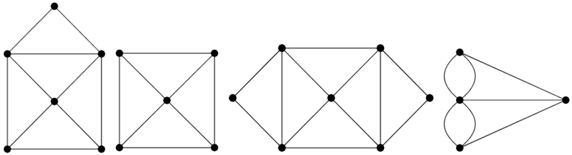

Question

- *Which of these graphs have Euler paths or circuits?*
- *Is there a connection between the degrees of each vertex and your answer above?*
- *How would you draw a graph that's guaranteed to have an Euler circuit?*

Euler paths and circuits

Theorem

- *A connected graph has an Euler circuit if and only if the degree of each vertex is even.*

Euler paths and circuits

Theorem

- *A connected graph has an Euler circuit if and only if the degree of each vertex is even.*
- *A connected graph has an Euler path if and only if there are at most two vertices with odd degree.*

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Suppose instead of traveling on each bridge in Konigsberg you wanted instead to travel to each landmass exactly once.

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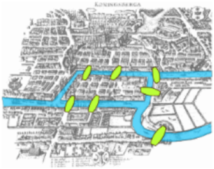
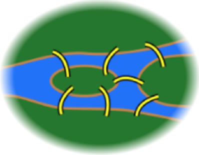
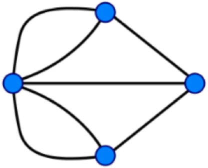
- A **Hamilton path** is a path which visits each vertex of a graph exactly once.
- A **Hamilton cycle** is a Hamilton path which starts and stops at the same vertex.

Hamilton paths

 →  → 

Hamilton paths

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Is there a Hamilton path for Konigsberg?

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How can we determine if there is a Hamilton path for a general graph G?

Euler and Hamilton paths

Recall K_n is the graph with n vertices where each pair of distinct vertices is connected by a unique edge.

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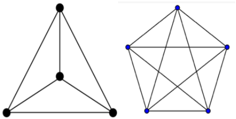

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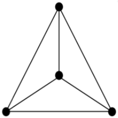

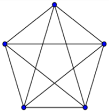

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- *For which n does K_n have an Euler path? An Euler circuit?*

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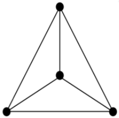
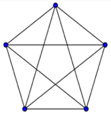

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- *For which n does K_n have a Hamilton path? A Hamilton cycle?*

Planar graphs and platonic solids

Recall that a graph is **planar** if its edges only intersect at vertices and is **connected** if theres a path from each vertex to every other vertex.

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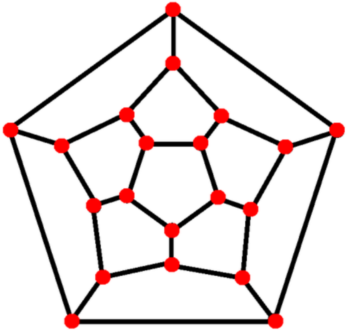

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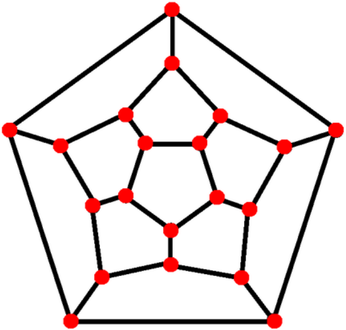

Let v be the number of vertices, e the number of edges, and f the number of faces or regions.

Planar graphs and platonic solids

Draw several planar connected graphs and compute v, e, and f for each of them.

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Question

What is the Euler characteristic of the graphs you drew?

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Recall a **tree** is a graph with no cycles.

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Planar graphs and platonic solids

Recall a **tree** is a graph with no cycles.

Question

What is the Euler characteristic of a tree?

Notice that all trees have $v = e + 1$.
What is f for a tree? $f = 1$ for all trees.
Therefore all trees have Euler characteristic

$$\chi = v - e + f = 2.$$

Planar graphs and platonic solids

Theorem

The Euler characteristic of every planar connected graph is 2.

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- We can do this until we are left with a tree.

