Prime numbers

Nashville Math Club

September 17, 2019

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- Can take too long
- There are many interesting questions it doesn't answer!

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Answer: The first 16 primes:

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Question: What kind of prime gaps can occur in general?

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Why does this finish the proof? □

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Question: Can you find examples of primes p such that $p + 2$ and $p + 4$ are both also prime? How many?

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Theorem

The "average" prime gap for p_n is $\ln p_n$.

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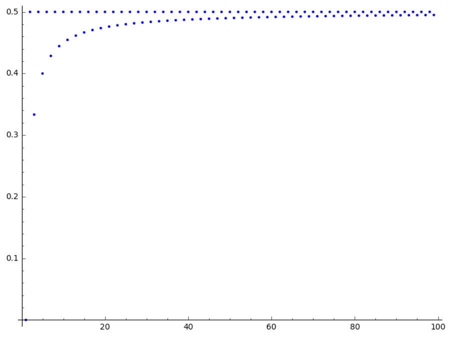

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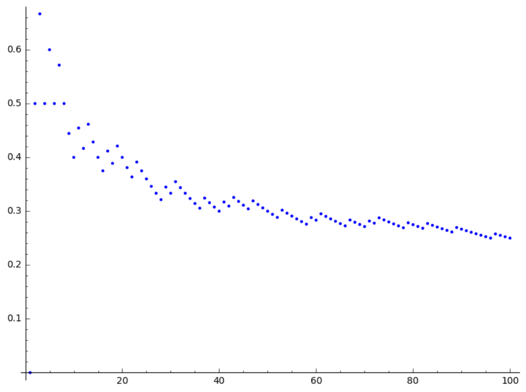

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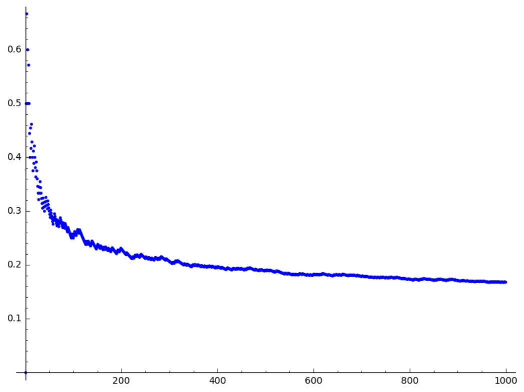

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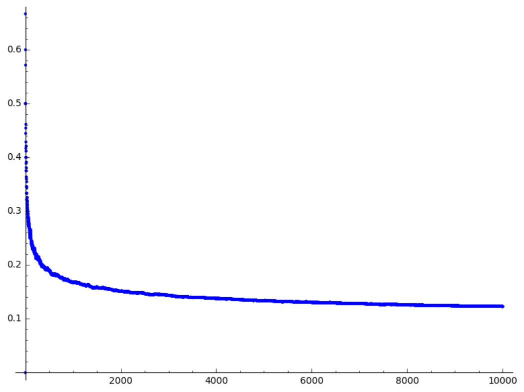

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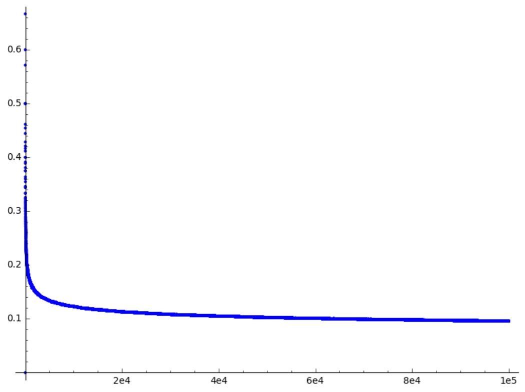

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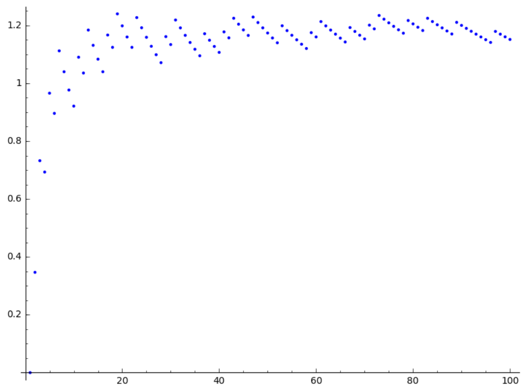

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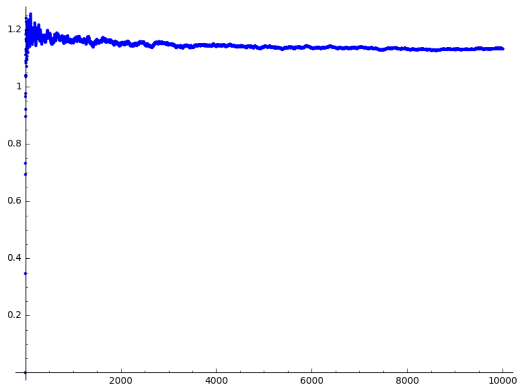

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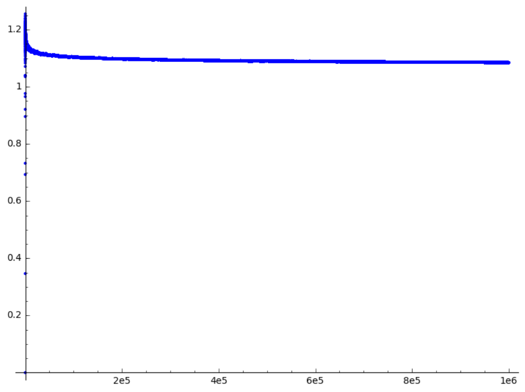

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Question: What does this mean?
Answer: $\frac{1}{\ln n}$ is a better and better approximation for the proportion $\frac{\pi(n)}{n}$ as n gets larger.
Alternate answer: $\frac{n}{\ln n}$ is a better and better approximation for the number of primes $\pi(n)$ up to n as n gets larger.

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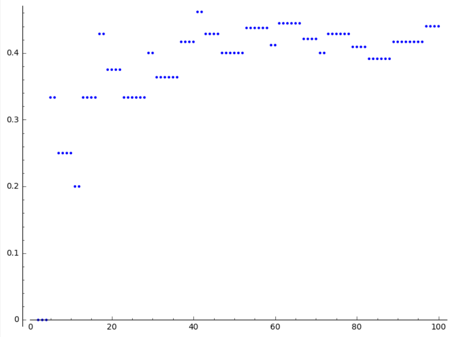

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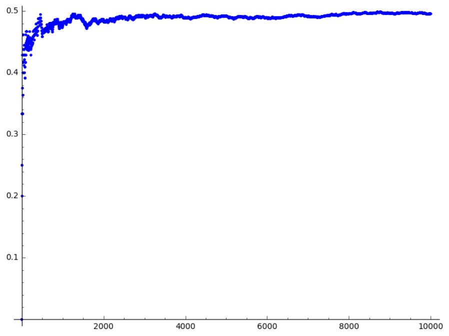

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This is known as *Chebyshev's bias*.