Sums of squares

Nashville Math Club

September 22, 2020

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- So about $\frac{\sqrt{X}}{X} = \frac{1}{\sqrt{X}}$ are.
- So up to a million, about 0.1% are, up to a trillion, about 1 in a million are. So 0% of numbers are perfect squares.

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- This explains 3 and 7. What about 6? The answer has to do with **prime factorizations**.

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- Natural first step: which **prime numbers** are $\square + \square$?

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- So we just have to show primes of the form $4n + 1$ are.
- Any ideas about how to do this?

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A One-Sentence Proof That Every Prime $p \equiv 1 \pmod 4$ Is a Sum of Two Squares

D. Zagier
Department of Mathematics, University of Maryland, College Park, MD 20742

The involution on the finite set $S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$ defined by

$$(x,y,z) \mapsto \begin{cases} (x + 2z,\ z,\ y - x - z) & \text{if } x < y - z \\ (2y - x,\ y,\ x - y + z) & \text{if } y - z < x < 2y \\ (x - 2y,\ x - y + z,\ y) & \text{if } x > 2y \end{cases}$$

has exactly one fixed point, so $|S|$ is odd and the involution defined by $(x, y, z) \mapsto (x, z, y)$ also has a fixed point. $\square$

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- We will try to discover our own proof.

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- For example, any odd number $2n + 1$ is a difference of two squares. Solve $x + y = 2n + 1$, $x - y = 1$ to get $x = n + 1$, $y = n$. Thus, $2n + 1 = (x + y)(x - y) = (n + 1)^2 - n^2$.

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- This doesn't seem to work for us. We need a **bigger number system**.

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- Really great article giving pictures to explain this: "The Fundamental Theorem of Algebra for Artists".

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- What is $(3+5i)(7-2i)$?

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Practicing with complex numbers

- Try computing some complex numbers yourself! Do the following:
- Compute $(3+5i)-(7-2i)$. Answer: Add **real and imaginary parts**: $(3-7)+(5+2)i=-4+7i$.
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$$\frac{3+5i}{7-2i}=\frac{(3+5i)(7+2i)}{(7-2i)(7+2i)}=\frac{21+35i+6i+10i^2}{49-4i^2}=\frac{11+41i}{53}$$

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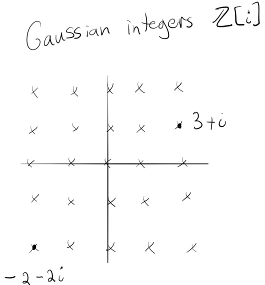

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- Ok, let's check: Its the secret behind our strange identity!

$$N((a+bi)(c+di)) = N((ac-bd) + (ad+bc)i)$$

$$= (ac-bd)^2 + (ad+bc)^2$$

$$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 = (a^2+b^2)(c^2+d^2) = N(a+bi)N(c+di).$$

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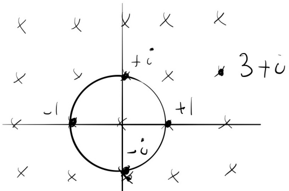

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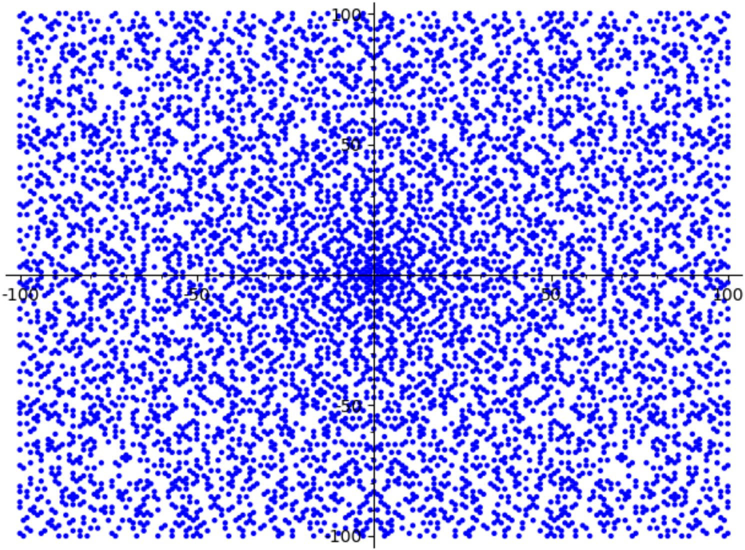
- But lattice you are always at distance less than 1 from a lattice point! Maximal distance is diagonal of square: $\sqrt{2}/2 < 1$.
- This is extremely special! For example, $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ means that the theorem is false for $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

A picture of the Gaussian primes



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Example

$5096 = 2^3 \cdot 13^1 \cdot 7^2$ is a sum of two squares (14^2 and 70^2), but $35672 = 2^3 \cdot 13^1 \cdot 7^3$ is **not**.

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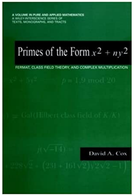

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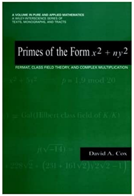

- What other questions can you think of?