Planar graphs and platonic solids

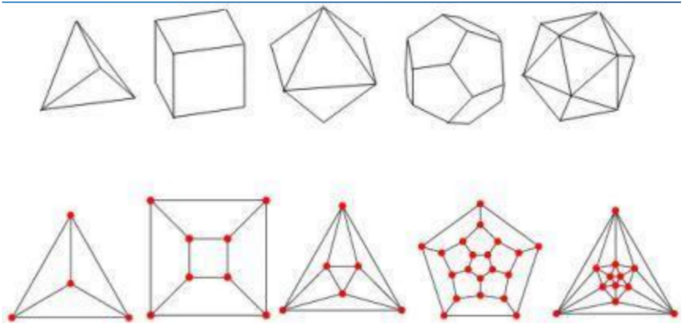
Theorem

The Euler characteristic of every planar connected graph is 2.

Proof

- If you have a connected planar graph that isn't a tree, remove an edge that completes a cycle.
- What does this do to the Euler characteristic?
- We can do this until we are left with a tree.
- Therefore the Euler characteristic of all connected planar graphs is 2.

Planar graphs and platonic solids



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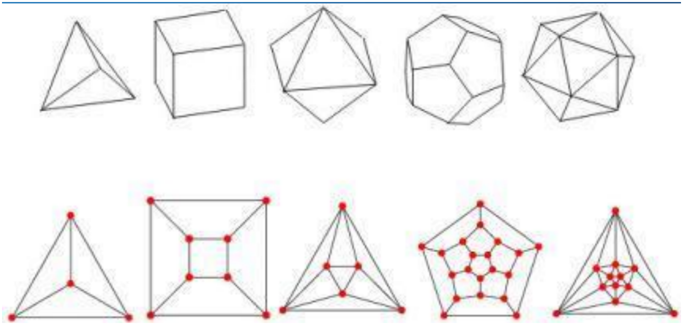

Each platonic solid has a corresponding planar connected graph

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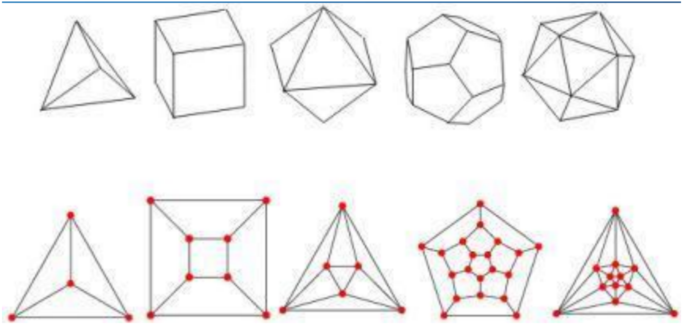

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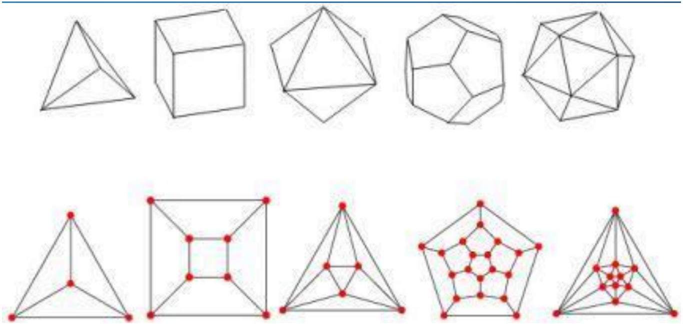

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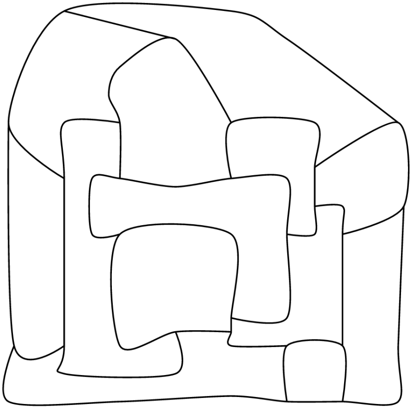
Graph colorings



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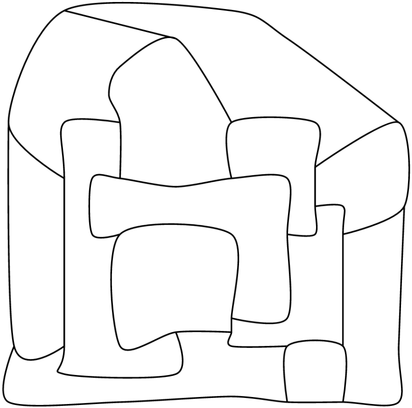
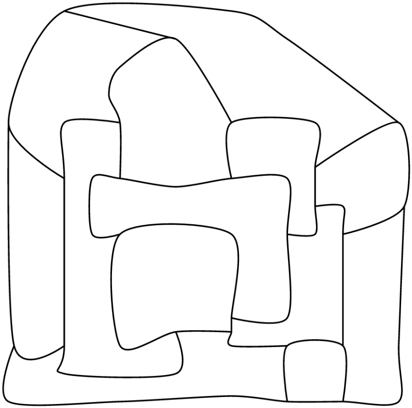


Mapmakers want to color this map, but want each adjacent region to have a different color.

Graph colorings



Mapmakers want to color this map, but want each adjacent region to have a different color.
How many colors are needed?

Graph colorings

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- Given a graph G, a coloring of the vertices is called a **vertex coloring**

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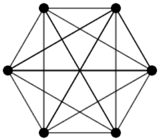
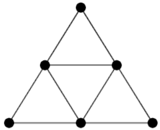
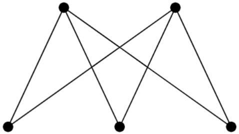
The smallest number of colors needed to get a proper vertex coloring is called the **chromatic number** of the graph.

Graph colorings



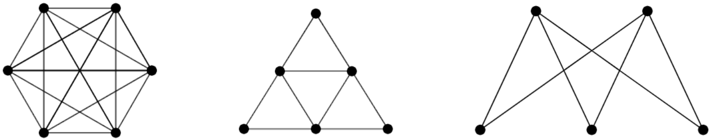




Question

- *What are the chromatic numbers of each of these graphs?*

Graph colorings



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- *What are the chromatic numbers of each of these graphs?*
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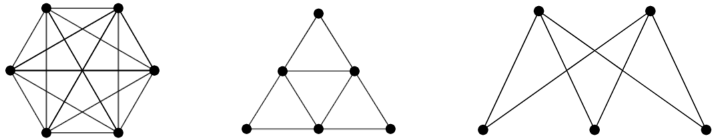

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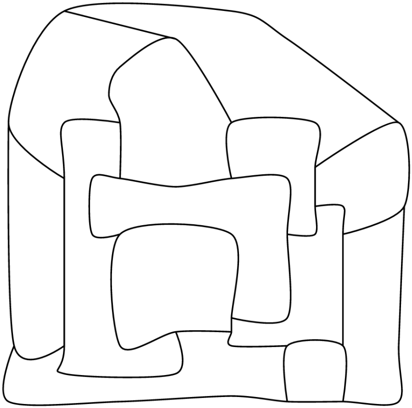
- *What are the chromatic numbers of each of these graphs?*
- *For each positive integer n, is there a graph with chromatic number equal to n?*
- *What is the chromatic number of a tree?*

Graph colorings



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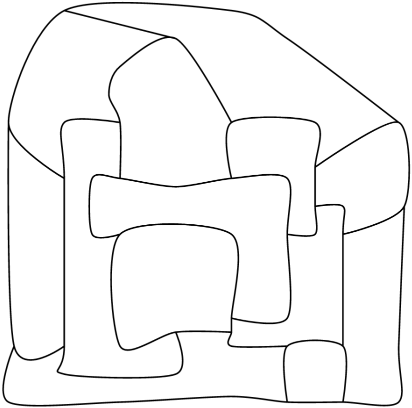

If we represent each region by a vertex and adjacent regions are connected by an edge what properties does our graph have?

Graph colorings

Theorem (The four color theorem)

If G is a planar graph, then its chromatic number is less than or equal to 4.

Graph colorings

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If G is a planar graph, then its chromatic number is less than or equal to 4.

Therefore every map can be colored with 4 or less colors!

Graph colorings

Question

What is the smallest number of colors that can be used to color the vertices of a cube so that no adjacent vertices have the same color?